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Research Article

# Lacunary Statistical Convergence of Order $\alpha$ for Generalized Difference Sequences in Linear Partial Metric Space

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**Abstract.** In the present study, with the use of generalized difference operator $\Delta^p$, we have the notion of $\Delta^p$-lacunary statistical $\varphi$-convergence and $\Delta^p$-lacunary strongly $\varphi$-Cesàro summability of order $\alpha$, in partial metric space $(X,\varphi)$, where $\varphi$ is a partial metric on $X$. We also analyse these notions with the fusion of modulus function. In addition we also establish some relationship between these concepts.

**Keywords.** Difference sequence spaces, Lacunary statistical convergence, Partial metric space, Modulus function

**Mathematics Subject Classification (2020).** 46A45, 40A05, 40A35



## 1. Introduction

The present work concerns mainly with the concepts of difference sequence spaces, statistical convergence, lacunary statistical convergence, and partial metric space. In the field of sequence space most of the work is dominated by the sequence of scalars. Through this study, we contribute to sequence space by using sequences via an arbitrary non-empty set $X$, equipped with a partial metric. To go through, let us first recall some basic tools.

Kizmaz [15], added to the field of sequence spaces a new idea of difference sequence spaces by introducing $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ (termed as difference sequence spaces) as follows:

$$E(\Delta) = \{u = (u_m) \in \omega : (\Delta u_m) \in E\}, \quad \text{for } E \in \{\ell_\infty, c, c_0\},$$

where $c_0$, $c$, $\ell_\infty$ are Banach spaces of null, convergent and bounded sequences of scalars, normed by $\|u\|_\infty = \sup_m |u_m|$ and $\omega$ is the space of scalar sequences.

It is observed that $E(\Delta)$ are Banach spaces with the norm

$$\|u\|_\Delta = |u_1| + \|\Delta u\|_\infty, \quad \text{for } u = (u_m) \in E(\Delta) \text{ where } \Delta u = (\Delta u_m) = (u_m - u_{m+1}).$$

Et and Çolak [8] generalized the above concept as follows:

$$E(\Delta^n) = \{u = (u_m) \in \omega : (\Delta^n u_m) \in E\}, \quad \text{for } E \in \{\ell_\infty, c, c_0\},$$

where $\Delta^n u = (\Delta^n u_m) = (\Delta^{n-1} u_m - \Delta^{n-1} u_{m+1})$, for all $m \in \mathbb{N}$ and $\Delta^0 u_m = u_m$. These spaces turn out to be complete when equipped with the norm

$$\|u\|_{\Delta^n} = \sum_{i=1}^{n} |u_i| + \|\Delta^n u\|_\infty, \quad \text{for } u = (u_m) \in E(\Delta^n).$$

In 1951, statistical convergence of real sequences was introduced in short by Fast [9]. Later on, this concept was studied as "convergence in density" by Buck [5] in 1953. It is also a part of monograph by Zygmund [38] and referred as almost convergence. Stienhaus [33], and Schoenberg [30] introduced and studied this concept independently in connection with summability of sequences. Later on this concept of statistical convergence and its various extensions have been explored by many more mathematicians and now it is so vigour and broad that find its applications in different areas of mathematics, such as measure theory, trigonometric series, Fourier series and others. Statistical convergence has its main pillar as natural density, defined as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \operatorname{card}(\{m \in A : m \le n\})$$

provided the limit exists. It is easily verified that $\delta(A) = 0$, for finite subset $A$ of $\mathbb{N}$ and $\delta(A) + \delta(\mathbb{N} - A) = 1$ for every $A \subseteq \mathbb{N}$. For a detailed account of natural density, one may peep into Niven and Zuckerman [23].

**Definition 1.1** ([28]). A scalar sequence $\langle u_m \rangle$ is convergent statistically to $u_0 \in \mathbb{R}$ if for $\varepsilon > 0$,

$$\delta(\{m \in \mathbb{N} : |u_m - u_0| \ge \varepsilon\}) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{card}(\{m \le n : |u_m - u_0| \ge \varepsilon\}) = 0,$$

and $u_0$ is referred as statistical limit of $\langle u_m \rangle$. We write $u_m \to u_0(S)$ and by $S(c)$ we denote the set of all statistically convergent real sequences.

With the passage of time, various generalization of this notion, have been studied by many more mathematicians. One may refer to Bhardwaj and Bala [2], Bhardwaj and Gupta [4], Connor [6, 7], Fridy [11], Gupta and Bhardwaj [14], Li [16], Mohiuddine and Aiyub [19], Mohiuddine and Alghamdi [20], Pehlivan and Fisher [25], Rath and Tripathy [26], Sengül and Et [31], Sharma and Kumar [32], Tripathy [34], Tripathy and Baruah [35], Tripathy and Dutta [36], and Tripathy *et al*. [37].

Before proceeding for lacunary statistical convergence, we recall lacunary sequence and lacunary density.

Following Freedman *et al.* [10], a lacunary sequence $\theta = \langle m_r\rangle_{r=0}^{\infty}$ is an increasing sequence such that $m_r - m_{r-1} \to \infty$, where $m_0 = 0$, $m_r \geq 0$. Here we notate $J_r = (m_{r-1}, m_r]$, $l_r = m_r - m_{r-1}$ and $t_r = \frac{m_r}{m_{r-1}}$.

There is a strong relation between the space $|\sigma_1|$ of strongly Cesàro summable sequences where

$$|\sigma_1| = \left\{\langle u_m\rangle : \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} |u_m - u_0| = 0 \text{ for some scalar } u_0\right\}$$

and the space $N_\theta$, where

$$N_\theta = \left\{\langle u_m\rangle : \lim_{r\to\infty} \frac{1}{l_r}\sum_{m\in J_r} |u_m - u_0| = 0 \text{ for some scalar } u_0\right\}.$$

Fridy and Orhan [13] in 1993 studied a new variant of statistical convergence, named as lacunary statistical convergence that is in the same relation with statistical convergence as $N_\theta$ with $|\sigma_1|$.

**Definition 1.2** ([12]). A real valued sequence $\langle u_m\rangle$ is lacunary statistical convergent to $u_0$ or we can say $u_m \to u_0(S_\theta)$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r} \operatorname{card}(\{m \in J_r : |u_m - u_0| \geq \varepsilon\}) = 0.$$

By $S_\theta(c)$ we notate the class of all lacunary statistical convergent sequences of reals.

Motivating from the definition of absolute value function, i.e., $|a|$

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

Nakano [21] in 1953, structured the image of modulus function. By Ruckle [27] and Maddox [17], a modulus function is a map $f : [0,\infty) \to [0,\infty)$ such that the following holds:

($M_1$) $f(\xi) = 0$ iff $\xi = 0$;

($M_2$) $f(\xi + \eta) \leq f(\xi) + f(\eta)$, for all $\xi \geq 0$, $\eta \geq 0$;

($M_3$) $f$ is monotonically increasing;

($M_4$) $\lim_{\xi\to 0^+} f(\xi) = f(0)$.

As an example, $f_1(\xi) = \frac{\xi}{1+\xi}$ and $f_2(\xi) = \xi^p$, $(0 < p \leq 1)$ are modulus functions where $f_1$ is bounded and $f_2$ is unbounded. It is observed that sum of two modulus functions is again a modulus function.

The credit of introducing the idea of partial metric space goes to Matthews [18] in 1994. Initially the concept of partial metric space was used in the field of computer science, but now it has been extensively used in biological sciences, information science and fixed point theory etc.

**Definition 1.3** ([18]). Let $X \neq \emptyset$. A function $\varphi : X \times X \to \mathbb{R}$ satisfying the following:

($\varphi_1$) $0 \leq \varphi(u,u) \leq \varphi(u,v)$,

($\varphi_2$) $\varphi(u,u) = \varphi(u,v) = \varphi(v,v) \iff u = v$,

($\varphi_3$) $\varphi(u,v) = \varphi(v,u)$,

($\varphi_4$) $\varphi(u,v) \le \varphi(u,w) + \varphi(w,v) - \varphi(w,w)$, for all $u,v,w \in X$, is said to be a partial metric on $X$ and $(X,\varphi)$ is called a partial metric space.

It can be observed in view of axiom ($\varphi_1$) of partial metric space, $|\varphi(u_m,u) - \varphi(u,u)|$ and $\varphi(u_m,u) - \varphi(u,u)$ are the same thing, for any sequence $\langle u_m \rangle$ in $X$ and $u \in X$.

In comparison to a metric on $X$, we can say a metric $\varphi$ is precisely a partial metric $\varphi : X \times X \to \mathbb{R}$ such that, for all $u \in X, \varphi(u,u) = 0$, that is, in the definition of partial metric space, only one side axiom of metric is preserved, i.e., for all $u,v \in X$, $\varphi(u,v) = 0 \Rightarrow u = v$ and other half that is, $u = v \Rightarrow \varphi(u,v) = 0$ need not hold good. For a detailed description of partial metric space, one may refer Bayram *et al.* [1], Bhardwaj and Kumar [3], Neil [22], Nuray [24], and Samet *et al.* [29].

Nuray [24], and Bayram *et al.* [1] stepped into partial metric space via statistical convergence and introduced notion of statistical convergence in partial metric space. We call this notion as statistical $\varphi$-convergence.

**Definition 1.4** ([1,24]). A sequence $\langle u_m \rangle$ in partial metric $(X,\varphi)$ is said to be statistically $\varphi$-convergent to some $u_0 \in X$ if for given $\varepsilon > 0$, $\delta(\{m \in \mathbb{N} : \varphi(u_m,u_0) \ge \varphi(u_0,u_0) + \varepsilon\}) = 0$ and we write it as $u_m \xrightarrow{\varphi} u_0(S)$. By $S(c^{\varphi})$, we notate the class of all statistically $\varphi$-convergent sequence from $(X,\varphi)$.

Throughout the paper, $(X,\varphi)$ and $f$ will denote the partial metric space and modulus function, respectively. In order to justify difference in $X$, we are considering here, $X$ as a linear space, i.e., by partial metric space we mean linear partial metric space.

## 2. Lacunary Statistical $\varphi$-convergence and Strongly $\varphi$-Cesàro Summability of Order $\alpha$ of Difference Sequences

Here we study the lacunary statistical $\varphi$-convergence of order $\alpha$ for difference sequences of an arbitrary partial metric space $(X,\varphi)$ and its relation with lacunary strongly $\varphi$-Cesàro summability of order $\alpha$.

**Definition 2.1.** Let $\langle u_m \rangle$ be sequence in $(X,\varphi)$ and $u_0 \in X$. If for given $\varepsilon > 0$, $\exists$ a positive integer $m_0$ such that following holds:

$$\varphi(\Delta^p u_m, u_0) \le \varphi(u_0,u_0) + \varepsilon, \quad \text{for all } m \ge m_0,$$

then we say $\langle u_m \rangle$ is $(\Delta^p,\varphi)$-convergent to $u_0$. We write $c^{\varphi}(\Delta^p)$ for the class of all $(\Delta^p,\varphi)$-convergent sequences.

**Definition 2.2.** A sequence $\langle u_m \rangle$ in partial metric space $(X,\varphi)$ is said to be $(\Delta^p,\varphi)$-bounded if $\exists$ some $u_0 \in X$ and $M > 0$ such that $\varphi(\Delta^p u_m, u_0) < \varphi(u_0,u_0) + M$, for all $m \ge 1$. We write $\ell_\infty^{\varphi}(\Delta^p)$ as the class of all $(\Delta^p,\varphi)$-bounded sequences.

**Definition 2.3.** A sequence $\langle u_m \rangle$ in partial metric $(X,\varphi)$ is said to be $\Delta^p$-statistically $\varphi$-convergent to some $u_0 \in X$ if for given $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}(\{m \le n : \varphi(\Delta^p u_m, u_0) \ge \varphi(u_0,u_0) + \varepsilon\}) = 0$$

and we write it as $u_m \xrightarrow{\varphi} u_0(S(\Delta^p))$. By $S(\Delta^p, c^{\varphi})$, we notate the class of all $\Delta^p$-statistically $\varphi$-convergent sequence from $(X,\varphi)$.

**Remark 2.4.** From the core of statistical convergence,

(i) It is not difficult to prove $c^{\varphi}(\Delta^p) \subsetneqq \ell_{\infty}^{\varphi}(\Delta^p)$, which is analogous in scalar sense, i.e., $c \subsetneqq \ell_{\infty}$.

(ii) As usual convergence implies statistical convergence so it is routine verification that

$$c^{\varphi}(\Delta^p) \subset S(\Delta^p, c^{\varphi}).$$

**Definition 2.5.** A sequence $\langle u_m \rangle$ in partial metric space $(X,\varphi)$ is said to be $\Delta^p$-statistically $\varphi$-convergent of order $\alpha$ $(0 < \alpha \le 1)$ to $u_0 \in X$ if for $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{n^{\alpha}} \operatorname{card}(\{m \le n : \varphi(\Delta^p u_m, u_0) \ge \varphi(u_0,u_0) + \varepsilon\}) = 0.$$

We write $u_m \xrightarrow{\varphi} u_0(S^{\alpha}(\Delta^p))$ and

$$S^{\alpha}(\Delta^p, c^{\varphi}) = \{\langle u_m \rangle : u_m \in X,\ u_m \xrightarrow{\varphi} u_0(S^{\alpha}(\Delta^p)),\ \text{for some } u_0 \in X\}.$$

For $\alpha = 1$, we call a $\Delta^p$-statistically $\varphi$-convergent sequence of order $\alpha$ simply as a $\Delta^p$-statistically $\varphi$-convergent sequence and corresponding space is denoted by $S(\Delta^p, c^{\varphi})$.

**Definition 2.6.** Let $\theta = \langle m_r \rangle$ be a lacunary sequence and $0 < \alpha \le 1$. A sequence $\langle u_m \rangle$ in partial metric space $(X,\varphi)$ is said to be $\Delta^p$-lacunary statistically $\varphi$-convergent of order $\alpha$ to $u_0 \in X$ if for $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r^{\alpha}} \operatorname{card}(\{m \in J_r : \varphi(\Delta^p u_m, u_0) \ge \varphi(u_0,u_0) + \varepsilon\}) = 0.$$

We write $u_m \xrightarrow{\varphi} u_0(S_{\theta}^{\alpha}(\Delta^p))$ and

$$S_{\theta}^{\alpha}(\Delta^p, c^{\varphi}) = \{\langle u_m \rangle : u_m \in X,\ u_m \xrightarrow{\varphi} u_0(S_{\theta}^{\alpha}(\Delta^p)),\ \text{for some } u_0 \in X\}.$$

**Definition 2.7.** Let $\langle u_m \rangle$ be a sequence in partial metric space $(X,\varphi)$ and $0 < \alpha \le 1$. The sequence $\langle u_m \rangle$ is $\Delta^p$-strongly $\varphi$-Cesàro summable of order $\alpha$ to $u_0 \in X$ if

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{m=1}^{n} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)| = 0.$$

We notate $|\sigma_1|^{\alpha}(\Delta^p, c^{\varphi})$ for the set of all $\Delta^p$-strongly $\varphi$-Cesàro summable sequences of order $\alpha$, i.e.,

$$|\sigma_1|^{\alpha}(\Delta^p, c^{\varphi}) = \left\{ \langle u_m \rangle : \lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{m=1}^{n} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)| = 0, \text{for some } u_0 \in X \right\}.$$

**Definition 2.8.** Let $\langle u_m \rangle$ be a sequence in partial metric space $(X,\varphi)$ and $0 < \alpha \le 1$. The sequence $\langle u_m \rangle$ is $\Delta^p$-lacunary strongly $\varphi$-Cesàro summable of order $\alpha$ to $u_0 \in X$ if

$$\lim_{r\to\infty} \frac{1}{l_r^{\alpha}} \sum_{m\in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)| = 0.$$

We use the notation $N_{\theta}^{\alpha}(\Delta^p, c^{\varphi})$ for the set of all $\Delta^p$-lacunary strongly $\varphi$-Cesàro summable sequences of order $\alpha$, i.e.,

$$N_{\theta}^{\alpha}(\Delta^p, c^{\varphi}) = \left\{ \langle u_m \rangle : \lim_{r\to\infty} \frac{1}{l_r^{\alpha}} \sum_{m\in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)| = 0,\ \text{for some } u_0 \in X \right\}$$

and we write $u_m \xrightarrow{\varphi} u_0(N_{\theta}^{\alpha}(\Delta^p))$.

**Theorem 2.9.** *Let $\alpha,\beta\in(0,1]$ such that $\alpha\le\beta$. Then $S^{\alpha}_{\theta}(\Delta^p,c^{\varphi})\subset S^{\beta}_{\theta}(\Delta^p,c^{\varphi})$, inclusion is strict.*

*Proof.* For given $\varepsilon>0$, we have

$$\begin{aligned}0&\le\frac{1}{l_r^{\beta}}\operatorname{card}(\{m\in J_r:|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|\ge\varepsilon\})\\&\le\frac{1}{l_r^{\alpha}}\operatorname{card}(\{m\in J_r:|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|\ge\varepsilon\}).\end{aligned}$$

Taking limit $r\to\infty$, we obtained required result.

For inclusion to be strict, consider the following example:

Let $X=\mathbb{R}$ with partial metric $\varphi$ defined as $\varphi(\xi,\eta)=|\xi-\eta|$, $\xi,\eta\in\mathbb{R}$. Construct a sequence $u=\langle u_m\rangle$ such that

$$\Delta^p u_m=\left.\begin{cases}[\sqrt{l_r}], & \text{at the first } [\sqrt{l_r}] \text{ integers on } J_r,\\ 0, & \text{otherwise,}\end{cases}\right\}\quad\text{for all } r=1,2,3,\dots.$$

This implies, $\operatorname{card}(\{m\in J_r:|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|\ge\varepsilon\})\le[\sqrt{l_r}]$. Thus, if we consider $\frac{1}{2}<\beta\le1$, then we have

$$\lim_{r\to\infty}\frac{1}{l_r^{\beta}}\operatorname{card}(\{m\in J_r:|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|\ge\varepsilon\})\le\lim_{r\to\infty}\frac{[\sqrt{l_r}]}{l_r^{\beta}}\longrightarrow0.$$

On the other hand for $0<\alpha<\frac{1}{2}$, we have

$$\lim_{r\to\infty}\frac{1}{l_r^{\alpha}}\operatorname{card}(\{m\in J_r:|\varphi(\Delta^p u_m,0)-\varphi(0,0)|\ge\varepsilon\})\le\lim_{r\to\infty}\frac{[\sqrt{l_r}]}{l_r^{\alpha}}\nrightarrow0.$$

Hence inclusion is proper for $\alpha<\beta$ with $0<\alpha<\frac{1}{2}$ and $\frac{1}{2}<\beta\le1$. □

**Corollary 2.10.** *Let $\theta=\langle u_m\rangle$ be lacunary sequence and $0<\alpha\le\beta\le1$. Then, we have following:*

(i) $S^{\alpha}_{\theta}(\Delta^p,c^{\varphi})=S^{\beta}_{\theta}(\Delta^p,c^{\varphi})$ *iff* $\alpha=\beta$,

(ii) $S^{\alpha}_{\theta}(\Delta^p,c^{\varphi})=S_{\theta}(\Delta^p,c^{\varphi})$ *iff* $\alpha=1$.

**Theorem 2.11.** *For $\alpha\in(0,1]$, we have the following:*

(i) *$N^{\alpha}_{\theta}(\Delta^p,c^{\varphi})\subset S^{\alpha}_{\theta}(\Delta^p,c^{\varphi})$, i.e., every $\Delta^p$-lacunary strongly $\varphi$-Cesàro summable sequence of order $\alpha$ is $\Delta^p$-lacunary statistically $\varphi$-convergent of order $\alpha$ to same limit and inclusion is proper.*

(ii) *If $\lim_{r\to\infty}\frac{l_r}{l_r^{\alpha}}=1$, then $S^{\alpha}_{\theta}(\Delta^p,c^{\varphi})\cap\ell^{\varphi}_{\infty}(\Delta^p)=N^{\alpha}_{\theta}(\Delta^p,c^{\varphi})\cap\ell^{\varphi}_{\infty}(\Delta^p)$.*

*Proof.* (i) For $\varepsilon>0$ and $0<\alpha\le1$, we have

$$\begin{aligned}\frac{1}{l_r^{\alpha}}\sum_{m\in J_r}|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|&\ge\frac{1}{l_r^{\alpha}}\sum_{\substack{m\in J_r\\|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|>\varepsilon}}|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|\\&\ge\varepsilon.\frac{1}{l_r^{\alpha}}\operatorname{card}(\{m\in J_r:|\varphi(\Delta^p u_m,u_0)-\varphi(u_0,u_0)|\ge\varepsilon\}.\end{aligned}$$

and hence the result follows by above inequality.

For proper inclusion, consider the following example:

Let $X = \mathbb{R}$ and $\varphi$ be the partial metric defined by $\varphi(\xi,\eta) = |\xi - \eta|$, $\xi,\eta \in \mathbb{R}$. Construct a sequence $u = \langle u_m \rangle$ such that

$$\Delta^p u_m = \left.\begin{cases} 1,2,\ldots,[l_r^{\frac{\alpha}{2}}], & \text{at the first } [l_r^{\frac{\alpha}{2}}] \text{ integers on } J_r, \\ 0, & \text{otherwise,} \end{cases}\right\} \text{ for all } r = 1,2,3\ldots,$$

where $[\cdot]$ denote the greatest integer function. Then for every $\varepsilon > 0$,

$$\frac{1}{l_r^\alpha}\text{card}(\{m \in J_r : |\varphi(\Delta^p u_m, 0) - \varphi(0,0)| \geq \varepsilon\}) \leq \frac{1}{l_r^\alpha}[l_r^{\frac{\alpha}{2}}] \to 0 \text{ as } r \to \infty,$$

and so $\langle u_m \rangle \in S_\theta^\alpha(\Delta^p, c^\varphi)$.

On the other hand,

$$\begin{aligned}\frac{1}{l_r^\alpha}\sum_{m\in J_r}|\varphi(\Delta^p u_m,0) - \varphi(0,0)| &= \frac{1}{l_r^\alpha}\sum_{m\in J_r}|\Delta^p u_m - 0| \\ &= \frac{1}{l_r^\alpha}[1+2+\ldots+[l_r^{\frac{\alpha}{2}}]] \\ &= \frac{1}{l_r^\alpha}\left(\frac{[l_r^{\frac{\alpha}{2}}]([l_r^{\frac{\alpha}{2}}]+1)}{2}\right) \to \frac{1}{2} \neq 0 \text{ as } r \to \infty,\end{aligned}$$

and this implies $\langle u_m \rangle \notin N_\theta^\alpha(\Delta^p, c^\varphi)$.

(ii) Let $\langle u_m \rangle \in S_\theta^\alpha(\Delta^p, c^\varphi) \cap \ell_\infty^\varphi(\Delta^p)$. Consider

$$\begin{aligned}&\frac{1}{l_r^\alpha}\sum_{m\in J_r}|\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| \\ &= \frac{1}{l_r^\alpha}\left(\sum_I |\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| + \sum_{II}|\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)|\right),\end{aligned}$$

here $\sum_I$ represent the sum over $m \in J_r$ with $|\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| < \varepsilon$ and $\sum_{II}$ is the sum over those $m \in J_r$ for which $|\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| \geq \varepsilon$.

As $\langle u_m \rangle \in \ell_\infty^\varphi(\Delta^p)$, so there exist some $b \in X$ and $M > 0$ such that

$$\varphi(\Delta^p u_m, b) < \varphi(b,b) + M, \quad \text{for all } m \geq 1.$$

Now $\varphi(\Delta^p u_m,u_0) \leq \varphi(\Delta^p u_m,b) + \varphi(b,u_0) - \varphi(b,b) \leq \varphi(b,b) + M + \varphi(u_0,b) - \varphi(b,b)$.

Let $\lambda = M + \varphi(u_0,b) - \varphi(u_0,u_0)$, then we have, $\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0) \leq \lambda$, for all $k \geq 1$, this implies

$$\begin{aligned}&\frac{1}{l_r^\alpha}\sum_{m\in J_r}|\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| \\ &\leq \frac{1}{l_r^\alpha}\left(\sum_I |\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| + \lambda \cdot \text{card}(\{m \in J_r : |\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| \geq \varepsilon\right) \\ &\leq \frac{\varepsilon}{l_r^\alpha} l_r + \frac{\lambda}{l_r^\alpha}\text{card}(\{m \in J_r : |\varphi(\Delta^p u_m,u_0) - \varphi(u_0,u_0)| \geq \varepsilon\})\end{aligned}$$

and taking limit $r \to \infty$ in above inequality and using $\lim_{r\to\infty}\frac{l_r}{l_r^\alpha} = 1$, the result follows. □

**Theorem 2.12.** *For $0 < \alpha \le 1$, the following holds:*

(i) *If* $\underline{\lim}\, t_r > 1$, *then* $S^\alpha(\Delta^p, c^\varphi) \subseteq S^\alpha_\theta(\Delta^p, c^\varphi)$.

(ii) *If* $\overline{\lim} \frac{m_r}{m^\alpha_{r-1}} < \infty$, *then* $S^\alpha_\theta(\Delta^p, c^\varphi) \subseteq S^\alpha(\Delta^p, c^\varphi)$.

(iii) *If* $\underline{\lim} \frac{l^\alpha_r}{m_r} > 0$, *then we have* $S(\Delta^p, c^\varphi) \subseteq S^\alpha_\theta(\Delta^p, c^\varphi)$.

*Proof.* (i) Let $\underline{\lim}\, t_r > 1$. Then, for sufficiently large $r$, $\exists\ \delta > 0$ such that $t_r > 1 + \delta$. As $t_r = \frac{m_r}{m_{r-1}}$, so $\frac{l_r}{m_{r-1}} \ge \delta$. This gives $\frac{m_{r-1}}{l_r} \le \frac{1}{\delta}$. After adding 1 to both sides, we have $\frac{m_r}{l_r} \le \frac{1+\delta}{\delta}$, i.e., $\frac{1}{m^\alpha_r} \ge \frac{\delta^\alpha}{(1+\delta)^\alpha} \frac{1}{l^\alpha_r}$.

For $\varepsilon > 0$ and sufficiently large $r$, we have

$$\begin{aligned}
&\frac{1}{m^\alpha_r} \operatorname{card}(\{m < m_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\}) \\
&\quad \ge \frac{1}{m^\alpha_r} \operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\}) \\
&\quad \ge \frac{\delta^\alpha}{(1+\delta)^\alpha} \frac{1}{l^\alpha_r} \operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\})
\end{aligned}$$

and hence the result.

(ii) Let $\limsup_r \frac{m_r}{m^\alpha_{r-1}} < \infty$. Then, there exists $M > 0$ such that $\frac{m_r}{m^\alpha_{r-1}} < M$, for all $r \ge 1$. Let us suppose, that $\langle u_m \rangle \in S^\alpha_\theta(\Delta^p, c^\varphi)$. Then, for $u_0 \in X$ and $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l^\alpha_r} \operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\}) = 0, \text{ i.e., } \lim_{r\to\infty} \frac{M_r}{l^\alpha_r} = 0,$$

where $M_r = \operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\})$. So for given $\varepsilon > 0$, $\exists\ r_0 \in \mathbb{N}$ such that $\frac{M_r}{l^\alpha_r} < \varepsilon$, for all $r > r_0$. Let $T = \sup\{M_r : 1 \le r \le r_0\}$ and $n$ be any integer satisfying $m_{r-1} < n \le m_r$. Then, we have

$$\begin{aligned}
&\frac{1}{n^\alpha} \operatorname{card}(\{m \le n : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\}) \\
&\quad \le \frac{1}{m^\alpha_{r-1}} \operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\}) \\
&\quad = \frac{1}{m^\alpha_{r-1}} \{M_1 + M_2 + \ldots + M_{r_0} + M_{r_0+1} + \ldots + M_r\} \\
&\quad \le \frac{r_0 T}{m^\alpha_{r-1}} + \frac{1}{m^\alpha_{r-1}} \{M_{r_0+1} + M_{r_0+2} + \ldots + M_r\} \\
&\quad = \frac{r_0 T}{m^\alpha_{r-1}} + \frac{1}{m^\alpha_{r-1}} \left\{ l_{r_0+1} \frac{M_{r_0+1}}{l_{r_0+1}} + l_{r_0+2} \frac{M_{r_0+2}}{l_{r_0+2}} + \ldots + l_r \frac{M_r}{l_r} \right\},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{n^\alpha} \operatorname{card}(\{m \le n : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \ge \varepsilon\}) \\
&\quad \le \frac{r_0 T}{m^\alpha_{r-1}} + \frac{1}{m^\alpha_{r-1}} \left( \sup_{r > r_0} \frac{M_r}{l_r} \right) \{l_{r_0+1} + l_{r_0+2} + \ldots + l_r\} \\
&\quad \le \frac{r_0 T}{m^\alpha_{r-1}} + \frac{1}{m^\alpha_{r-1}} \varepsilon (m_r - m_{r_0})
\end{aligned}$$

$$= \frac{r_0 T}{m_{r-1}^{\alpha}} + \varepsilon\left(\frac{m_r}{m_{r-1}^{\alpha}} - \frac{m_{r_0}}{m_{r-1}^{\alpha}}\right)$$
$$\leq \frac{r_0 T}{m_{r-1}^{\alpha}} + \varepsilon\left(\frac{m_r}{m_{r-1}^{\alpha}}\right)$$
$$\leq \frac{r_0 T}{m_{r-1}^{\alpha}} + \varepsilon \cdot M,$$

and hence by applying limit in above inequality we get the result.

(iii) Let $\varepsilon > 0$ and $u_0 \in X$. Then, we have

$$\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \geq \varepsilon\} \subseteq \{m \leq m_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \geq \varepsilon\},$$

this implies

$$\frac{1}{m_r}\operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \geq \varepsilon\})$$
$$\leq \frac{1}{m_r}\operatorname{card}(\{m \leq m_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \geq \varepsilon\}),$$

i.e.,

$$\frac{l_r^{\alpha}}{m_r}\frac{1}{l_r^{\alpha}}\operatorname{card}(\{m \in J_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \geq \varepsilon\})$$
$$\leq \frac{1}{m_r}\operatorname{card}(\{m \leq m_r : |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)| \geq \varepsilon\}).$$

Taking limit $r \to \infty$ in above inequality, we get result. □

**Theorem 2.13.** *For* $0 < \alpha \leq 1$, *following holds:*

(i) *If* $\underline{\lim}\, t_r > 1$, *then* $|\sigma_1|^{\alpha}(\Delta^p, c^{\varphi}) \subseteq N_{\theta}^{\alpha}(\Delta^p, c^{\varphi})$.

(ii) *If* $\overline{\lim}\, \frac{m_r}{m_{r-1}^{\alpha}} < \infty$, *then* $N_{\theta}^{\alpha}(\Delta^p, c^{\varphi}) \subseteq |\sigma_1|^{\alpha}(\Delta^p, c^{\varphi})$.

(iii) *If* $\underline{\lim}\, \frac{l_r^{\alpha}}{m_r} > 0$, *then we have* $|\sigma_1|(\Delta^p, c^{\varphi}) \subset N_{\theta}^{\alpha}(\Delta^p, c^{\varphi})$.

*Proof.* (i) Let $u \in |\sigma_1|^{\alpha}(\Delta^p, c^{\varphi})$. Then, for given $\varepsilon > 0$ and sufficiently large $r$, as in part (i) of Theorem 2.12, we have

$$\frac{1}{m_r^{\alpha}} \geq \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}}\frac{1}{l_r^{\alpha}}.$$

Now consider

$$\frac{1}{m_r^{\alpha}}\left(\sum_{m \leq m_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)|\right) \geq \frac{1}{m_r^{\alpha}}\left(\sum_{m \in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)|\right)$$
$$\geq \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}}\frac{1}{l_r^{\alpha}}\left(\sum_{m \in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)|\right)$$

and hence result follows by taking limit $r \to \infty$ in above inequality.

(ii) Let $u \in N_{\theta}^{\alpha}(\Delta^p, c^{\varphi})$ and $n$ be any integer such that $m_{r-1} < n \leq m_r$. Then, we have

$$\frac{1}{n^{\alpha}}\left(\sum_{k=1}^{n} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)|\right) < \frac{1}{m_{r-1}^{\alpha}}\left(\sum_{k=1}^{m_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)|\right).$$

Taking $M_r = \sum_{m\in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)|$ and proceeding same as that in part (ii) of Theorem 2.12, we get required inclusion.

(iii) Let $u \in |\sigma_1|(\Delta^p, c^\varphi)$ and $u_0 \in X$. Then, we have

$$\frac{1}{m_r}\left(\sum_{m\in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)|\right) \le \frac{1}{m_r}\left(\sum_{m=1}^{m_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)|\right),$$

i.e.,

$$\frac{l_r^\alpha}{m_r}\frac{1}{l_r^\alpha}\left(\sum_{m\in J_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)|\right) \le \frac{1}{m_r}\left(\sum_{k=1}^{m_r} |\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)|\right).$$

Taking limit $r \to \infty$ and using $\underline{\lim} \frac{l_r^\alpha}{m_r} > 0$ in above inequality, we get required inclusion. □

## 3. Lacunary Summability via Modulus Function

In the present section we gave a direction to the lacunary strongly $\varphi$-Cesàro summability of order $\alpha$ (discussed in Section 2) towards a more general concept, with the fusion of modulus function $f$ and hence having the notion of $(\Delta^p, f)$-lacunary strongly $\varphi$-Cesàro summability of order $\alpha$ in the form of following:

**Definition 3.1.** Let $\theta = \langle m_r\rangle$ be a lacunary sequence, $\alpha \in (0,1]$ and $f$ be a modulus function. We say that the sequence $\langle u_m\rangle$ is $(\Delta^p, f)$-lacunary strongly $\varphi$-Cesàro summable of order $\alpha$ to $u_0 \in X$ if

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)) = 0$$

and class of all such sequences is denoted by $N_\theta^\alpha(\Delta^p, f, c^\varphi)$, i.e.,

$$N_\theta^\alpha(\Delta^p, f, c^\varphi) = \left\{\langle u_m\rangle : \lim_{r\to\infty} \frac{1}{l_r^\alpha} \sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)) = 0, \text{ for some } u_0 \in X\right\}.$$

**Theorem 3.2.** *Let $\theta$ be lacunary sequence. Then, $N_\theta(\Delta^{p-1}, f, c^\varphi) \subset N_\theta(\Delta^p, f, c^\varphi)$ and inclusion is proper. In general $N_\theta(\Delta^i, f, c^\varphi) \subset N_\theta(\Delta^p, f, c^\varphi)$, for $i = 1,2,\dots,p-1$ and inclusion is proper.*

*Proof.* Using the standard technique proof is easy and hence omitted. □

**Theorem 3.3.** *For $\alpha \in (0,1]$, $N_\theta^\alpha(\Delta^p, f, c^\varphi) \subseteq S_\theta^\alpha(\Delta^p, c^\varphi)$.*

*Proof.* Let $u = \langle u_m\rangle \in N_\theta^\alpha(\Delta^p, f, c^\varphi)$. Then, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)) = 0.$$

Now

$$\sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0)) \ge \sum_{\substack{m\in J_r \\ \varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0) > \varepsilon}} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0))$$
$$\ge f(\varepsilon)\operatorname{card}(\{m \in J_r : \varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0) \ge \varepsilon\}$$

and hence the result follows by multiplying the above inequality $\frac{1}{l_r^\alpha}$ and taking limit $r \to \infty$. □

Theorem 3.3, for modulus function $f(u) = u$, agrees with Theorem 2.11 and we have

**Corollary 3.4.** $N_\theta^\alpha(\Delta^p, c^\varphi) \subseteq S_\theta^\alpha(\Delta^p, c^\varphi)$.

**Lemma 3.5** ([25])**.** *For modulus $f$ and $0 < \delta < 1$, we have for each $t > \delta$,*

$$f(t) \le 2f(1)\delta^{-1}t.$$

**Theorem 3.6.** *For modulus $f$ and $\lim_{r\to\infty} \frac{l_r}{l_r^\alpha} = 1$, $N_\theta^\alpha(\Delta^p, c^\varphi) \subseteq N_\theta^\alpha(\Delta^p, f, c^\varphi)$.*

*Proof.* Let $u = \langle u_m \rangle \in N_\theta^\alpha(\Delta^p, c^\varphi)$. Then, we have $M_r = \frac{1}{l_r^\alpha} \sum_{m\in J_r} (\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \to 0$ as $r \to \infty$ for some $u_0 \in X$. Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$, (this is possible due to continuity of modulus function $f$ at 0). Then, we have

$$\begin{aligned}\frac{1}{l_r^\alpha} \sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) &= \frac{1}{l_r^\alpha} \sum_{\substack{m\in J_r \\ \varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0) \le \delta}} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \\ &\quad + \frac{1}{l_r^\alpha} \sum_{\substack{m\in J_r \\ \varphi(\Delta^p u_m, u_0) - \varphi(u_0,u_0) > \delta}} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \\ &\le \frac{1}{l_r^\alpha}(l_r \cdot \varepsilon) + 2f(1)\delta^{-1} M_r \quad \text{(using Lemma 3.5)}\end{aligned}$$

and the result follows. □

**Theorem 3.7.** *Let $f_1$ and $f_2$ be modulus functions and $\lim_{r\to\infty} \frac{l_r}{l_r^\alpha} = 1$. Then following holds:*

(i) $N_\theta^\alpha(\Delta^p, f_2, c^\varphi) \subset N_\theta^\alpha(\Delta^p, f_1 \circ f_2, c^\varphi)$.

(ii) $N_\theta^\alpha(\Delta^p, f_1, c^\varphi) \cap N_\theta^\alpha(\Delta^p, f_2, c^\varphi) \subset N_\theta^\alpha(\Delta^p, f_1 + f_2, c^\varphi)$.

*Proof.* (i) Let $u = \langle u_m \rangle \in N_\theta^\alpha(\Delta^p, f_2, c^\varphi)$. As $f_1$ (being modulus function) is continuous at 0, so for given $\varepsilon > 0$ we may choose $\delta > 0$ $(0 < \delta < 1)$ such that $f_1(t) < \varepsilon$ for $0 \le t \le \delta$. Write $z_m = f_2(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0))$. Now

$$\begin{aligned}\frac{1}{l_r^\alpha} \sum_{m\in J_r} f_1(z_m) &= \frac{1}{l_r^\alpha} \sum_{\substack{m\in J_r \\ z_m \le \delta}} f_1(z_m) + \frac{1}{l_r^\alpha} \sum_{\substack{m\in J_r \\ z_m > \delta}} f_1(z_m) \\ &\le \frac{1}{l_r^\alpha}(l_r \varepsilon) + \frac{1}{l_r^\alpha} 2f_1(1)\delta^{-1} \sum_{m\in J_r} z_m \quad \text{(using Lemma 3.5)}\end{aligned}$$

and the result follows.

(ii) Let $y_m = \varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)$. Proof of this part follows by the inequality

$$(f_1 + f_2)(y_m) \le f_1(y_m) + f_2(y_m).$$

□

**Corollary 3.8.** *If $\lim_{r\to\infty} \frac{l_r}{l_r^\alpha} = 1$, then $N_\theta^\alpha(\Delta^p, c^\varphi) \subset N_\theta^\alpha(\Delta^p, f, c^\varphi)$.*

## 4. Results on Lacunary Refinement

The present section concludes the paper by showing various inclusion relations, which arises for varying lacunary sequences $\theta$.

**Definition 4.1.** By lacunary refinement $\theta^* = \langle m_r^* \rangle$ of a lacunary sequence $\theta = \langle m_r \rangle$ we mean $J_r^* \supseteq J_r$ where $J_r^* = (m_{r-1}^*, m_r^*]$ and $J_r = (m_{r-1}, m_r]$.

We use $l_r^* = m_r^* - m_{r-1}^*$ throughout this section.

**Theorem 4.2.** *$\langle u_m \rangle \notin N_\theta(\Delta^p, f, c^\varphi)$ implies $\langle u_m \rangle \notin N_{\theta^*}(\Delta^p, f, c^\varphi)$.*

*Proof.* Let $\langle u_m \rangle \notin N_\theta(\Delta^p, f, c^\varphi)$. Then, for any $u_0 \in X$,

$$\lim_{r\to\infty} \frac{1}{l_r} \sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \neq 0.$$

So there exists $\varepsilon > 0$ and a subsequence $\langle m_{r_j} \rangle$ of $\langle m_r \rangle$ such that

$$\frac{1}{l_{r_j}} \sum_{m\in J_{r_j}} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \geq \varepsilon.$$

Writing $J_{r_j} = J_{t+1}^* \cup J_{t+2}^* \cup \ldots \cup J_{t+p}^*$, then we have

$$\frac{\sum_{J_{t+1}^*} f(\varphi(\Delta^p u_i, u_0) - \varphi(u_0, u_0)) + \ldots + \sum_{J_{t+p}^*} f(\varphi(\Delta^p u_i, u_0) - \varphi(u_0, u_0))}{l_{t+1}^* + \ldots + l_{t+p}^*} \geq \varepsilon$$

implies for some $j$, we have $\frac{1}{l_{t+j}^*} \sum_{J_{t+j}^*} f(\varphi(\Delta^p u_i, u_0) - \varphi(u_0, u_0)) \geq \varepsilon$ and hence

$$\langle u_m \rangle \notin N_{\theta^*}(\Delta^p, f, c^\varphi).$$ □

**Theorem 4.3.** *Let $0 < \alpha \leq \beta \leq 1$ and $\liminf_{r\to\infty} \frac{l_r}{l_r^*} > 0$. Then $S_{\theta^*}^\alpha(\Delta^p, f, c^\varphi) \subseteq S_\theta^\beta(\Delta^p, f, c^\varphi)$.*

*Proof.* As $J_r^* \supseteq J_r$ for all $r \in \mathbb{N}$, so for $\varepsilon > 0$, we have

$$\{m \in J_r^* : f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \geq \varepsilon\} \supseteq \{m \in J_r : f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \geq \varepsilon\}.$$

This implies

$$\begin{aligned}
&\frac{1}{l_r^{*\alpha}} \operatorname{card}(\{m \in J_r^* : f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \geq \varepsilon\}) \\
&\quad \geq \frac{1}{l_r^{*\alpha}} \operatorname{card}(\{m \in J_r : f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \geq \varepsilon\}) \\
&\quad \geq \frac{1}{l_r^{*\beta}} \operatorname{card}(\{m \in J_r : f\left(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)\right) \geq \varepsilon\}) \\
&\quad = \frac{l_r^\beta}{l_r^{*\beta}} \frac{1}{l_r^\beta} \operatorname{card}(\{m \in J_r : f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) \geq \varepsilon\}).
\end{aligned}$$

Hence the proof. □

**Corollary 4.4.** *If $0 < \alpha \leq 1$ and $\underline{\lim} \frac{l_r}{l_r^*} > 0$, then*

(i) $S_{\theta^*}^\alpha(\Delta^p, f, c^\varphi) \subseteq S_\theta(\Delta^p, f, c^\varphi)$.

(ii) $S_{\theta^*}^\alpha(\Delta^p, c^\varphi) \subseteq S_\theta^\beta(\Delta^p, c^\varphi)$.

**Theorem 4.5.** *Let $0 < \alpha \leq \beta \leq 1$. If $\underline{\lim}_{r\to\infty} \frac{l_r}{l_r^*} > 0$, then we have $N_{\theta^*}^\alpha(\Delta^p, f, c^\varphi) \subseteq N_\theta^\beta(\Delta^p, f, c^\varphi)$*

*Proof.* Let $u \in N^{\alpha}_{\theta^*}(\Delta^p, f, c^{\varphi})$. Then, for given $\varepsilon > 0$ and $u_0 \in X$, we have

$$\frac{1}{l_r^{*\alpha}}\left(\sum_{m\in J_r^*} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0))\right)$$
$$= \frac{1}{l_r^{*\alpha}}\left(\sum_{k\in J_r^*-J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0)) + \sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0))\right)$$
$$\geq \frac{l_r^{\beta}}{l_r^{*\alpha}}\frac{1}{l_r^{\beta}}\left(\sum_{m\in J_r} f(\varphi(\Delta^p u_m, u_0) - \varphi(u_0, u_0))\right)$$

and hence the result follows by taking limit $r \to \infty$ and using $\underline{\lim}\, \frac{l_r}{l_r^*} > 0$ in above inequality. □

**Corollary 4.6.** *If* $\underline{\lim}_{r\to\infty} \frac{l_r}{l_r^*} > 0$, *then following holds:*

(i) $N^{\alpha}_{\theta^*}(\Delta^p, f, c^{\varphi}) \subseteq N_{\theta}(\Delta^p, f, c^{\varphi})$.

(ii) $N^{\alpha}_{\theta^*}(\Delta^p, c^{\varphi}) \subseteq N^{\alpha}_{\theta}(\Delta^p, c^{\varphi})$.

## 5. Conclusion

We have introduced the notions of lacunary statistical convergence and strong Cesàro summability of order $\alpha$, for higher order difference sequence spaces in context to the linear partial metric space $(X, \varphi)$ in the present paper. It is also proved that for $(\Delta^p, \varphi)$-bounded sequences, both these notions coincide, i.e., every lacunary statistical $\varphi$-convergent sequence of order $\alpha$ is lacunary strongly $\varphi$-Cesàro summable and conversely. Beside this, various inclusion relations have been established. In Section 3, lacunary strongly $\varphi$-Cesàro summability of order $\alpha$ is studied with the aid of modulus function $f$, hence having the notion of $(\Delta^p, f)$-lacunary strongly $\varphi$-Cesàro summability of order $\alpha$. The last section shows inclusion among the already existing spaces which arises for varying lacunary sequence $\theta$.

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The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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