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Research Article

Connected Certified Domination in Graphs: Properties and Algorithmic Aspects

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Abstract. A certified dominating set D of a graph $\Gamma = (V_\Gamma, E_\Gamma)$ denoted by γ_{cer}-set, is the subset of V_Γ such that $|N(u) \cap (V_\Gamma - D)|$ is either 0 or 2, $\forall\ u \in D$. A set $\mathcal{D}_c \subseteq V_\Gamma$ is called a connected certified dominating set (γ^c_{cer}-set) if $\mathcal{D}_c$ is γ_{cer}-set and $\Gamma[\mathcal{D}_c]$ is connected. Also, if $\mathcal{D}_c$ has no proper subset, then it is a smallest γ^c_{cer}-set, and the cardinality of the smallest γ^c_{cer}-set is called as connected certified domination number (CCDN) of the graph Γ represented by $\gamma^c_{cer}(\Gamma)$. In this article we continue the study of connected certified domination. Herein, we classify graphs with larger values of CCDN and then we will study some properties of connected certified domination. Moreover, we will provide upper bounds and Nordhaus-Gaddum results for the CCDN. Additionally, we will prove that the connected certified domination problem is NP-complete for star convex bipartite graphs, comb convex bipartite graphs, and planar graphs.

Keywords. Dominating set, Certified Dominating Set (CFDS), Connected Certified Dominating Set (CCDS), Nordhaus-Gaddum results

Mathematics Subject Classification (2020). 05CXX, 05C69

1. Introduction

The idea of certified domination was presented by Dettlaff *et al.* [4] in 2020, and further studies are going on this domination parameter (see, for example, Dettlaff *et al.* [4, 5], Goswami *et al.* [11], Ilyass and Goswami [17], Jakkepalli *et al.* [18], Lone and Goswami [21], Raj and Kumari [24] for recent articles on this topic). A CFDS D of a graph Γ signified by γ_{cer}-set, is the subset of V_Γ such that $|N(u) \cap (V_\Gamma - D)|$ is either 0 or 2, $\forall\ u \in D$, and the cardinality of the smallest γ_{cer}-set is so-called the certified domination number (CFDN) of Γ signified by $\gamma_{cer}(\Gamma)$. A set $\mathcal{D}_c \subseteq V_\Gamma$ is a connected certified dominating set (CCDS) denoted by γ^c_{cer}-set, if $\mathcal{D}_c$ is γ_{cer}-set and the subgraph $\Gamma[\mathcal{D}_c]$ induced by $\mathcal{D}_c$ is connected. Also, if $\mathcal{D}_c$ has no proper subset, then it is a smallest γ^c_{cer}-set and the cardinality of the minimal γ^c_{cer}-set is termed as connected certified domination number (CCDN) of the graph Γ denoted by γ^c_{cer}-(Γ). Over the last two decades, a significant amount of study has been conducted on domination theory and its various parameters. By imposing certain constraints on the dominating set, several domination-related parameters have been appropriately defined in domination theory. We refer to Haynes excellent book entitled "*Domination in Graphs*" [14] and a survey by Haynes *et al*. [15]. In this article, we continue the study of connected certified domination which is introduced by Ilyass and Goswami [17] and studied by Lone and Goswami [21]. In Section 2 of this paper, we have discussed some properties of CCDN and characterized graphs with larger values of CCDN. In Section 3 we have discussed some upper bounds for CCDN followed by Nordhaus Gaddum results in Section 4. In the last section of this paper, we discussed the complexity results on the connected certified domination problem.

1.1 Definitions and Notations

Unless stated otherwise, all the graphs covered in the study are connected and undirected graphs without multiple edges or loops. In general, we will follow Haynes [14] and West [25] for notations and other terminologies.

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be any graph. The complement $\bar{\Gamma}$ of the graph $\Gamma = (V_\Gamma, E_\Gamma)$ is defined on the vertex set V_Γ of Γ, where an edge $xy \in \bar{\Gamma}$ if and only if $xy \notin \Gamma$. Throughout the article, $\Delta(\Gamma)$ and $\delta(\Gamma)$ will be used to represent the maximal and the minimal degree of the graph Γ, respectively. Also, $\delta^*(\Gamma)$ and $\Delta^*(\Gamma)$ will represent $\min\{\delta(v), \delta(\bar{\Gamma})\}$ and $\max\{\Delta(\Gamma), \Delta(\bar{\Gamma})\}$. We will use L_Γ and $S_1(\Gamma)$($S_2(\Gamma)$, respectively) to represent the set of leaves and weak supports (strong supports, respectively) of graph Γ.

Let D be a dominating set of a graph Γ. An element of D that has all neighbors in D is said to be shadowed with respect to D (shadowed for short), an element of D that has exactly one neighbor in $V_\Gamma \setminus D$ is said to be half-shadowed (HS) with respect to D (half-shadowed for short), while an element of D having at least two neighbors in $V_\Gamma \setminus D$ is said to be illuminated with respect to D (illuminated for short) (Dettlaff *et al*. [4]). Notice that if D is γ^c_{cer}-set of any graph Γ, then D has no HS element.

2. Graphs with Larger Values of CCDN

We have observed that $\gamma^c_{cer}(\Gamma) \le n$ and $\gamma^c_{cer}(\Gamma) \ne n-1$, for any graph Γ of order n. There are also some graphs with $\gamma^c_{cer}(\Gamma) = n$, for example, cycle graph C_4. Thus it's natural to characterize all graphs with $\gamma^c_{cer}(\Gamma) = n$ which is shown in this section. More specifically, in this section, we will find the precise CCDN of the path graph, cycle graph, and corona product of graphs. In particular, we prove that, $\gamma^c_{cer}(\Gamma) = n$ if and only if Γ is some cycle graph C_n, or the path graph P_n,or the corona of some graph, $\forall\ n \ge 4$.

Before we get into our main findings, let us go over some important terminology. The corona product of graphs H_1 and H_2 is the graph $\Gamma = H_1 \circ H_2$ obtained from the disjoint union of H_1 and $|V_{H_1}|$ copies of H_2 in which the k^{th} vertex of H_1 is joined to all the vertices of the kth copy of H_2. For every $u \in V_\Gamma$, H_u will represent the copy of H whose vertices are connected one by one to the vertex u. Furthermore, using the join $u + H_u$, $u \in V_\Gamma$, we represent the subgraph of the corona $\Gamma \circ H$ as $u + H_u$. Also, we denote Γ^+ to be the graph obtained from the corona of Γ with any trivial graph H, in which vertex $u' \in H$ is connected with every vertex of Γ, that is, $\Gamma^+ = \Gamma \circ H$, where H is any trivial graph.

We begin this section with some properties of CCDN.

Proposition 2.1. *Let Γ be an islote free graph and $|\Gamma| \geq 3$, then $\gamma^c_{cer}(\Gamma) = 1$ if and only if Γ has a universal vertex.*

Proposition 2.2. *If Γ is an islote free graph of order n,then $\gamma^c_{cer}(\Gamma) \neq n-1$, $\forall$ n.*

Proposition 2.3. *Every vertex in $S_2(\Gamma)$ of graph Γ belongs to every γ^c_{cer}-set of Γ.*

Proof. Let D_c be a γ^c_{cer}-set of Γ, let $s_1 \in S_2(\Gamma)$ be a strong support vertex of Γ, and let $l_1 \in L_\Gamma$ is such that $l_1 \subseteq N_\Gamma(s_1)$. If $s_1 \notin D_c$, then $l_1 \in D_c$. But then l_1 would have only one neighbor in $V_\Gamma \setminus D_c$, and D_c would not be a γ^c_{cer}-set. □

Observation 2.4. $S_1(\Gamma) + L_1(\Gamma) \in \gamma^c_{cer}(\Gamma)$*-set.*

Proposition 2.5. *If $\mathcal{D}_c$ is γ^c_{cer}-set of any graph Γ, then $\mathcal{D}_c$ has no HS element.*

Proof. Let Γ be any connected graph and $\mathcal{D}_c$ be its γ^c_{cer}-set. Suppose there exists a vertex $u \in \mathcal{D}_c$ such that u is HS, then u has exactly one neighbor in $V_\Gamma \setminus \mathcal{D}_c$, a contradiction to the fact that $\mathcal{D}_c$ is $\gamma^c_{cer}(\Gamma)$-set and every vertex in $\mathcal{D}_c$ must dominate 0 or atleast 2 vertices in $V_\Gamma \setminus \mathcal{D}_c$. □

Next, we characterize graphs with larger values of CCDN and we start with the following result.

Theorem 2.6. *Let Γ be any islote free graph of order n and H be any graph of order $m \geq 2$. Then $\mathcal{D}_c \subseteq V_{\Gamma\circ H}$ is a γ^c_{cer}-set in $\Gamma \circ H$ if and only if $V_{u+H_u} \cap \mathcal{D}_c$ is a γ^c_{cer}-set of $u + H_u$ for every $u \in V_\Gamma$.*

Proof. Let $\mathcal{D}_c$ be a γ^c_{cer}-set of $\Gamma \circ H$ and let $u \in V_\Gamma$. If $u \in \mathcal{D}_c$ then u is a γ^c_{cer}-set of $u + H_u$. It follows that $V_{(u+H_u)} \cap \mathcal{D}_c$ is a γ^c_{cer}-set of $u + H_u$. Suppose that $u \notin \mathcal{D}_c$ and let $v \in V_{u+H_u} \setminus \mathcal{D}_c$ with $u \neq v$. Since $\mathcal{D}_c$ is a γ^c_{cer}-set of $\Gamma \circ H$, then $\exists w \in \mathcal{D}_c$ such that $vw \in E_{(\Gamma\circ H)}$. Then $w \in V_{H_u} \cap \mathcal{D}_c$ and $vw \in E_{(u+H_u)}$. This proves that $V_{u+H_u} \cap \mathcal{D}_c$ is a γ^c_{cer}-set of $u + H_u$.
Conversely, suppose that $V_{(u+H_u)} \cap \mathcal{D}_c$ is a γ^c_{cer}-set of $u + H_u$ for every $u \in V_\Gamma$. Then, clearly, D_c is a γ^c_{cer}-set of $\Gamma \circ H$. □

Corollary 2.7. *Let Γ and H be any isolate free graphs such that $|V_\Gamma| = n$ and $|V_H| = m \geq 2$. Then $\gamma^c_{cer}(\Gamma \circ H) = n$.*

Proof. Let $\mathcal{D}_c = V_\Gamma$. Then $V_{u+H_u} \cap \mathcal{D}_c = u$ is a γ^c_{cer}-set of $u + H_u$ for every $u \in V_\Gamma$. By Theorem 2.6, $\mathcal{D}_c$ is a γ^c_{cer}-set of $\Gamma \circ H$. □

Corollary 2.8. *Let Γ be an islote free graph of order n and H be any trivial graph then $\gamma^c_{cer}(\Gamma \circ H) = 2n$.*

Proof. Let Γ be an islote free graph of order n and H be any trivial graph. Then $\Gamma \circ H = \Gamma^{+}$ is a graph of order $2n$. By definition, every vertex of Γ^{+} is either a weak support vertex or pendant vertex.

Now, let $\mathcal{D}_c$ be a minimum γ_{cer}^{c}-set of Γ^{+}. Therefore, by Observation 2.4, $S_1(\Gamma)+L_1(\Gamma)=\mathcal{D}_c$, which implies $V_{(\Gamma^{+})}=\mathcal{D}_c$ and hence we conclude that $\gamma_{cer}^{c}(\Gamma \circ H)=\gamma_{cer}^{c}(\Gamma^{+})=2n$. □

Proposition 2.9. *If Γ is a path graph P_n of order n, then*

$$\gamma_{cer}^{c}(\Gamma)=\begin{cases} 1, & \text{for } n=1 \text{ or } n=3,\\ 2, & \text{for } n=2,\\ n, & \forall\ n\geq 4.\end{cases}$$

Proof. Let $\Gamma=P_n$ be a path graph of order n. Let $V_{P_n}=(v_1,v_2,\dots,v_n)$ be the vertex set of P_n. The result is obvious for $n=1,2,3$.

For $n\geq 4$. Let $\mathcal{D}_c$ be a γ_{cer}^{c}-set of P_n. We will show that $\mathcal{D}c=V_{P_n}$. Suppose on the contrary that $|\mathcal{D}_c|\leq n-2$ (since $|\mathcal{D}_c|=n-1$ is not possible by Proposition 2.2), which implies that there exists at least one vertex $v\in\mathcal{D}_c$ such that v is illuminated and $\deg_{\Gamma}(v)\geq 3$, which is not possible since $\Gamma=P_n$ is a path graph on n-vertices with $n\geq 4$ and has no vertex of degree greater than 2. □

Proposition 2.10. *If Γ is a cyclic graph C_n of order n, then*

$$\gamma_{cer}^{c}(\Gamma)=\begin{cases} 1, & \text{for } n=3,\\ n, & \forall\ n\geq 4.\end{cases}$$

Observation 2.11. *Let Γ be an isolate free graph order n, then γ_{cer}^{c}-set of Γ is always an induced path and it is an n-vertex induced path in the case when the graph Γ is C_n, or P_n, or corona of some graph.*

Theorem 2.12. *If Γ is an isolate free graph of order $n\geq 4$, then $\gamma_{cer}^{c}(\Gamma)=n$ if and only if Γ is either C_n, or P_n, or the corona of some graph.*

Proof. Let Γ be an isolate free graph of order $n\geq 4$ and let $\mathcal{D}_c$ be a γ_{cer}^{c}-set of Γ such that $|\mathcal{D}_c|=|V_{\Gamma}|=n$, that implies γ_{cer}^{c}-set $\mathcal{D}_c$ is an n-vertex induced path, and γ_{cer}^{c}-set $\mathcal{D}_c$ of a graph Γ is an n-vertex induced path whenever Γ is C_n, or P_n, or corona of some graph.

Conversely, if we suppose Γ to be C_n, or P_n, or the corona of some graph, then by Corollary 2.7, Proposition 2.9 and Proposition 2.10 $\gamma_{cer}^{c}(\Gamma)=n$. □

3. Upper Bounds on CCDN

In this context, we shall examine the upper bounds on CCDN. Dettlaff *et al.* [4] have discussed the upper bounds on CFDN in their paper. They have presented an upper bound on $\gamma_{cer}(\Gamma)$ in relation with the $\gamma(\Gamma)$ and $|S_1(\Gamma)|$ of the graph Γ. Similarly, we will prove the upper bound on $\gamma_{cer}^{c}(\Gamma)$ with respect to the connected domination number $\gamma_c(\Gamma)$ and $|S_1(\Gamma)|$, that is, we will prove that $\gamma_{cer}^{c}(\Gamma)\leq\gamma_c(\Gamma)+|S_1(\Gamma)|$. We shall denote the set of leaf neighbors of $S_1(\Gamma)$ by $L_1(\Gamma)$ and of $S_2(\Gamma)$ by $L_2(\Gamma)$.

Theorem 3.1. *If Γ is an isolate free graph of order n, then $\gamma_{cer}^{c}(Z)\leq\gamma_c(\Gamma)+|S_1(\Gamma)|$.*

Proof. If Γ is an isolate free graph of order $n \leq 3$, then the result is obvious.
Assume that Γ is an isolate free graph of order $n \geq 4$. Let $\mathcal{D}_c$ be a γ_c-set of Γ that reduce the number of HS vertices and such that $\mathcal{D}_c$ does not contain any leaf of Γ. (Notice that such $\mathcal{D}_c$ always exists as Γ is a connected graph and $|V_\Gamma| \geq 4$.) Let $V_{hs} \subseteq \mathcal{D}_c$ be the set of all HS vertices of $\mathcal{D}_c$ and L_Γ is the set of leaves in Γ.

Now, if $V_{hs} = \phi$, then $\gamma^c_{cer}(\Gamma) = \gamma_c(\Gamma) \leq \gamma_c(\Gamma) + |S_1(\Gamma)|$, and if $V_{hs} \neq \phi$ then we have the following claims:

Claim 1. If $u \in V_{hs}$, then $\deg_\Gamma(u) \geq 2$ and $u \notin S_2(\Gamma)$.
From the choice of $\mathcal{D}_c$, that is from the assumption that $\mathcal{D}_c \cap L_\Gamma = \phi$, the inequality $\deg_\Gamma(u) \geq 2$ will follow immediately.
For the second property, on the contrary, suppose that $u \in S_2(\Gamma)$, since u has at least two neighbors in L_Γ and from the assumption $L_\Gamma \subseteq V_\Gamma - \mathcal{D}_c$ which means u would not be HS, a contradiction to our assumption. Hence $u \notin S_2(\Gamma)$.
We now prove that all vertices in V_{hs} are weak supports. On the contrary, suppose that there is a HS vertex $u \in V_{hs} \setminus S_1(\Gamma)$ and let w be the unique neighbor of u in $V_\Gamma \setminus \mathcal{D}_c$. Since $u \notin S_1(\Gamma)$ or $S_2(\Gamma)$ (by assumption and *Claim* 1), that implies u is not a leaf.

Claim 2. The set $N_\Gamma(v) - \{u\} \subseteq V_\Gamma - D_c$.
If $N_\Gamma(v) - u$ is not a subset of $V_\Gamma - \mathcal{D}_c$ then $\mathcal{D}_c - u$ would be a smaller γ_c-set of Γ, a contradiction to $\mathcal{D}_c$. Hence $N_\Gamma(v) - \{u\} \subseteq V_\Gamma - \mathcal{D}_c$.

Claim 3. No vertex in the set $N_\Gamma(u) \setminus \{v\}$ is shadowed.
If a vertex $t \in N_\Gamma(u) - \{v\}$ was shadowed, then $\mathcal{D}_c \setminus \{t\}$ would be a smaller γ_c-set (than $\mathcal{D}_c$) of Γ.
Since no neighbor of u is a leaf (by *Claim* 1), and therefore from *Claim* 2 and *Claim* 3, we conclude that $(\mathcal{D}_c - \{u\}) \cup \{v\}$ would be a γ_c-set of Γ with a lesser number of HS vertices, a contradiction. Hence, the set V_{hs} of HS vertices contains weak supports of Γ only.
Now see that if we add all the leaves adjacent to HS weak supports to $\mathcal{D}_c$ then the resultant set say $\mathcal{D}'_c$ will be a γ_c-set of Γ with no HS vertices, which implies $\mathcal{D}'_c$ is a γ^c_{cer}-set of Γ.
Therefore, $\gamma^c_{cer}(\Gamma) \leq |\mathcal{D}'_c| = |\mathcal{D}_c| + |V_{hs}| = \gamma_c(\Gamma) + |S_1(\Gamma)|$, that implies $\gamma^c_{cer}(\Gamma) \leq \gamma_c(\Gamma) + |S_1(\Gamma)|$. □

Theorem 3.2. *For any isolate free graph Γ of order $n \geq 3$ $\gamma^c_{cer}(\Gamma) \leq n - |L_2(\Gamma)|$.*

Proof. It is easy to observe that for any tree T of order $n \geq 3$ the set of non-leaf vertices and all the leaves that are adjacent to the weak support vertices of T is the unique minimal $\gamma^c_{cer}(T)$-set. Therefore, $\gamma^c_{cer}(T) \leq n - |L_2(\Gamma)|$. Let D_c be the minimal γ^c_{cer}-set of any isolate free graph Γ of order $n \geq 3$ and $S_2(\Gamma)$ be the set of strong support vertices of the graph Γ. We know that the CCDN of any graph Γ is $\gamma^c_{cer}(\Gamma) \leq n$ and by Proposition 2.3 every vertex in $S_2(\Gamma)$ of graph Γ belongs to every single γ^c_{cer}-set of Γ, i.e., $S_2(\Gamma) \subseteq \mathcal{D}_c$ and hence $\gamma^c_{cer} - (\Gamma) \leq n - |L_2(\Gamma)|$, where $L_2(\Gamma)$ is the set of leaf neighbors of $S_2(\Gamma)$. □

4. Nordhaus-Gaddum Results for CCDN

The tight bounds (upper or lower) on the sum and product of a graph's chromatic number appeared for the first time in 1965 in a paper published by Nordhaus and Gaddum [22]. Since then, similar outcomes have been reported for a variety of parameters (see, for example, Chartrand and Mitchem [2] and Füredi *et al*. [8]). Nordhaus-Gaddum type results concerning domination related parameters in graphs have been investigated in several papers (see, for example, Cockayne *et al*. [3], Erfang *et al*. [7], Goddard and Henning [9], Goddard *et al*. [10], Harary and Haynes [12], Hattingh *et al*. [13], Henning *et al*. [16], Khoeilar *et al*. [20], and Payan and Xuong [23]). We recommend Chapter 10 of Haynes *et al*. [14] book for an outline of Nordhaus-Gaddum type results for domination-related parameters. For several graph invariants, together with different domination parameters, Nordhaus-Gaddum inequalities have been proved. Aouchiche and Hansen's outstanding survey includes a wide pool of Nordhaus-Gaddum type results up to the year 2013 (Aouchiche and Hansen [1]). In this part of our paper, we present Nordhaus-Gaddum type inequalities for CCDN of graphs.

We obtain the following observation by computing all the graphs of order at most 5.

Observation 4.1. *Let* Γ *and* $\bar{\Gamma}$ *be any isolate free graphs of order* $n \le 5$. *Then:*

(i) $(\gamma^c_{cer}(\Gamma)+\gamma^c_{cer}(\bar{\Gamma}), \gamma^c_{cer}(\Gamma)\gamma^c_{cer}(\bar{\Gamma})) = (2,1)$ *if* $n=1$.

(ii) $(\gamma^c_{cer}(\Gamma)+\gamma^c_{cer}(\bar{\Gamma}), \gamma^c_{cer}(\Gamma)\gamma^c_{cer}(\bar{\Gamma})) = \{(8,16),(7,10),(10,25)\}$ *if* $n=4,5$.

Remark. For $n = 2,3$. Let Γ be an isolate free graph of order 2 (3, respectively), then, in this case, its complement $\bar{\Gamma}$ would be a null graph and hence $\gamma^c_{cer}(\bar{\Gamma})$ is not defined for graphs of order 2 and 3. Therefore, Nordhaus-Gaddum type inequalities cannot be defined for graphs of order $n=2$ and 3.

Finally, we have the following result.

Theorem 4.2. *If* Γ *and* $\bar{\Gamma}$ *are any isolate-free graphs of order* $n \ge 6$ *and* $\delta^*(Z) = \min\{\delta(\Gamma), \delta(\bar{\Gamma})\}$, *then*

(i) $\gamma^c_{cer}(\Gamma)+\gamma^c_{cer}(\bar{\Gamma}) \le n+2$,

(ii) $(\gamma^c_{cer}(\Gamma)\gamma^c_{cer}(\bar{\Gamma}) \le 2n$.

In addition to it the following statements are equivalent:

(i) $\gamma^c_{cer}(\Gamma)+\gamma^c_{cer}(\bar{\Gamma}) = n+2$,

(ii) $\gamma^c_{cer}(\Gamma)\gamma^c_{cer}(\bar{\Gamma}) = 2n$.

(iii) Γ *or* $\bar{\Gamma}$ *is:*

- (a) *corona of some graphs,*
- (b) *cyclic graph* C_n, *or*
- (c) *path graph* P_n.

Proof. Part A. Let Γ and $\bar{\Gamma}$ be connected graphs of order $n \ge 6$. Let $\mathcal{D}_c$ and $\bar{\mathcal{D}}_c$ be the smallest γ^c_{cer}-set of Γ and $\bar{\Gamma}$ respectively and let $\delta^*(\Gamma) = \min\{\delta(\Gamma), \delta(\bar{\Gamma})\}$.

Case 1. If $\delta^*(\Gamma) \ge 2$, then $\Delta^*(\Gamma) \le n-3$. This also implies $\gamma^c_{cer}(\Gamma) > 1$ and $\gamma^c_{cer}(\bar{\Gamma}) > 1$. Thus, since by $\gamma^c_{cer}(\Gamma) \le n$ and $\gamma^c_{cer}(\bar{\Gamma}) \le n$, it is enough to show that $\gamma^c_{cer}(\bar{\Gamma}) \le 2$.

Let $x, y \in V_\Gamma$ such that x, y are non-adjacent in Γ. Suppose that $\deg_\Gamma(x) = n-3$ and $\deg_\Gamma(y) = 2$, that is, x and y are the vertices in Γ having maximum and minimum degrees, respectively. Then in complement of Γ which is the graph $\bar{\Gamma}$, $\deg_{\bar{\Gamma}}(x) = 2$ and $\deg_{\bar{\Gamma}}(y) = n-3$, that implies

x is dominating 2 vertices and y is dominating $n-3$ vertices in $\bar{\Gamma}$. As assumed x and y are non-adjacent in Γ therefore, in $\bar{\Gamma}x$ will dominate all those vertices which are in the $N_\Gamma[y]$ and y will dominate vertices that are in $N_\Gamma[x]$. Also, the subgraph induced by x and y in $\bar{\Gamma}$ is connected and both x,y are dominating at least 2 vertices in $\bar{\Gamma}$. That implies $\{x,y\} \in \bar{\mathcal{D}}_c$.
Also, $N_{\bar{\Gamma}}[x] \cup N_{\bar{\Gamma}}[y] = V_{\bar{\Gamma}}$, and since $\gamma^c_{cer}(\bar{\Gamma}) > 1$ therefore, $\gamma^c_{cer}(\bar{\Gamma}) \le 2$.

Case 2. Assume that $\delta^*(\Gamma) = \min\{\delta(\Gamma), \delta(\bar{\Gamma})\} = 1$. Then $\Delta^*(\Gamma) = n-2$, implies that $\gamma^c_{cer}(\Gamma) > 1$ and $\gamma^c_{cer}(\bar{\Gamma}) > 1$. Thus, since $\gamma^c_{cer}(\Gamma) \le n$ and $\gamma^c_{cer}(\bar{\Gamma}) \le n$, it suffices to show that $\gamma^c_{cer}(\Gamma) = 2$ or $\gamma^c_{cer}(\bar{\Gamma}) = 2$. Suppose that $\delta(\Gamma) = 1$. Let s be a support vertex in Γ and l_1, l_2 be the leaves adjacent to s. Let $\deg_\Gamma(s) = 3$ and suppose t is the only neighbor of s other than l_1 and l_2. we consider two cases; $\deg_\Gamma(t) = n-3$, and $\deg_\Gamma(t) \le n-4$.

(i) $\deg_\Gamma(t) = n-3$. Since, $\deg_\Gamma(s) = 3$ therefore $N_\Gamma[\{s,t\}] = N_\Gamma[s] \cup N_\Gamma[t] = V_\Gamma \setminus N_\Gamma[t] \cup N_\Gamma[t] = V_\Gamma$, that implies $\{s,t\} \subseteq [V_\Gamma] \cap N_\Gamma[t]$, $\{s,t\} \subseteq [V_\Gamma] \cap N_\Gamma[s]$, and $\gamma^c_{cer}(\Gamma) > 1$, we conclude that $\{s,t\}$ is the smallest γ^c_{cer}-set of Γ and $\gamma^c_{cer}(\Gamma) = 2$.

   Similarly, $\{l_1, t\}$ will be the smallest γ^c_{cer}-set of $\bar{\Gamma}$. As $N_{\bar{\Gamma}}[l_1] = V_\Gamma \setminus \{s\}$ and $N_{\bar{\Gamma}}[t] = V_\Gamma \setminus N_\Gamma[t] = V_\Gamma \setminus \{l_1, l_2\}$ and since $\gamma^c_{cer}(\bar{\Gamma}) > 1$ therefore, $\gamma^c_{cer}(\bar{\Gamma}) = 2$.

(ii) $\deg_\Gamma(t) \le n-4$. Let v_1, v_2 be any two elements in $V_\Gamma \setminus \{N_\Gamma[t]\}$. In this case, as s and t are adjacent in Γ, $\{l_1, t\}$ will be the smallest γ^c_{cer}-set of $\bar{\Gamma}$, since $N_{\bar{\Gamma}}[l_1, t] = V_\Gamma$, $\{v_1, v_2\} \subseteq N_{\bar{\Gamma}}[l_1] \cap V_Z \setminus \{l_1, t\}$ and $\{v_1, v_2\} \subseteq N_{\bar{\Gamma}}[t] \cap V_\Gamma \setminus \{l_1, t\}$. From this, it again follows that $\gamma^c_{cer}(\bar{\Gamma}) = 2$.

Case 3. If $\delta^*(\Gamma) = 0$. Then Γ or $\bar{\Gamma}$ will have isolated vertices which will be a contradiction to our assumption. Hence the case $\delta^*(\Gamma) = 0$ is not possible.

Part B. We now verify that (i), (ii), and (iii) are equivalent.
Let Γ be a graph of order $n \ge 6$ such that $\gamma^c_{cer}(\Gamma) + \gamma^c_{cer}(\bar{\Gamma}) = n+2$ and $\gamma^c_{cer}(\Gamma)\gamma^c_{cer}(\bar{\Gamma}) = 2n$, respectively. From this assumption and part A above it follows that $\delta^*(\Gamma) = 1$. Therefore, $\gamma^c_{cer}(\Gamma) = n$ or $\gamma^c_{cer}(\bar{\Gamma}) = n$, as we have already proved that $\gamma^c_{cer}(\Gamma) = 2$ or $\gamma^c_{cer}(\bar{\Gamma}) = 2$. It will follow from this and Theorem 2.12 that Γ or $\bar{\Gamma}$ is either C_n, or P_n, or the corona of some graph. We proved the implication (i)⇒(ii) and (ii)⇒(iii).
Similarly, if Γ is C_n, or P_n, or the corona of some graph, then by Corollary 2.7, Proposition 2.9 and Proposition 2.10, $\gamma^c_{cer}(\Gamma) = n$. From the fact that C_n, or P_n, or the corona of some graph has no isolate vertices and Γ is the graph of order $n \ge 6$, it implies that $\gamma^c_{cer}(\bar{\Gamma}) \ge 2$. Since $\delta(\Gamma) = 1$, as in Part A, we proved that $\gamma^c_{cer}(\bar{\Gamma}) = 2$. Therefore, $\gamma^c_{cer}(\Gamma) + \gamma^c_{cer}(\bar{\Gamma}) = n+2$ and $\gamma^c_{cer}(\Gamma)\gamma^c_{cer}(\bar{\Gamma}) = 2n$. This proves (iii)⇒(i) and (iii)⇒(ii). □

5. Connected Certified Domination Problem (CCDP)

Instance: A graph $\Gamma = (V_\Gamma, E_\Gamma)$ and a positive integer g.

Question: Does there exist a CCDS D_c of G with $|D_c| \le g$?

In this section, we will prove that the CCDP for star convex bipartite graphs, comb convex bipartite graphs, and planar graphs is NP-complete by reduction from the exact cover by 3-sets (X3C) (Karp [19]), which is given below.

5.1 Complexity Results

Exact Cover by 3-Sets (X3C)

Instance: A set M with $|\mathcal{M}| = 3r$ and a collection $\mathcal{S}$ of 3-element subsets of $\mathcal{M}$.

Question: Does $\mathcal{S}$ contain an exact cover for $\mathcal{M}$, i.e., a sub-collection $\mathcal{S}' \subset \mathcal{S}$ such that every element in $\mathcal{M}$ occurs in exactly one member of $\mathcal{S}'$?

Take into account an instance of $X3C$ problem with $\mathcal{M} = \{m_1, m_2, \dots, m_3r\}$ and $\mathcal{S} = \{S_1, S_2, \dots, S_q\}$. The graph $\mathcal{K}$ with vertex set $\mathcal{M} \cup \mathcal{S}$, where m_i is adjacent to $\mathcal{S}_j$ if and only if $m_i \in \mathcal{S}_j$, is called the incidence graph of the given instance of $X3C$.

Theorem 5.1. *For star convex bipartite graphs the CCDP is NP-complete.*

Proof. Clearly CCDP is in NP. Assume that $\mathcal{I}$ is an instance of the $X3C$ and let $\mathcal{K}$ denote the incidence graph associated with $\mathcal{I}$. Let Γ be the graph derived from $\mathcal{K}$ by introducing an edge between vertices x and y and connecting vertex x to all the vertices in set $\mathcal{S}$. Clearly, Γ forms a bipartite graph with the bipartition $\mathcal{M} \cup \{x\}$ and $\mathcal{S} \cup \{y\}$. Let $\mathcal{T}$ represent the star with vertex set $V(\mathcal{T}) = \mathcal{M} \cup \{x\}$, where x serves as its central vertex. Let $y_1 \in \mathcal{S} \cup \{y\}$. Clearly, $x \in N(y_1)$ and the induced subgraph $\mathcal{T}[N(y_1)]$ is a subtree of $\mathcal{T}$. Hence Γ is a star convex bipartite graph. Note that the formation of Γ can be accomplished within polynomial time. We assert that there exists a solution for the instance $\mathcal{I}$ of $X3C$ if and only if $\mathcal{G}$ possesses a CCDS D_c with $|D_c| \le r+1$. If S' is a solution of $\mathcal{I}$, then $D_c = S' \cup \{x\}$ is connected dominating set of Γ and every vertex of D_c dominates exactly 3 vertices of $\mathcal{M}$. Hence D_c is a CCDS of Γ with $|D_c| = r+1$.
Conversely, let D'_c be the CCDS of Γ with $|D_c| = r+1$. As x is a support vertex of the graph Γ, then by Proposition 2.3 $x \in D'_c$. If $D'_c \cap \mathcal{M} \neq \phi$, then for dominating the vertices of $\mathcal{M} - (D'_c \cap \mathcal{M})$, at least $\frac{(3r - |D'_c \cap \mathcal{M}|)}{3}$ vertices are required. Therefore $|D'_c| \ge \frac{(3r - |D'_c \cap \mathcal{M}|)}{3} + |D'_c \cap \mathcal{M}| > r$, which is a contradiction. Hence $D'_c \cap \mathcal{M} = \phi$ and $\{S_j | S_j \in D'_c\}$ is a solution of the instance $\mathcal{I}$ of X3C. □

Theorem 5.2. *For split graphs the CCDP is NP-complete.*

Proof. Clearly, CCDP is in NP. Assume that $\mathcal{I}$ is an instance of the X3C and let $\mathcal{K}$ denote the incidence graph associated with $\mathcal{I}$. Let Γ be the split graph derived from $\mathcal{K}$ by making the induced graph $\Gamma[S]$ complete. Clearly, the construction of Γ can be accomplished within polynomial time. We assert that there exists a solution for the instance $\mathcal{I}$ of X3C if and only if Γ possesses a CCDS D_c with $|D_c| \le r$. Let S' be a solution of $\mathcal{I}$, then $D_c = S_j | S_j \in S'$ is a connected dominating set of Γ and every vertex of D_c dominates exactly 3 vertices of $\mathcal{M}$. Therefore D_c is a CCDS of Γ and $|D_c| = r$.
Conversely, let D'_c be the CCDS of Γ with $|D_c| \le r$. As in Theorem 5.1, it can be proved that $D'_c \cap \mathcal{M} = \phi$ and $\{S_j \mid S_j \in D'_c\}$ is a solution of the instance $\mathcal{I}$ of X3C. □

Theorem 5.3. *For comb convex bipartite graphs the CCDP is NP-complete.*

Proof. Clearly, CCDP is in NP. Assume that $\mathcal{I}$ is an instance of the X3C and let $\mathcal{K}$ denote the incidence graph associated with $\mathcal{I}$. Consider Γ, a graph derived from the incidence graph $\mathcal{K}$ shown below:

(i) Add a set $\mathcal{M}' = \{m'_1, m'_2, \dots, m'_{3r}\}$ of $3r$ vertices such that $\mathcal{M}' \cup S$ is a complete bipartite graph with bipartition $\mathcal{M}'$ and S.

(ii) Include three vertices x, x' and y and the edges xy, xS_i and $x'S_i$ for all $S_i \in S$.

It is evident that Γ forms a bipartite graph with the bipartition $\mathcal{M} \cup \mathcal{M}' \cup \{x, x'\}$ and $S \cup \{y\}$. Now, let $\mathcal{T}$ represents tree with $V(\mathcal{T}) = \mathcal{M} \cup \mathcal{M}' \cup \{x, x'\}$ such that $(m'_1, m'_2, \dots, m'_{3r})$ is a path in $\mathcal{T}$ and $m_i m'_i, xx' \in E(\mathcal{T})$, for all i, $1 \le i \le 3r$. Certainly $\mathcal{T}$ is a comb. Suppose that $u \in S \cup \{y\}$. Then

$$N(u) = \begin{cases} \mathcal{M}' \cup \{x'\}, & \text{if} \quad u \in S, \\ a, & \text{otherwise.} \end{cases}$$

Since $\mathcal{T}[N(u)]$ is clearly a subtree of $\mathcal{T}$, it follows that Γ is a comb convex bipartite graph. Similar to the proof of Theorem 5.1, we can show that instance $\mathcal{I}$ of X3C has a solution if and only if Γ has CCDS D_c with $|D_c| \le r+1$. □

The problem X3C, subject to the additional constraint that the incidence graph K is planar, is referred to as the Planar-X3C problem. It has been demonstrated in [6] that the Planar-X3C problem is NP-complete. Utilizing the Planar-X3C problem and a proof analogous to that of Theorem 5.1, we derive the following theorem.

Theorem 5.4. *For planar graphs the CCDP is NP-complete.*

5.2 Construction of a Connected Certified Dominating Set

The subsequent algorithm presents a procedure for constructing a CCDS from any connected dominating set within a specified graph.

Algorithm: Construction of CCDS

Input: A simple and undirected graph Γ

Output: A CCDS D'_c of Γ.

a: $D'_c \leftarrow$ Any connected dominating set D_c of Γ

b: Let $Y \leftarrow \{u : u \in D_c, |N(u) \cap (V \setminus D_c)| = 1\}$

c: **While** $Y \neq \Phi$ **do**

d: Let $u \in Y$ and $v = N(u) \cap (V \setminus D_c)$

e: **If** $d(v) = 1$ **then**

f: $D'_c \leftarrow D'_c \cup \{v\}$

g: **else**

h: $D'_c \leftarrow D'_c \setminus \{u\} \cup \{w\}$

i: **end if**

$Y \leftarrow \{u : u \in D_c, |N(u) \cup (V \setminus D_c)| = 1\}$

j: end while

k: return D_c

6. Conclusion

The research has introduced and explored the idea of a connected certified dominating set, abbreviated CCDS, in the context of connected undirected graph Γ. The main focus of the authors has been on various crucial areas: studying properties of CCDN of graphs, characterizing graphs with larger values of CCDN, and exploring bounds for CCDN of graphs. In addition to it, the authors has established Nordhaus-Gaddum type inequalities for CCDN of graphs and they proved that the connected certified domination problem is NP complete for Star Convex Bipartite graphs, Comb Convex Bipartite graphs, and planar graphs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Applied Mathematics* **161**(4-5) (2013), 466 – 546, DOI: 10.1016/j.dam.2011.12.018.

[2] G. Chartrand and J. Mitchem, Graphical theorems of the Nordhaus-Gaddum class, in: *Recent Trends in Graph Theory*, M. Capobianco, J. B. Frechen and M. Krolik (editors), Lecture Notes in Mathematics, Vol. 186, Springer, Berlin — Heidelberg, DOI: 10.1007/BFb0059422.

[3] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total domination in graphs, *Networks* **10**(3) (1980), 211 – 219, DOI: 10.1002/net.3230100304.

[4] M. Dettlaff, M. Lemańska, J. Topp, R. Ziemann and P. Żyliński, Certified domination, *AKCE International Journal of Graphs and Combinatorics* **17**(1) (2020), 86 – 97, DOI: 10.1016/j.akcej.2018.09.004.

[5] M. Dettlaff, M. Lemańska, M. Miotk, J. Topp, R. Ziemann and P. Żyliński, Graphs with equal domination and certified domination numbers, *Opuscula Mathematica* **39**(6) (2019), 815 – 827, DOI: 10.7494/OpMath.2019.39.6.815.

[6] M. E. Dyer and A. M. Frieze, Planar 3DM is NP-complete, *Journal of Algorithms* **7**(2) (1986), 174 – 184, DOI: 10.1016/0196-6774(86)90002-7.

[7] S. Erfang, D. Chuangyin and K. Liying, A note on Nordhaus-Gaddum inequalities for domination, *Discrete Applied Mathematics* **136**(1) (2004), 83 – 85, DOI: 10.1016/S0166-218X(03)00200-2.

[8] Z. Füredi, A. V. Kostochka, R. Škrekovski, M. Stiebitz and D. B. West, Nordhaus-Gaddum-type theorems for decompositions into many parts, *Journal of Graph Theory* **50**(4) (2005), 273 – 292, DOI: 10.1002/jgt.20113.

[9] W. Goddard and M. A. Henning, Nordhaus-Gaddum bounds for independent domination, *Discrete Mathematics* **268**(1-3) (2003), 299 – 302, DOI: 10.1016/S0012-365X(03)00032-3.

[10] W. Goddard, M. A. Henning and H. C. Swart, Some Nordhaus-Gaddum-type results, *Journal of Graph Theory* **16**(3) (1992), 221 – 231, DOI: 10.1002/jgt.3190160305.

[11] V. S. Goswami, A. Ilyass and K. K. Bhagwat, Certified domination number of some graphs, *AIP Conference Proceedings* **2735** (2023), 040029, DOI: 10.1063/5.0140648.

[12] F. Harary and T. W. Haynes, Nordhaus-Gaddum inequalities for domination in graphs, *Discrete Mathematics* **155**(1-3) (1996), 99 – 105, DOI: 10.1016/0012-365X(94)00373-Q.

[13] J. H. Hattingh, E. Jonck, E. J. Joubert and A. R. Plummer, Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs, *Discrete Mathematics* **308**(7) (2008), 1080 – 1087, DOI: 10.1016/j.disc.2007.03.061.

[14] T. W. Haynes, *Domination in Graphs, Volume 2: Advanced Topics*, 1st edition, Routledge, 520 pages (2017), DOI: 10.1201/9781315141428.

[15] T. W. Haynes, S. T. Hedetniemi and M. A. Henning (editors), *Topics in Domination in Graphs*, Vol. 64, Springer Cham., viii + 545 pages (2020), DOI: 10.1007/978-3-030-51117-3.

[16] M. A. Henning, E. J. Joubert and J. Southey, Nordhaus-Gaddum bounds for total domination, *Applied Mathematics Letters* **24**(6) (2011), 987 – 990 DOI: 10.1016/j.aml.2011.01.011.

[17] A. Ilyass and V. S. Goswami, Connected certified domination number of certain graphs, *AIP Conference Proceedings* **2735** (2023), 040028, DOI: 10.1063/5.0140647.

[18] P. K. Jakkepalli, S. Arumugam, H. Khandelwal and V. S. Reddy, Algorithmic aspects of certified domination in graphs, *Communications in Combinatorics and Optimization* **7**(2) (2022), 247 – 255, DOI: 10.22049/CCO.2021.27302.1226.

[19] R. M. Karp, Reducibility among combinatorial problems, in: *Complexity of Computer Computations*, *50 Years of Integer Programming 1958-2008*, M. Jünger *et al*. (editors), Springer, Berlin, Heidelberg, pp. 219 – 241 (2010), DOI: 10.1007/978-3-540-68279-0_8.

[20] R. Khoeilar, H. Karami, M. Chellali, S. M. Sheikholeslami and L. Volkmann, Nordhaus-Gaddum type results for connected and total domination, *RAIRO-Operations Research* **55** (2021), S853 – S862, DOI: 10.1051/ro/2020025.

[21] A. I. Lone and V. Goswami, Connected certified domination edge critical and stable graphs, *Acta Universitatis Sapientiae Informatica* **15**(1) (2023), 25 – 37, DOI: 10.2478/ausi-2023-0003.

[22] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, *The American Mathematical Monthly* **63**(3) (1956), 175 – 177, DOI: 10.2307/2306658.

[23] C. Payan and N. H. Xuong, Domination-balanced graphs, *Journal of Graph Theory* **6**(1) (1982), 23 – 32, DOI: 10.1002/jgt.3190060104.

[24] S. D. Raj and S. G. S. Kumari, Certified domination number in product of graphs, *Turkish Journal of Computer and Mathematics Education (TURCOMAT)* **11**(3) (2020), 1166 – 1170, DOI: 10.17762/turcomat.v11i3.10298.

[25] D. B. West, *Introduction to Graph Theory*, 2nd edition, Pearson Education, India (2002).