Communications in Mathematics and Applications
Vol. 14, No. 4, pp. 1385–1394, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i4.2575

http://www.rgnpublications.com

***Special Issue*:**
Recent Trends in Applied and Computational Mathematics
Proceedings of the Third International Conference on Recent Trends in Applied and Computational Mathematics (ICRTACM-2022)
School of Applied Sciences, Department of Mathematics, Reva University, Bangaluru, India, 10th & 11th October, 2022
***Editors*: M. Vishu Kumar, A. Salma, B. N. Hanumagowda and U. Vijaya Chandra Kumar**

Research Article

Linear Study of Ferromagnetic Convection in Nanofluids Under the Effect of Variable Viscosity

S. Rajashree*[1] and N. P. Chandrashekara[2]

[1] *Department of Mathematics, Government First Grade College, Lingasuguru 584122, Karnataka, India*
[2] *Department of Mathematics, B. N. M. Institute of Technology (affiliated to VTU Belagavi), Bengaluru 560070, Karnataka, India*
***Corresponding author:** rsgmath@gmail.com

Received: February 23, 2023 **Accepted:** June 4, 2023

Abstract. Rayleigh-Bénard ferroconvective problem is considered in a Newtonian nanofluid with Fe_3O_4-magnetite, as a nanoparticle dispersed in the medium, under the effect of variable viscosity. Employing double fourier series, we arrive at the system of differential equations well known as generalized Lorenz Model both in linear and non-linear forms. In the current paper, linear stabillity analysis is considered and graphs have been plotted for stationary nanofluid Rayleigh number (R_{nfs}) versus variable viscosity and wavenumber for variant values of buoyancy, non-buoyancy magnetization parameters (BMP and NBMP, respectively) and variable viscosity, and the same has been discussed in detail.

Keywords. Convection, Ferromagnetic nanoliquid, Variable viscosity, Lorenz model

Mathematics Subject Classification (2020). 76M25, 76R10, 76R50, 76W05

1. Introduction

We come across several works on convection in Newtonian nanofluids that are not ferromagnetic in nature like Chamkha *et al*. [6], Agarwal and Bhadauria [1], Buongiorno [5], Kim *et al*. [11], Oztop and Abu-Nada [16], Putra *et al*. [17], Siddheshwar *et al*. [23], Siddheshwar and Meenakshi [22], and Tzou [25, 26] and references cited therein. Chamkha *et al*. [6] explores the literature on MHD convection of nanofluids in various geometries and presents the available physical properties in the paper. The non-linear thermal stability of a horizontal layer in a nanofluid which incorporates the effect of Brownian motion with thermophoresis can be seen in the works of Agarwal and Bhadauria [1], and Tzou [25, 26]. Buongiorno [5] describes the effect of thermophoresis in nanofluids mechanistically and develops a new correlation structure for heat transfer. Kim *et al*. [11] describes the effect of nanoparticle addition on the convective instability and concludes that the heat transfer coefficient of a nanofluid is enhanced by all parameters with respect to the volume fraction of the nanoparticles. Oztop and Abu-Nada [16] deals with the natural convection in partially heated rectangular enclosures filled with nanofluids and found that aspect ratio is one of the parameter in enhancing heat transfer. Putra *et al*. [17] deals with the natural convection of nanofluids in horizontal cylinder and investigates the dependence of heat transfer enhancement on various parameters such as geometry, concentration and material of the nanoparticles. Detailed discussion is made on the onset of convection and the amount of heat transfer in Newtonian nanoliquids compared to that in the absence of nanoparticles by Siddheshwar *et al*. [23], and Siddheshwar and Meenakshi [22].

Also, we can see many problems on convection in ferromagnetic liquids that does not involve nanoparticles in it, e.g., Auernhammer and Brand [3], Alam *et al*. [2], Gotoh and Yamada [8], Kaloni and Lou [10], Laroze *et al*. [13], Maruthamanikandan [14], Odenbach [15], Shivakumara *et al*. [20], Siddheshwar and Abraham [21], Stiles and Kagan [24], Yamaguchi *et al*. [27] and references cited therein. But there are very few works on convection which consists of both ferro and nano, namely, Krauzina *et al*. [12], Sheikholeslami and Chamkha [18], Sheikholeslami [19] in which the effect of variable viscosity has not been included.

In this paper, we consider the above type of problem, that is, convection in ferromagnetic nanofluids using Lorenz model, under the effect of variable viscosity which depends on both magnetic field as well as temperature. The linear stability analysis is carried out in the current problem and the plots are drawn for the variation of R_{nfs} versus variable viscosity and wavenumber. After the study of linear stability, we have also reached to the stage of Lorenz model in both linear and nonlinear forms, of which the later enables us to determine the heat transport in the forthcoming research works.

2. Mathematical Formulation

We consider a depth, d of ferro-nanofluid layer with nano-sized Fe_3O_4-magnetite particles dispersed in the Newtonian system, parallel to the horizontal plane of large extent, subject to a temperature gradient along z-axis and gravity acting in downward direction ($\vec{g} = -g\widehat{k}$).

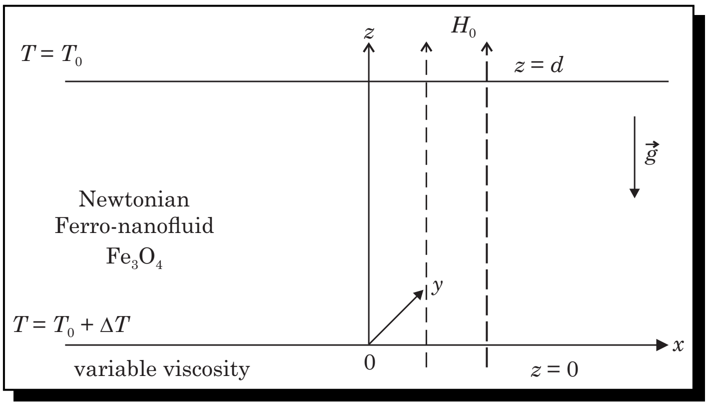

Figure 1. Physical configuration

We mainly focus on the study of two-dimensional flows only (independent of y), and the viscosity considered in the problem is temperature and magnetic field dependent. The magnetic fluid properties are assumed to be those of an electrically non-conducting superparamagnet and the properties of nanofluids are extracted from the previous studies (Brinkman model [4], Hamilton-Crosser model [9] and Mixture theory). $H = H_0\widehat{k}$ is an external magnetic field applied vertically along z-axis and H_0 is the uniform magnetic field. The imposed temperatures at the lower and upper boundaries are, $T_{z=0} = T_0+\Delta T$ and $T_{z=d} = T_0$, respectively. Under the assumption of the Boussinesq approximation and small scale convective motions, following are the equations, governing the current problem:

Equation of continuity:

$$\nabla \cdot \vec{q} = 0 \tag{2.1}$$

Equation of momentum:

$$\rho_{nf}\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla p + \nabla\cdot(\mu_{nf}(\vec{H},T)(\nabla\vec{q} + \nabla\vec{q}^{\,Tr})) + \mu_0(\vec{M}\cdot\nabla)\vec{H} - [\rho_{nf} - (\rho\beta)_{nf}(T-T_0)]g\widehat{k} \tag{2.2}$$

Equation of energy:

$$\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T = \alpha_{nf}\nabla^2 T \tag{2.3}$$

Maxwell's equations:

$$\nabla\cdot B = 0,\ \ \nabla\times\vec{H} = 0,\ \ B = \mu_0(\vec{M}+\vec{H}) \tag{2.4}$$

Magnetic equation of state:

$$\vec{M} = M_0 + \chi_m(\vec{H} - H_0) - K(T - T_0) \tag{2.5}$$

where $\vec{q}$ is the velocity vector, t is the time, p is the pressure, μ_0 is the magnetic permeability, M is the magnetization, B is the magnetic induction, Tr is the transpose, M_0 is the average value of magnetization, K is the pyromagnetic coefficient and χ_m is the magnetic susceptibility.

Variable viscosity of nanofluid:

$$\mu_{nf}(\vec{H},T) = \mu^*(H_0,T_0)[e^{-\delta_T(T_b-T_0)+\delta_H(\vec{H}_b-H_0)}], \tag{2.6}$$

where δ_T, $\delta_H > 0$ are very small, $\mu^*(H_0, T_0)$ is the reference viscosity at $H = H_0$ and $T = T_0$.

The properties of ferro-nanofluids are obtained using the below:

Phenomenological laws:

$$\frac{\mu_{nf}}{\mu_f} = \frac{1}{(1-\chi)^{2.5}} \qquad \text{(Brinkman model [4])} \tag{2.7}$$

$$\frac{k_{nf}}{k_f} = \frac{\left(\frac{k_{np}}{k_f}+2\right) - 2\chi\left(1-\frac{k_{np}}{k_f}\right)}{\left(\frac{k_{np}}{k_f}+2\right) + \chi\left(1-\frac{k_{np}}{k_f}\right)} \qquad \text{(Hamilton-Crosser model [9])} \tag{2.8}$$

Mixture theory:

$$\alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}},$$
$$\frac{\rho_{nf}}{\rho_f} = (1-\chi) + \chi\frac{\rho_{np}}{\rho_f},$$
$$\frac{(\rho C_p)_{nf}}{(\rho C_p)_f} = (1-\chi) + \chi\frac{(\rho C_p)_{np}}{(\rho C_p)_f},$$
$$\frac{(\rho\beta)_{nf}}{(\rho\beta)_f} = (1-\chi) + \chi\frac{(\rho\beta)_{np}}{(\rho\beta)_f}, \tag{2.9}$$

where μ_f — variable viscosity, k_f — thermal conductivity, ρ_f — density, C_{pf} — heat capacity, and β_f — thermal expansion coefficient of the basefluid. Similarly, μ_{nf}, k_{nf}, ρ_{nf}, C_{pnf}, β_{nf} and μ_{np}, k_{np}, ρ_{np}, C_{pnp}, β_{np} holds for nanofluid and nanoparticle respectively, α_{nf} — thermal diffusivity of nanofluid and χ — nanoparticle volume fraction.

Consider the solution of basic state in the below form:

$$\vec{q}_b = (0,0),\ p = p_b(z),\ \rho = \rho_b(z),\ \vec{g} = -g\hat{k},\ \vec{H}_b = H_b\hat{k},\ \vec{M}_b = M_b\hat{k},\ T_b = T_0 + \Delta T\left(1-\frac{z}{d}\right). \tag{2.10}$$

Using the above in Maxwell's equation (2.4):

$$H_b(z) + M_b(z) = c, \tag{2.11}$$

where c is the constant of integration.

Using equations (2.10) and (2.11) in magnetic equation of state (2.5) and solving for H_b and M_b, we have

$$\vec{H}_b = \left[H_0 + \frac{K\Delta Tz}{(1+\chi_m)d}\right]\hat{k},\ \ \vec{M}_b = \left[M_0 - \frac{K\Delta Tz}{(1+\chi_m)d}\right]\hat{k}, \tag{2.12}$$

and consider the superimposed perturbed state in the below form:

$$\vec{q} = \vec{q}_b + \vec{q}\,',\ p = p_b + p',\ \rho = \rho_b + \rho',\ \vec{H}_b = H_b\hat{k} + \vec{H}',\ \vec{M}_b = M_b\hat{k} + \vec{M}',\ T = T_b + T'. \tag{2.13}$$

Now, we shall introduce stream function, ψ (due to consideration of two-dimensional flows) as follows:

$$u' = -\frac{\partial\psi}{\partial z},\ \ w' = \frac{\partial\psi}{\partial x}, \tag{2.14}$$

which satisfies the continuity equation (2.1).

Eliminating the pressure in the equation of momentum and non-dimensionalizing the resulting equation along with the energy equation, we have

$$\frac{1}{\mathrm{Pr}_{nf}}\left[\frac{\partial}{\partial\tau}(\nabla^2\psi)+J(\psi,\nabla^2\psi)\right]$$
$$=R_{nf}M_1a^2J\left(\theta,\frac{\partial\varphi}{\partial z}\right)+a\left[f(z)\nabla^4\psi+2\frac{\partial}{\partial z}(f(z))(\nabla^2\psi)-\frac{\partial^2}{\partial z^2}(f(z))\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}\right)\right]$$
$$+R_{nf}a^2\left[(1+M_1)\frac{\partial\theta}{\partial x}-M_1\frac{\partial^2\varphi}{\partial x\partial z}\right], \tag{2.15}$$

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial\psi}{\partial x}+a\nabla^2\theta-J(\psi,\theta). \tag{2.16}$$

Non-dimensionalizing the magnetic equation of state and using the resultant in Maxwell's equation, we have

$$M_3\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{\partial\theta}{\partial z}=0, \tag{2.17}$$

where

$\mathrm{Pr}_{nf}=\dfrac{\mu_{nf}}{\rho_{nf}\alpha_{nf}}$, is the nanofluid Prandtl number,

$R_{nf}=\dfrac{(\rho\beta)_{nl}g\Delta Td^3}{\alpha_{nl}\mu_{nl}}$, is the nanofluid Rayleigh number,

$M_1=\dfrac{\mu_0K^2\Delta T}{(\rho\beta)_{nl}gd(1+\chi_m)}$, is the buoyancy magnetization parameter,

$M_3=\frac{\left(1+\frac{M_0}{H_0}\right)}{(1+\chi_m)}$, is the non-buoyancy magnetization parameter,

t, θ = dimensionless time and temperature respectively,

$J(m,n)=\dfrac{\partial m}{\partial x}\dfrac{\partial n}{\partial z}-\dfrac{\partial m}{\partial z}\frac{\partial n}{\partial x}$, is the Jacobian,

$\nabla^2\psi=\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial z^2}$, is the Laplacian,

a = ratio of thermal diffusivity of nanofluid to basefluid,

$$\nabla^4\psi=\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial z^2}+\frac{\partial^4\psi}{\partial z^4},$$

φ = magnetic scalar potential,

$f(z)=e^{-V(1-z)}$,

$$V=\left(\delta_T-\frac{\delta_HK}{(1+\chi_m)}\right)\Delta T. \tag{2.18}$$

Boundary conditions: We consider the following boundary conditions (Finlayson [7]):

$$\theta=\psi=\frac{\partial^2\psi}{\partial z^2}=\frac{\partial\varphi}{\partial z}=0,\ \text{at } z=0 \text{ and } 1. \tag{2.19}$$

3. Linear Stability Analysis

We shall assume the following solutions, which satisfies the aforesaid boundary conditions:

$$\psi(x,z) = \psi_0 \sin(\kappa x)\sin(\pi z), \tag{3.1}$$

$$\theta(x,z) = \theta_0 \cos(\kappa x)\sin(\pi z), \tag{3.2}$$

$$\varphi(x,z) = \varphi_0 \cos(\kappa x)\cos(\pi z), \tag{3.3}$$

where κ is the wavenumber.

Using the solutions (3.1)-(3.3) in dimensionless equations (2.15)-(2.17), we arrive at the stationary nanofluid Rayleigh number, R_{nfs}, for the onset of convection:

$$R_{nfs} = \frac{2\delta^2(\kappa^2 M_3 + \pi^2)}{\kappa^2[\kappa^2(1+M_1)M_3 + \pi^2]} F(V), \tag{3.4}$$

where

$$F(V) = \delta^4 V_1 - 2\delta^2 V_2 + (k^2 - \pi^2)V_3, \tag{3.5}$$

$$V_1 = \int_0^1 f(z)\sin^2(\pi z)dz, \tag{3.6}$$

$$V_2 = \int_0^1 \frac{\partial}{\partial z}[f(z)]\sin^2(\pi z)dz, \tag{3.7}$$

$$V_3 = \int_0^1 \frac{\partial^2}{\partial z^2}[f(z)]\sin^2(\pi z)dz, \tag{3.8}$$

and $\delta^2 = (\pi^2 + \kappa^2)$.

4. Lorenz Model

As per the boundary condition (2.19), one can assume the functions as shown below:

$$\psi(x,z,\tau) = -\frac{1}{\kappa} a_1(\tau)\sin(\kappa x)\sin(\pi z), \tag{4.1}$$

$$\theta(x,z,\tau) = a_2(\tau)\cos(\kappa x)\sin(\pi z) + a_3(\tau)\sin(2\pi z), \tag{4.2}$$

$$\varphi(x,z,\tau) = a_4(\tau)\cos(\kappa x)\cos(\pi z) + a_5(\tau)\cos(2\pi z), \tag{4.3}$$

where $a_1(\tau)$, $a_2(\tau)$, $a_3(\tau)$, $a_4(\tau)$ and $a_5(\tau)$ are amplitudes of convection.

Using (4.1)-(4.3) in equations (2.15)-(2.17), we arrive at a system of ordinary differential equations known as generalized Lorenz model for linear study:

$$\frac{1}{2\,\mathrm{Pr}_{nf} F(V)} a_1'(\tau) = -\frac{a a_1(\tau)}{\delta^2} - r a^2 a_2(\tau), \tag{4.4}$$

$$a_2'(\tau) = -a\delta^2 a_2(\tau) - a_1(\tau), \tag{4.5}$$

$$a_3'(\tau) = -4a\pi^2 a_3(\tau), \tag{4.6}$$

where $M_{13} = \frac{\pi\kappa^2 M_1 M_3}{\pi^2 + \kappa^2(1+M_1)M_3}$ and $r = \frac{R_{nf}}{R_{nfs}}$.

Lorenz model for non-linear study is as:

$$\left.\begin{aligned} &\tfrac{1}{2\mathrm{Pr}_{nf} F(V)} a_1'(\tau) = -\tfrac{a a_1(\tau)}{\delta^2} - r a^2 a_2(\tau)(1 - M_{13} a_3(\tau)), \\ &a_2'(\tau) = -a\delta^2 a_2(\tau) - a_1(\tau) - \pi a_1(\tau) a_3(\tau), \\ &a_3'(\tau) = -4a\pi^2 a_3(\tau) + \tfrac{\pi}{2} a_1(\tau) a_2(\tau). \end{aligned}\right\} \tag{4.7}$$

5. Results and Discussions

The variations of R_{nfs} has been plotted versus variable viscosity and wave number (see Figures 2–4). Figures 2 and 3 show the variation of R_{nfs} with variable viscosity for variant values of BMP and NBMP respectively and it is found that R_{nfs} decreases with increase in both BMP and NBMP which in turn indicates the destabilization of the system and early onset of convection.

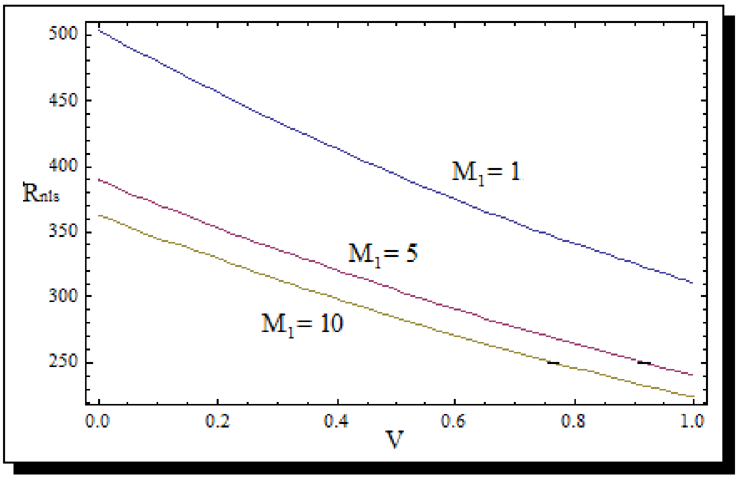

Figure 2. Plot of R_{nfs} vs variable viscosity, V for variant values of BMP, M_1

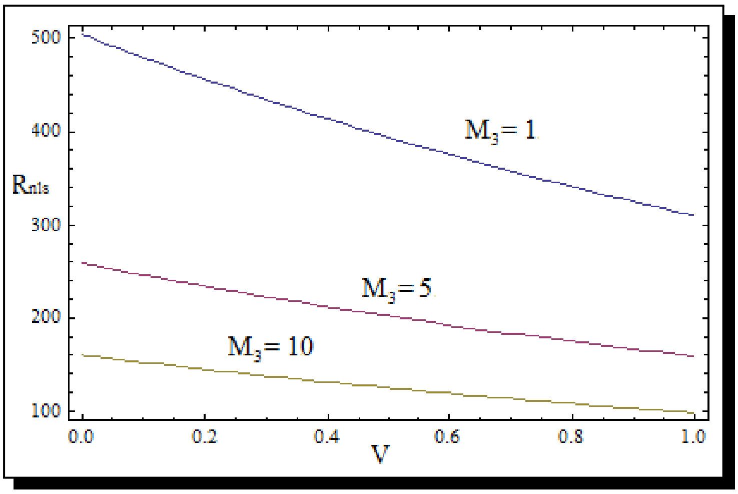

Figure 3. Plot of R_{nfs} vs variable viscosity, V for variant values of NBMP, M_3

Figure 4 shows the variation of R_{nfs} with wavenumber for variant values of variable viscosity and for the fixed values of BMP and NBMP. Positive values of variable viscosity parameter means the superiority of temperatura dependent viscosity where as negative values of variable viscosity parameter means the superiority of magnetic field dependent viscosity. From the graph it is clear that as variable viscosity parameter increases, termal Rayleigh number decreases thereby indicating the early onset of convection. Comparably, when temperaturea dependent viscosity rules, there is an early onset of convection and when magnetic field dependent viscosity rules, there is a delay in onset of convection. So, it becomes evident that variable viscosity parameter can be used to regulate the stabilization of the fluid system. Henceforth, one can conclude that, the early onset of convection takes place in temperatura dominance viscosity compared to magnetic-field dominance viscosity.

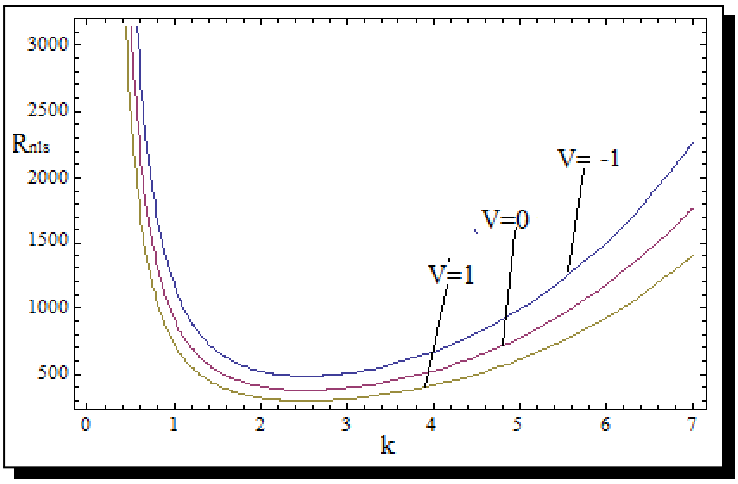

Figure 4. Plot of R_{nfs} vs wave number, κ for variant values of variable viscosity, V

In the absence of variable viscosity and magnetic field, $V = 0$ and M_1, $M_3 = 0$, respectively. $V = 0$ implies $F(V) = \frac{\delta^4}{2}$ and $M_1, M_3 = 0$ implies $M_{13} = 0$, both when applied in the Lorenz model of non-linear sort (4.4)-(4.6), reverts back to Classical Lorenz system.

Acknowledgement

Both the authors are grateful to Prof. P. G. Siddheshwar and BNM Institute of Technology, Bengaluru (affiliated to VTU, Belagavi) for their research support. S. Rajashree thank the Department of Collegiate Education, Government of Karnataka for research support. We also wish to thank the reviewers for their comments.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Agarwal and B. S. Bhadauria, Convective heat transport by longitudinal rolls in dilute nanoliquids, *Journal of Nanofluids* **3**(4) (2014), 380 – 390, DOI: 10.1166/jon.2014.1110.

[2] E. Md. M. Alam, S. Huq, Md. S. Uddin and M. M. Rahman, Effect of sinusoidal thermal boundary condition on unsteady magnetohydrodynamics convection in a square enclosure filled with Fe_3O_4-water ferrofluid, *International Journal of Statistics and Applied Mathematics* **4**(6) (2019), 111 – 127, URL: http://www.mathsjournal.com/pdf/2019/vol4issue6/PartB/4-6-4-854.pdf.

[3] G. K. Auernhammer and H. R. Brand, Thermal convection in a rotating layer of a magnetic fluid, *The European Physical Journal B – Condensed Matter and Complex Systems* **16** (2000), 157 – 168, DOI: 10.1007/s100510070261.

[4] H. C. Brinkman, The viscosity of concentrated suspensions and solutions, *The Journal of Chemical Physics* **20** (1952), 571 – 571, DOI: 10.1063/1.1700493.

[5] J. Buongiorno, *Convective transport in nanofluids*, *ASME Journal of Heat and Mass Transfer* **128**(3) (2006), 240 – 250, DOI: 10.1115/1.2150834.

[6] A. J. Chamkha, S. K. Jena and S. K. Mahapatra, MHD convection of nanofluids: A review, *Journal of Nanofluids* **4**(3) (2015), 271 – 292, DOI: 10.1166/jon.2015.1166.

[7] B. A. Finlayson, Convective instability of ferromagnetic fluids, *Journal of Fluid Mechanics* **40**(4) (1970), 753 – 767, DOI: 10.1017/S0022112070000423.

[8] K. Gotoh and M. Yamada, Thermal convection in a horizontal layer of magnetic fluids, *Journal of the Physical Society of Japan* **51**(9) (1982), 3042 – 3048, DOI: 10.1143/JPSJ.51.3042.

[9] R. L. Hamilton and O. K. Crosser, Thermal conductivity of heterogeneous two-component systems, *Industrial and Engineering Chemistry Fundamentals* **1** (1962), 187 – 191, DOI: 10.1021/i160003a005.

[10] P. N. Kaloni and J. X. Lou, Convective instability of magnetic fluids, *Physical Review E* **70** (2004), 026313, DOI: 10.1103/PhysRevE.70.026313.

[11] J. Kim, Y. T. Kang and C. K. Choi, Analysis of convective instability and heat transfer characteristics of nanofluids, *Physics of Fluids* **16** (2004), 2395 – 2401, DOI: 10.1063/1.1739247.

[12] M. T. Krauzina, A. A. Bozhko, G. F. Putin and S. A. Suslov, Intermittent flow regimes near the convection threshold in ferromagnetic nanofluids, *Physical Review E* **51** (2015), 013010, DOI: 10.1103/PhysRevE.91.013010.

[13] D. Laroze, P. G. Siddheshwar and H. Pleiner, Chaotic convection in a ferrofluid, *Communications in Nonlinear Science and Numerical Simulation* **18**(9) (2013), 2436 – 2447, DOI: 10.1016/j.cnsns.2013.01.016.

[14] S. Maruthamanikandan, *Instabilities in Ferromagnetic, Dielectric and Other Complex Liquids*, PhD Thesis, Bangalore University, India (2005), URL: https://people.bath.ac.uk/ensdasr/PAPERS/Ph.D.%20Thesis%20-%20Mani.pdf.

[15] S. Odenbach, Microgravity experiments on thermomagnetic convection in magnetic fluids, *Journal of Magnetism and Magnetic Materials* **149**(1-2) (1995), 155 – 157, DOI: 10.1016/0304-8853(95)00360-6.

[16] H. F. Oztop and E. Abu-Nada, Numerical study of natural convection in partially heated rectangular enclosures filled with nanofluids, *International Journal of Heat and Fluid Flow* **29**(5) (2008), 1326 – 1336, DOI: 10.1016/j.ijheatfluidflow.2008.04.009.

[17] N. Putra, W. Roetzel and S. K. Das, Natural convection of nano-fluids, *Heat Mass Transfer* **39** (2003), 775 – 784, DOI: 10.1007/s00231-002-0382-z.

[18] M. Sheikholeslami and A. J. Chamkha, Flow and convective heat transfer of a ferro-nanofluid in a double-sided lid-driven cavity with a wavy wall in the presence of a variable magnetic field, *Numerical Heat Transfer, Part A: Applications* **69**(10) (2016), 1186 – 1200, DOI: 10.1080/10407782.2015.1125709.

[19] M. Sheikholeslami, Influence of coulomb forces on Fe_3O_4–H_2O nanofluid thermal improvement, *International Journal of Hydrogen Energy* **42**(2) (2017), 821 – 829, DOI: 10.1016/j.ijhydene.2016.09.185.

[20] I. S. Shivakumara, N. Rudraiah and C. E. Nanjundappa, Effect of non-uniform basic temperature gradient on Rayleigh–Benard–Marangoni convection in ferrofluids, *Journal of Magnetism and Magnetic Materials* **248**(3) (2002), 379 – 395, DOI: 10.1016/S0304-8853(02)00151-8.

[21] P. G. Siddheshwar and A. Abraham, Effect of time-periodic boundary temperatures/body force on Rayleigh–Benard convection in a ferromagnetic fluid, *Acta Mechanica* **161** (2003), 131 – 150, DOI: 10.1007/s00707-002-1004-z.

[22] P. G. Siddheshwar and N. Meenakshi, Amplitude equation and heat transport for Rayleigh–Bénard convection in newtonian liquids with nanoparticles, *International Journal of Applied and Computational Mathematics* **3** (2017), 271 – 292, DOI: 10.1007/s40819-015-0106-y.

[23] P. G. Siddheshwar, C. Kanchana, Y. Kakimoto and A. Nakayama, Steady finite-amplitude Rayleigh–Bénard convection in nanoliquids using a two-phase model: Theoretical answer to the phenomenon of enhanced heat transfer, *ASME Journal of Heat and Mass Transfer* **139**(1) (2017), 012402, DOI: 10.1115/1.4034484.

[24] P. J. Stiles and M. Kagan, Thermoconvective instability of a horizontal layer of ferrofluid in a strong vertical magnetic field, *Journal of Magnetism and Magnetic Materials* **85**(1-3) (1990), 196 – 198, DOI: 10.1016/0304-8853(90)90050-Z.

[25] D. Y. Tzou, Instability of nanofluids in natural convection, *ASME Journal of Heat and Mass Transfer* **130**(7) (2008), 072401, DOI: 10.1115/1.2908427.

[26] D. Y. Tzou, Thermal instability of nanofluids in natural convection, *International Journal of Heat and Mass Transfer* **51**(11-12) (2008), 2967 – 2979, DOI: 10.1016/j.ijheatmasstransfer.2007.09.014.

[27] H. Yamaguchi, I. Kobori, Y. Uehata and K. Shimada, Natural convection of magnetic fluid in a rectangular box, *Journal of Magnetism and Magnetic Materials* **201**(1-3) (1999), 264 – 267, DOI: 10.1016/S0304-8853(99)00022-0.