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Research Article

Restrained and Total Restrained Domination of Ladder Graphs

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Abstract. Telle and Proskurowksi introduced restrained domination as a vertex partition problem in partial k-tress (Algorithms for vertex partitioning problems on partial k-trees, *SIAM Journal on Discrete Mathematics* **10**(4) (1997), 529 – 550). For a graph $G(V,E)$, a restrained domination number is the minimum cardinality of a subset $\mathfrak{D}$ of V such that for every vertex $v \in \bar{\mathfrak{D}}$ there is a vertex in $\mathfrak{D}$ as well as in $\bar{\mathfrak{D}}$ adjacent to v. If $\mathfrak{D}$ satisfies an additional condition that every vertex of V has a neighbor in $\mathfrak{D}$, then $\mathfrak{D}$ is said to be a total restrained dominating set. Minimum cardinality of $\mathfrak{D}$ is said to be total restrained domination number of graph G. In this paper we have obtained domination, restrained, total and total restrained domination number of some ladder graphs.

Keywords. Domination, Total domination, Restrained domination

Mathematics Subject Classification (2020). 05C69

1. Introduction

Interconnection network problems can be modelled by graphs by denoting websites as vertices and the interconnecting links as edges, hence graph theoretical concepts can be used to understand complicated network problems and are used to arrive at the solution. One such well known graph theoretical problem is the concept of domination. A wide range of research has been done on variants of domination due to its theoretical concepts in tackling many practical situations.

Let $G(V,E)$ denote a simple connected, finite graph. For any two vertices $a,b \in V$, b is said to be a neighbor of a if ab is an edge in G. A subset $\mathfrak{D}$ of V is said to be a dominating set of G, if every vertex in $\bar{\mathfrak{D}} = V - \mathfrak{D}$ has a neighbor in $\mathfrak{D}$. If every vertex in $\bar{\mathfrak{D}}$ has a neighbor in both $\mathfrak{D}$ and $\bar{\mathfrak{D}}$, then $\mathfrak{D}$ is said to be a restrained dominating set. Suppose every vertex of V has a neighbor in $\mathfrak{D}$ then $\mathfrak{D}$ is a total dominating set. If $\mathfrak{D}$ satisfies the condition of restrained as well as total then $\mathfrak{D}$ is called a total restrained dominating set. Then minimum cardinality of each of the above possible set of $\mathfrak{D}$ are respectively called as domination number $\gamma(G)$, restrained domination number $\gamma_r(G)$, total domination number $\gamma_t(G)$ and total restrained domination number $\gamma_{tr}(G)$, respectively.

In a prison, the aim is to watch each prisoner by at least one guard and to protect the right of each prisoner, he needs to be observed by at least one of the other prisoners. This should be achieved with minimum number of guards to minimize the cost. This problem can be answered using the concept of restrained domination number of a graph representing the jail network (jail rooms, other points where prisoners and guards to be kept). Here guards represent the vertices of restrained dominating set $\mathfrak{D}$ and the prisoners represent vertices of $\bar{\mathfrak{D}}$. For more details on domination and related work refer Domke *et al*. [3], Domke *et al*. [4], Hattingh and Plummer [5], Hattingh and Joubert [6–8], Henning [13]. Similarly, we can answer many such situations using the concepts of domination. For more survey work on total restrained domination, we refer Cockayne *et al*. [1], Cyman and Raczek [2], Hattingh *et al*. [9–11], Haynes *et al*. [12], Henning and Martiz [14], Raczek and Cyman [15], and Telle and Proskurowski [16].

In this paper domination, restrained, total, total restrained domination of different ladder graphs are discussed and the exact value γ, γ_r, γ_t and γ_{tr} for the above graphs are obtained. Throughout the paper, let $\mathfrak{D}$, $\mathfrak{D}_r$, $\mathfrak{D}_t$, $\mathfrak{D}_{tr}$ denote a dominating, restrained dominating, total dominating and total restrained dominating set with minimum cardinality of ladder graphs.

Different ladder graphs can be obtained from two paths $u_1 - u_2 - \dots - u_n$ and $v_1 - v_2 - \dots - v_n$ ($n \geq 2$) are given in Table 1 with their edge set and graph structure.

Table 1

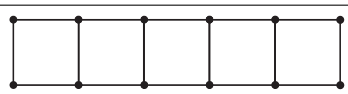
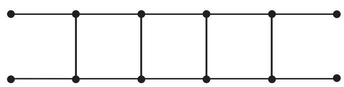
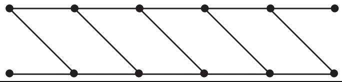
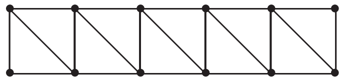
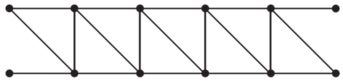
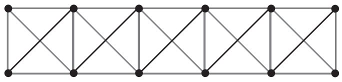
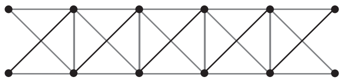
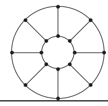
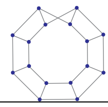
	Name of the graph G	Edge set $E(G)$	Example
1	Ladder graph L_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup$ $\{u_iv_i : 1 \leq i \leq n\}$	
2	Open Ladder OL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup$ $\{u_iv_i : 2 \leq i \leq n-1\}$	

Table *Contd*.

	Name of the graph G	Edge set $E(G)$	Example
3	Slanting Ladder SL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_{i+1} : 1 \le i \le n-1\}$	
4	Triangular Ladder TL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_i : 1 \le i \le n\} \cup$ $\{u_iv_{i+1} : 1 \le i \le n-1\}$	
5	Open Triangular Ladder OTL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_i : 2 \le i \le n\} \cup$ $\{u_iv_{i+1} : 1 \le i \le n-1\}$	
6	Diagonal Ladder DL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_i : 1 \le i \le n\} \cup$ $\{u_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_{i-1} : 2 \le i \le n\}$	
7	Open Diagonal Ladder ODL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_i : 2 \le i \le n-1\} \cup$ $\{u_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_{i-1} : 2 \le i \le n\}$	
8	Circular Ladder CL_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_i : 1 \le i \le n\} \cup \{u_1u_n, v_1v_n\}$	
9	Mobius Ladder Graph M_n	$\{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup$ $\{u_iv_i : 1 \le i \le n\} \cup \{u_1v_n,\ u_nv_1\}$	

2. Ladder and Open Ladder Graphs

In this section we discuss all the above defined domination variants for ladder and open ladder graphs and then give the exact values of each of the domination number.

Theorem 2.1. *For the ladder graph L_n $(n \ge 3)$,*

(i) $\gamma(L_n) = \gamma_r(L_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$,

(ii) $\gamma_t(L_n) = \gamma_{tr}(L_n) = 2\left\lceil \frac{n}{3} \right\rceil$.

Proof. For $n = 3,4$, it is easily verifiable:

- For $n = 3$,
$$\mathfrak{D} = \mathfrak{D}_r = \mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_2, v_2\} \text{ and } \gamma(L_3) = \gamma_r(L_3) = \gamma_t(L_3) = \gamma_{tr}(L_3) = 2.$$
- For $n = 4$,
$$\mathfrak{D} = \mathfrak{D}_r = \{u_1, v_3, v_4\} \text{ and } \gamma(L_4) = \gamma_r(L_4) = 3,$$
$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_1, u_2, v_3, v_4\} \text{ and } \gamma_t(L_4) = \gamma_{tr}(L_4) = 4.$$

For $n \ge 5$, results are discussed in two different cases depending on the value of n.

(i) For $\gamma(L_n)$ and $\gamma_r(L_n)$

Case 1: When $n \equiv 0,\, 2 \pmod 4$

$$\mathfrak{D} = \mathfrak{D}_r = \{v_i : 1 \le i \equiv 1 \pmod 4 \le n\} \cup \{u_i : 3 \le i \equiv 3 \pmod 4 \le n\} \cup \{x\}$$

where $x = \begin{cases} v_{n-1}, & \text{when } n \equiv 0 \pmod 4, \\ u_{n-1}, & \text{when } n \equiv 2 \pmod 4. \end{cases}$

Case 2: When $n \equiv 1,3 \pmod 4$

$$\mathfrak{D} = \mathfrak{D}_r = \{v_i : 1 \le i \equiv 1 \pmod 4 \le n\} \cup \{u_i : 3 \le i \equiv 3 \pmod 4 \le n\}.$$

In both the cases, $|\mathfrak{D}| = \left\lfloor \frac{n}{2} \right\rfloor + 1$.
Here, each vertex $w \in \bar{\mathfrak{D}}_r$ is dominated by exactly one vertex of $\mathfrak{D}_r$ except v_{n-2} and u_{n-2} in case of $n = 4k$ and $n = 4k+2$ ($k \in \mathbb{Z}^+$), respectively, which are dominated by two vertices of $\mathfrak{D}_r$. Also, each vertex w has atleast one neighbour in $\bar{\mathfrak{D}}_r$. In both the cases removal of a vertex from $\mathfrak{D}_r$ fails the condition of domination. Hence $\mathfrak{D}_r$ is a minimal dominating and a restrained dominating set.
Therefore, $\gamma(L_n) = \gamma_r(L_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

(ii) For $\gamma_t(L_n)$ and $\gamma_{tr}(L_n)$.

Case 1: When $n \equiv 0,1,3 \pmod 4$

$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{v_i, u_i : 2 \le i \equiv 2 \pmod 3 \le n\}$$

Case 2: When $n \equiv 2 \pmod 4$,

$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{v_i, u_i : 2 \le i \equiv 2 \pmod 3 \le n-2\} \cup \{v_n,\ u_n\}.$$

In both the cases, $|\mathfrak{D}_{tr}| = 2\left\lceil \frac{n}{3} \right\rceil$.
Each vertex $w \in V$ is dominated by exactly one vertex of $\mathfrak{D}_{tr}$ except v_{n-1} and u_{n-1} in *Case* 2 which are dominated by two vertices of $\mathfrak{D}_{tr}$. Also, each vertex w has atleast one neighbour in $\bar{\mathfrak{D}}_{tr}$. In both the cases removal of a vertex from $\mathfrak{D}_{tr}$ fails the condition of domination.
Hence, $\gamma(L_n) = \gamma_r(L_n) = 2\left\lceil \frac{n}{3} \right\rceil$. □

Lemma 2.1. *Every restrained dominating set of a graph contains all the pendant vertices of G.*

Proof. Let $\mathfrak{D}$ be a dominating set of G and v be a pendant vertex not in $\mathfrak{D}$. Then there is a vertex $u \in \mathfrak{D}$ such that vu is the pendant edge. Now, $v \in \bar{\mathfrak{D}}$ has no neighbours in $\bar{\mathfrak{D}}$ which fails the condition of restrained domination. Hence the result. □

Theorem 2.2. *For an open ladder graph OL_n ($n \ge 3$),*

(i) $\gamma(OL_n) = \begin{cases} 2, & \textit{when } n = 3, \\ \left\lfloor \frac{n}{2} \right\rfloor + 2, & \textit{when } n \ge 4. \end{cases}$

(ii) $\gamma_r(OL_n) = \begin{cases} 4, & \textit{when } n = 3,4, \\ \left\lfloor \frac{n}{2} \right\rfloor + 3, & \textit{when } n \ge 5. \end{cases}$

(iii) $\gamma_t(OL_n) = \begin{cases} 2, & \textit{when } n = 3, \\ 2\left(\left\lfloor \frac{n-4}{3} \right\rfloor + 2\right), & \textit{when } n \ge 4. \end{cases}$

(iv) $\gamma_{tr}(OL_n) = \begin{cases} 2n & \textit{when } n = 3,4, \\ 2\left(\left\lfloor \frac{n-4}{3} \right\rfloor + 4\right) & \textit{when } n \geq 5. \end{cases}$

Proof. For $n = 3,4$, it is easily verifiable

- For $n = 3$,

$$\mathfrak{D} = \mathfrak{D}_t = \{u_2, v_2\} \text{ and } \gamma(OL_3) = \gamma_t(OL_3) = 2,$$
$$\mathfrak{D}_r = \{u_1, v_1, u_3, v_3\} \text{ and } \gamma_r(OL_3) = 4,$$
$$\mathfrak{D}_{tr} = V(OL_3) \text{ and } \gamma_{tr}(OL_3) = 6.$$

- For $n = 4$,

$$\mathfrak{D} = \mathfrak{D}_t = \{u_2, u_3, v_2, v_3\} \text{ and } \gamma(OL_4) = \gamma_t(OL_4) = 4,$$
$$\mathfrak{D}_r = \{u_1, v_1, u_4, v_4\} \text{ and } \gamma_r(OL_4) = 4,$$
$$\mathfrak{D}_{tr} = V(OL_4) \text{ and } \gamma_{tr}(OL_4) = 8.$$

For $n \geq 5$:

(i) Let $\mathfrak{D} = \{u_1, u_i : 4 \leq i \equiv 0 \pmod 4 \leq n\} \cup \{v_i : 2 \leq i \equiv 2 \pmod 4 \leq n\} \cup \{x, y\}$ where $x = \begin{cases} \emptyset, & \text{when } n \equiv 0,1 \pmod 4, \\ u_n, & \text{when } n \equiv 2,3 \pmod 4 \end{cases}$ and $y = \begin{cases} v_n, & \text{when } n \equiv 0,1 \pmod 4, \\ \emptyset, & \text{when } n \equiv 2,3 \pmod 4, \end{cases}$ such that $|\mathfrak{D}| = \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Each vertex $w \in \bar{\mathfrak{D}}$ is dominating by exactly one vertex of $\mathfrak{D}$ except v_{n-1} and u_{n-1} in case of $n \equiv 0,1 \pmod 4$ and $n \equiv 2,3 \pmod 4$, respectively, which are dominated by two vertices of $\mathfrak{D}$ due to the existence of pendant vertices. Hence, $\gamma(OL_n) = \left\lfloor \frac{n}{2} \right\rfloor + 2$.

(ii) From Lemma 2.1, for restrained domination, let $\mathfrak{D}_r = S_1 \cup S_2 \cup S_3$, where $S_1 = \{u_1, v_1, u_n, v_n\}$, $S_2 = \{u_i : 3 \leq i \equiv 3 \pmod 4 < n\}$ and $S_3 = \{v_i : 5 \leq i \equiv 1 \pmod 4 < n\}$. Each vertex $w \in \bar{\mathfrak{D}}_r$ is dominated by exactly one vertex of $\mathfrak{D}_r$ except u_2 and v_{n-1} and u_2 and u_{n-1} in the case of $n \equiv 0,3 \pmod 4$ and $n \equiv 1,2 \pmod 4$, respectively, which are dominated by two vertices of $\mathfrak{D}_r$ and are due to the mandatory inclusion of pendant vertices. Also, each vertex w has atleast one neighbour in $\bar{\mathfrak{D}}_r$,

$$|\mathfrak{D}_r| = 4 + \left\lceil \frac{n-3}{4} \right\rceil + \left\lceil \frac{n-5}{4} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor + 3 = \gamma_r(OL_n).$$

(iii) In the graph OL_n to satisfy total domination condition $\mathfrak{D}_t$ must contain vertices u_2, v_2, and u_{n-1}, v_{n-1} which are dominating the pendant vertices. Let $\mathfrak{D}_t = S_1 \cup S_2$, where $S_1 = \{u_2, v_2, u_{n-1}, v_{n-1}\}$, $S_2 = \{u_i, v_i : 5 \leq i \equiv 2 \pmod 3 \leq n-2\}$. Each vertex $w \in V$ is dominated by exactly one vertex of $\mathfrak{D}_t$ except u_{n-2} and v_{n-2} in case of $n \equiv 0,3 \pmod 4$ which are dominated by two vertices of $\mathfrak{D}_t$ and are due to the mandatory inclusion of vertices adjacent to pendant vertices, $|\mathfrak{D}_t| = 2\left(2 + \left\lfloor \frac{n-4}{3} \right\rfloor\right) = \gamma_t(OL_n)$.

(iv) It is to be observed that with reference to the $\mathfrak{D}_r$ defined in (iii), each vertex $w \in \bar{\mathfrak{D}}_r$ has at least one neighbour in $\bar{\mathfrak{D}}_r$ except for the pendant vertices. By Lemma 2.1, $\mathfrak{D}_{tr} = \mathfrak{D}_r \cup \{u_1, v_1, u_n, v_n\}$ satisfies the condition of restrained domination which is also total domination, $|\mathfrak{D}_{tr}| = 2\left(4 + \left\lfloor \frac{n-4}{3} \right\rfloor\right) = \gamma_{tr}(OL_n)$. □

3. Results on Slanting Ladder Graph

In this section, we have obtained domination, restrained, total and total restrained domination number for slanting ladder graph.

Theorem 3.1. *For a slanting ladder graph SL_n $(n \geq 3)$,*

(i) $\gamma(SL_n) = \begin{cases} 2, & \text{for } n = 3,4, \\ 2\left\lfloor \frac{n}{4} \right\rfloor + 1, & \text{for } 5 \leq n \equiv 1 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil, & \text{for } 6 \leq n \equiv 0,2,3 \pmod 4. \end{cases}$

(ii) $\gamma_r(SL_n) = \begin{cases} n, & \text{for } n = 3,4,5, \\ 2\left(1 + \left\lfloor \frac{n}{4} \right\rfloor\right), & \text{for } 5 < n \equiv 1,2 \pmod 4, \\ 2 + \left\lceil \frac{n}{4} \right\rceil + \left\lfloor \frac{n}{4} \right\rfloor, & \text{for } 4 < n \equiv 0,3 \pmod 4. \end{cases}$

(iii) $\gamma_t(SL_n) = \begin{cases} 3, & \text{for } n = 3, \\ 2\left\lceil \frac{n}{3} \right\rceil, & \text{for } n \equiv 0,4,5 \pmod 6, \\ 4\left\lfloor \frac{n}{6} \right\rfloor + r, & \text{for } n \equiv r \pmod 6, \end{cases}$

where $r \in \{1,2,3\}$.

(iv) $\gamma_{tr}(SL_n) = 4 + 4\left\lfloor \frac{n}{6} \right\rfloor + c$,

where $c = \begin{cases} -2, & \text{for } n \equiv 0,1 \pmod 6, \\ 0, & \text{for } n \equiv 2,3,4 \pmod 6, \\ 1, & \text{for } n \equiv 5 \pmod 6, \end{cases}$

when $n \geq 5$.

Proof. For $n = 3,4$, it is easily verifiable:

- For $n = 3$,

$$\mathfrak{D} = \{u_2, v_2\} \text{ and } \gamma(SL_3) = 2,$$
$$\mathfrak{D}_t = \mathfrak{D}_r = \mathfrak{D}_{tr} = \{v_1, v_2, u_2, u_3\} \text{ and } \gamma_t(SL_3) = \gamma_r(SL_3) = \gamma_{tr}(SL_3) = 4.$$

- For $n = 4$,

$$\mathfrak{D} = \{u_3, v_2\} \text{ and } \gamma(SL_4) = 2,$$
$$\mathfrak{D}_t = \mathfrak{D}_r = \mathfrak{D}_{tr} = \{v_1, v_2, u_3, u_4\} \text{ and } \gamma_t(SL_4) = \gamma_r(Sl_4) = \gamma_{tr}(SL_4) = 4.$$

For $n \geq 5$,

(i) Consider,

$$\mathfrak{D} = \{u_i : i \equiv 3 \pmod 4\} \cup \{v_i : i \equiv 2 \pmod 4\} \cup \{x, y\},$$

where $x = \begin{cases} u_{n-1}, & n \equiv 1,2 \pmod 4, \\ \emptyset, & n \equiv 0,3 \pmod 4, \end{cases}$ and $y = \begin{cases} v_{n-1}, & n \equiv 2,3 \pmod 4, \\ \emptyset, & n \equiv 0,1 \pmod 4, \end{cases}$

$$|\mathfrak{D}| = \begin{cases} 2\left\lfloor \frac{n}{4} \right\rfloor + 1, & n \equiv 1 \pmod 4, \\ 2\left\lceil \frac{n}{4} \right\rceil, & n \equiv 0,2,3 \pmod 4. \end{cases}$$

Here, each vertex $w \in \bar{\mathfrak{D}}$ is dominated by exactly one vertex of $\mathfrak{D}$ except u_{n-2} in case of $n \equiv 2 (\text{mod} 4)$ which is dominated by two vertices. Hence the result.

(ii) For $n \geq 4$, from Lemma 2.1, for restrained domination $\mathfrak{D}_r$ must contain all the pendant vertices. Let

$$\mathfrak{D}_r = \{v_1, u_n\} \cup \{u_i : i \equiv 2 \ (\text{mod} 4) < n\} \cup \{v_i : 5 \leq i \equiv 1 \ (\text{mod} 4) \leq n\} \cup \{x\},$$

where $x = \begin{cases} u_{n-1}, & \text{when } n \equiv 0 \ (\text{mod} 4), \\ \emptyset, & \text{otherwise.} \end{cases}$

Each vertex $w \in \bar{\mathfrak{D}}_r$ is dominated by exactly one vertex of $\mathfrak{D}_r$ except u_{n-1} in case of $n \equiv 1 \ (\text{mod} 4)$ which is dominated by both u_n and v_n (from Lemma 2.1, u_n cannot be removed, removal of v_n makes v_{n-1} not dominated by any vertex of $\mathfrak{D}_r$). Also, each vertex w has atleast one neighbour in $\bar{\mathfrak{D}}_r$,

$$|\mathfrak{D}_r| = \begin{cases} 2\left(1 + \left\lfloor \frac{n}{4} \right\rfloor\right), & \text{for } n \equiv 1,2 \ (\text{mod} 4), \\ 2 + \left\lceil \frac{n}{4} \right\rceil + \left\lfloor \frac{n}{4} \right\rfloor, & \text{for } n \equiv 0,3 \ (\text{mod} 4). \end{cases}$$

Hence the result.

(iii) Proof is discussed in two cases based on the value of n,

$$\mathfrak{D}_t = \{u_i, u_{i+1} : i \equiv 4 \ (\text{mod} 6) < n\} \cup \{v_i, v_{i+1} : i \equiv 2 \ (\text{mod} 6) < n\} \cup \{x, y\},$$

where $x = \begin{cases} \emptyset, & \text{when } n \equiv 0,5 \ (\text{mod} 6), \\ u_{n-1}, & \text{when } n \equiv 1,2,3,4 \ (\text{mod} 6), \end{cases}$ and $y = \begin{cases} \emptyset, & \text{when } n \equiv 0,3,5 \ (\text{mod} 6), \\ v_n, & \text{when } n \equiv 2,4 \ (\text{mod} 6). \end{cases}$

From the above set $\mathfrak{D}_t$:

Case 1: When $n \equiv 0,4,5 \ (\text{mod} 6)$

$$|\mathfrak{D}_t| = \begin{cases} 2\left(\frac{n}{6} + \frac{n}{6}\right), & \text{for } n \equiv 0 \ (\text{mod} 6), \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 1 + 2\left\lceil \frac{n}{6} \right\rceil + 1, & \text{for } n \equiv 4 \ (\text{mod} 6), \\ 2\left\lceil \frac{n}{6} \right\rceil + 2\left\lceil \frac{n}{6} \right\rceil, & \text{for } n \equiv 5 \ (\text{mod} 6). \end{cases}$$

Combining the above results, $|\mathfrak{D}_t| = 4\left\lceil \frac{n}{6} \right\rceil = 2\left\lceil \frac{n}{3} \right\rceil$.

Here, all the vertices of V are dominated by exactly one vertex of $\mathfrak{D}_t$ except v_n in case of $n \equiv 4 \ (\text{mod} 6)$ which is dominated by both u_{n-1} and v_{n-1} vertices of $\mathfrak{D}_t$, removal of any one of these fails the condition for total domination. Hence $\gamma_t(SL_n) = 2\left\lceil \frac{n}{3} \right\rceil$.

Case 2: When $n \equiv 1,2,3 \ (\text{mod} 6)$

$$|\mathfrak{D}_t| = \begin{cases} 2\left\lfloor \frac{n}{6} \right\rfloor + 1 + 2\left\lfloor \frac{n}{6} \right\rfloor, & \text{for } n \equiv 1 \ (\text{mod} 6), \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 1 + 2\left\lfloor \frac{n}{6} \right\rfloor + 1, & \text{for } n \equiv 2 \ (\text{mod} 6), \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 2\left\lceil \frac{n}{6} \right\rceil + 1, & \text{for } n \equiv 3 \ (\text{mod} 6). \end{cases}$$

Combining the above, $|\mathfrak{D}_t| = 4\left\lfloor \frac{n}{6} \right\rfloor + r$ such that $r \in \{1,2,3\}$ and $n \equiv r \ (\text{mod} 6)$.

Here, all the vertices of V are dominated by exactly one vertex of $\mathfrak{D}_t$ except

- u_{n-2} in case of $n \equiv 1 \ (\text{mod} 6)$ is dominated by both u_{n-1} and u_{n-3}.
- u_{n-3} in case of $n \equiv 2 \ (\text{mod} 6)$ is dominated by both u_{n-1} and u_{n-3}.
- u_{n-2} in case of $n \equiv 3 \ (\text{mod} 6)$ is dominated by both u_{n-1} and v_{n-1}.

Removal of any one of these vertices fails the condition for domination. Hence the result.

(iv) Let $\mathfrak{D}_{tr} = S_1 \cup S_2 \cup S_3$ where $S_1 = \{v_1, v_2, u_n, u_{n-1}\}$, $S_2 = \{u_i, u_{i+1} \neq u_{n-1}, i \equiv 3 \pmod 6 \leq n-2\}$ and $S_3 = \{v_i, v_{i+1} : 7 \leq i \equiv 1 \pmod 6 \leq n\}$. From Lemma 2.1, $\mathfrak{D}_{tr}$ must contain S_1 and also all the vertices of V are dominated by exactly one vertex of $\mathfrak{D}_{tr}$ except.

(v) u_{n-1} and u_{n-2} in case of $n \equiv 0 \pmod 6$, mutually dominates each other and are also dominated by u_n and u_{n-3}, respectively.

(vi) $u_{n-2} \notin \mathfrak{D}_{tr}$ is dominated by u_{n-1} and u_{n-3}; $u_{n-1} \in \mathfrak{D}_{tr}$ is dominated by both v_n and u_n in case of $n \equiv 1 \pmod 6$.

(vii) v_n and u_{n-1} in case of $n \equiv 2 \pmod 6$ mutually dominates each other and are also dominated by v_{n-1} and u_n of $\mathfrak{D}_{tr}$, respectively.

(viii) v_n in case of $n \equiv 3 \pmod 6$ is dominated by two vertices of $\mathfrak{D}_{tr}$.

(ix) $u_{n-1} \in \mathfrak{D}_{tr}$ in case of $n \equiv 5 \pmod 6$ is dominated by two vertices of $\mathfrak{D}_{tr}$.

Removal of any one of these vertices from $\mathfrak{D}_{tr}$ fails the condition for total domination. Hence

$$\gamma_t(SL_n) = \begin{cases} 4 + 4\left\lfloor \frac{n}{6} \right\rfloor - 2, & \text{for } n \equiv 0,1 \pmod 6, \\ 4 + 4\left\lfloor \frac{n}{6} \right\rfloor + 0, & \text{for } n \equiv 2,3,4 \pmod 6, \\ 4 + 4\left\lfloor \frac{n}{6} \right\rfloor + 1, & \text{for } n \equiv 5 \pmod 6. \end{cases}$$

Thus,

$$\gamma_{tr}(SL_n) = 4 + 4\left\lfloor \frac{n}{6} \right\rfloor + c,$$

$$\text{where } c = \begin{cases} -2, & \text{for } n \equiv 0,1 \pmod 6, \\ 0, & \text{for } n \equiv 2,3, \pmod 6 \text{ and } n \geq 5, \\ 1, & \text{for } n \equiv 5 \pmod 6. \end{cases}$$

□

4. Results on Triangular and Open-Triangular Ladder Graphs

In this section, restrained and total restrained domination number of triangular and open triangular ladder graphs are discussed, and their values are given.

Theorem 4.1. *For a triangular ladder graph TL_n $(n \geq 3)$:*

(i) $\gamma(TL_n) = \gamma_r(TL_n) = \begin{cases} 2\left\lfloor \frac{n}{5} \right\rfloor + 1, & \textit{for } n \equiv 1,2 \pmod 5, \\ 2\left\lceil \frac{n}{5} \right\rceil, & \textit{for } n \equiv 0,3,4 \pmod 5, \end{cases}$

(ii) $\gamma_t(TL_n) = \gamma_{tr}(TL_n) = \begin{cases} 4\left\lceil \frac{n}{7} \right\rceil, & \textit{for } n \equiv 0,5,6 \pmod 7, \\ 4\left\lfloor \frac{n}{7} \right\rfloor + c, & \textit{for } n \equiv 1,2,3,4 \pmod 7, \end{cases}$

$$\textit{where } c = \begin{cases} 1, & n \equiv 1 \pmod 7, \\ 2, & n \equiv 2,3 \pmod 7, \\ 3, & n \equiv 4 \pmod 7. \end{cases}$$

Proof. (i) For $n \geq 3$: Let

$$\mathfrak{D} = \mathfrak{D}_r = \{v_i : i \equiv 2 \pmod 5 \leq n\} \cup \{u_j : j \equiv 4 \pmod 5 \leq n\} \cup \{x\},$$

$$\text{where } x = \begin{cases} u_n, & n \cong 1,3 \pmod 5, \\ \emptyset, & n \cong 0,2,4 \pmod 5. \end{cases}$$

Each vertex $w \in \bar{\mathfrak{D}}$ is dominated by exactly one vertex of $\mathfrak{D}$ except v_{n-2} and u_{n-2} in case of $n = 4k$ and $n = 4k+2$ ($k \in \mathbb{Z}^+$), respectively, which are dominated by two vertices of $\mathfrak{D}$. Also, each vertex w has atleast one neighbour in $\bar{\mathfrak{D}}$.

Hence, $|\mathfrak{D}| = |\mathfrak{D}_r| = \begin{cases} 2\left\lfloor \frac{n}{5} \right\rfloor + 1, & \text{when } n \equiv 1,2 \pmod 5, \\ 2\left\lceil \frac{n}{5} \right\rceil, & \text{when } n \equiv 0,3,4 \pmod 5). \end{cases}$

(ii) For $n = 3$,

$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{v_2, u_2\} \text{ and } \gamma_t(TL_3) = \gamma_{tr}(TL_3) = 2.$$

For $n = 4$,

$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{v_2, u_2, u_3\} \text{ and } \gamma_t(TL_4) = \gamma_{tr}(TL_4) = 3.$$

For $n \geq 5$: Let

$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{v_i, v_{i+1} : i \equiv 2 \pmod 7 \leq n)\} \cup \{u_i, u_{i+1} : i \equiv 4 \pmod 7 \leq n\} \cup \{x\},$$

where $x = \begin{cases} u_n, & n \cong 2 \pmod 7, \\ v_n, & n \cong 4,5 \pmod 7, \\ u_{n-1}, & n \cong 1 \pmod 7, \\ \emptyset, & n \cong 0,3,5,6 \pmod 7. \end{cases}$

Each vertex $w \in V$ is dominated by exactly one vertex of $\mathfrak{D}_t$ except v_{n-1} and u_{n-1} in *Case* 2 which are dominated by two vertices of $\mathfrak{D}_t$. Also, each vertex w has atleast one neighbour in $\bar{\mathfrak{D}}_t$. □

Theorem 4.2. *For an open triangular ladder graph OTL_n ($n \geq 3$):*

(i) $\gamma(OTL_n) = \begin{cases} 2\left\lfloor \frac{n}{5} \right\rfloor + 1, & \textit{when } n \equiv 1,2 \pmod 5, \\ 2\left\lceil \frac{n}{5} \right\rceil, & \textit{when } n \equiv 0,3,4 \pmod 5). \end{cases}$

(ii) $\gamma_r(OTL_n) = \begin{cases} 2\left(\left\lfloor \frac{n}{5} \right\rfloor + 1\right), & \textit{when } n \equiv 0,1 \pmod 5, \\ 2\left\lceil \frac{n}{5} \right\rceil + 1, & \textit{when } n \equiv 2,3 \pmod 5, \\ 2\left(\left\lceil \frac{n}{5} \right\rceil + 1\right), & \textit{when } n \equiv 4 \pmod 5). \end{cases}$

(iii) $\gamma_t(OTL_n) = \begin{cases} 4\left\lceil \frac{n}{7} \right\rceil, & \textit{when } n \equiv 0,5,6 \pmod 7, \\ 4\left\lfloor \frac{n}{7} \right\rfloor + k_1, & \textit{when } n \equiv 1,2,3,4 \pmod 7, \end{cases}$

where $k_1 = \begin{cases} 1, & \textit{when } n \equiv 1 \pmod 7, \\ 2, & \textit{when } n \equiv 2,3 \pmod 7, \\ 3, & \textit{when } n \equiv 4 \pmod 7). \end{cases}$

(iv) $\gamma_{tr}(OTL_n) = \begin{cases} 4\left\lceil \frac{n}{7} \right\rceil + k_1, & \textit{when } n \equiv 3,4,5,6 \pmod 7, \\ 4\left\lfloor \frac{n}{7} \right\rfloor + k_2, & \textit{when } n \equiv 0,1,2 \pmod 7, \end{cases}$

where $k_1 = \begin{cases} 0, & \textit{when } n \equiv 3,4,5 \pmod 7, \\ 1, & \textit{when } n \equiv 6 \pmod 7, \end{cases}$ *and* $k_2 = \begin{cases} 2, & \textit{when } n \equiv 0,1 \pmod 7, \\ 3, & \textit{when } n \equiv 2 \pmod 7). \end{cases}$

Proof. (i) Let $\mathfrak{D} = \{v_2, u_{n-1}\} \cup \{u_i, v_j \mid n \geq 6,\ 4 \leq i \equiv 4 \pmod 5 \leq n-2 \text{ and } 2 \leq j \equiv 2 \pmod 5 \leq n-1\}$. As OTL_n graph contains the pendant vertices v_1 and u_n to dominate these vertices $\mathfrak{D}$ should contain the vertices v_2 and u_{n-1}, respectively. It can be observed that all the vertices of $\bar{\mathfrak{D}}$ are dominated by exactly one vertex of $\mathfrak{D}$ except one or two vertices which cannot be avoided. Thus

$$|\mathfrak{D}| = \begin{cases} 2+\lfloor \frac{n}{5} \rfloor + 1 + \lfloor \frac{n}{5} \rfloor, & \text{when } n \equiv 1,2 \pmod 5, \\ 2+\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{5} \rfloor, & \text{when } n \equiv 0,3,4 \pmod 5, \end{cases}$$

$$= \begin{cases} 2\lfloor \frac{n}{5} \rfloor + 1, & \text{when } n \equiv 1,2 \pmod 5, \\ 2\lceil \frac{n}{5} \rceil, & \text{when } n \equiv 0,3,4 \pmod 5. \end{cases}$$

(ii) From Lemma 2.1, we must include the pendant vertices v_1, u_n,

$$\mathfrak{D}_r = \{v_1, u_n\} \cup \{u_i, v_j \mid 2 \le i \equiv 2 \pmod 5 \le n-1 \text{ and } 5 \le j \equiv 0 \pmod 5 \le n-1\} \cup \{x\},$$

where $x = \begin{cases} \phi, & \text{when } n \equiv 1,3 \pmod 5, \\ u_{n-1} \text{ or } v_{n-1}, & \text{when } n \equiv 0,2,4 \pmod 5). \end{cases}$

(iii) For $n = 3,4$, it is easily verifiable:

- For $n = 3$, $\mathfrak{D}_t = \{u_2, v_2\}$ and $\gamma(OTL_3) = 2$.
- For $n = 4$, $\mathfrak{D}_t = \{u_3, v_2, v_3\}$ and $\gamma(OTL_4) = 3$.

Since OTL_n contains two pendant vertices, to dominate these vertices both the pendant vertices or their adjacent vertices should be included in $\mathfrak{D}_t$. But to achieve minimum cardinality of $\mathfrak{D}_t$, $v_2, u_{n-1} \in \mathfrak{D}_t$. Also, to satisfy the condition of total domination $v_3, u_{n-2} \in \mathfrak{D}_t$. Thus,

$$\mathfrak{D}_t = \{v_2, v_3, u_{n-2}, u_{n-1}\} \cup \{u_i, u_{i+1}, v_j, v_{j+1} \mid n \ge 6,\ 5 \le i \equiv 5 \pmod 7 \le n-2 \text{ and } 9 \le j \equiv 2 \pmod 7 \le n-3\}.$$

(iv) Followed by the discussion in (ii) and (iii),

$$\mathfrak{D}_{tr} = \{v_1, v_2, u_{n-1}, u_n\} \cup \{u_i, u_{i+1}, v_j, v_{j+1} \mid 4 \le i \equiv 4 \pmod 7 \le n-2 \text{ and } 8 \le j \equiv 1 \pmod 7 \le n-2\} \cup \{x\},$$

where $x = \begin{cases} v_{n-1}, & \text{when } n \equiv 2 \pmod 7, \\ \phi, & \text{otherwise.} \end{cases}$

Hence the result. □

5. Result on Diagonal Ladder Circular Ladder, and Mobious Graphs

Since diagonal and open open-diagonal ladder graphs contains induced K_4 subgraphs, the results will be the same for both the graphs.

Theorem 5.1. *For a diagonal ladder graph DL_n $(n \ge 3)$:*

(i) $\gamma(DL_n) = \gamma_r(DL_n) = \lceil \frac{n}{3} \rceil$,

(ii) $\gamma_t(DL_n) = \gamma_{tr}(DL_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for } n \equiv 0,1,3 \pmod 4, \\ \frac{n}{2}+1, & \text{for } n \equiv 2 \pmod 4. \end{cases}$

Proof. For $n = 3,4,5$, results are easily verifiable:

- For $n = 3$,
$$\mathfrak{D} = \mathfrak{D}_r = \{u_2\} \text{ and } \gamma(DL_3) = \gamma_r(DL_3) = 1,$$
$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_2, v_2\} \text{ and } \gamma_r(DL_3) = \gamma_{tr}(DL_3) = 2.$$
- For $n = 4$,
$$\mathfrak{D} = \mathfrak{D}_r = \mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_2, u_3\} \text{ and } \gamma(DL_4) = \gamma_r(DL_4) = \gamma_t(DL_4) = \gamma_{tr}(DL_4) = 2.$$
- For $n = 5$,
$$\mathfrak{D} = \mathfrak{D}_r = \{v_2, u_4\} \text{ and } \gamma(DL_5) = \gamma_r(DL_5) = 2,$$
$$\mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_4, v_2, v_3\} \text{ and } \gamma_t(DL_5) = \gamma_{tr}(DL_5) = 3.$$

For $n \ge 6$, dominating sets are:

(i) $\mathfrak{D} = \mathfrak{D}_r = \{v_i \mid 2 \le i \le 2 \ (\mathrm{mod}\, 3) \le n-2\} \cup \{x\}$,
where $x = \begin{cases} \phi, & \text{when } n \equiv 0 \ (\mathrm{mod}\, 3), \\ v_{n-1}, & \text{when } n \equiv 1 \ (\mathrm{mod}\, 3), \\ v_n, & \text{when } n \equiv 2 \ (\mathrm{mod}\, 3). \end{cases}$

(ii) $\mathfrak{D}_t = \mathfrak{D}_{tr} = \{v_i, v_{i+1} \mid 2 \le i \le 2 \ (\mathrm{mod}\, 4) \le n \text{ and } (i+1) \le n\} \cup \{x\}$,
where $x = \begin{cases} v_{n-1}, & \text{when } n \equiv 2 \ (\mathrm{mod}\, 4), \\ \phi, & \text{otherwise.} \end{cases}$

Hence result. □

Theorem 5.2. *For a circular ladder graph CL_n $(n \ge 3)$,*

(i) $\gamma(CL_n) = \gamma_r(CL_n) = 2\left\lceil \frac{n}{4} \right\rceil$,

(ii) $\gamma_t(CL_n) = \gamma_{tr}(CL_n) = \begin{cases} 4\left\lceil \frac{n}{2} \right\rceil, & \text{when } n \equiv 0,4,5 \ (\mathrm{mod}\, 6), \\ 4\left\lfloor \frac{n}{2} \right\rfloor + x, & \text{when } n \equiv x \ (\mathrm{mod}\, 6), \end{cases}$
where $x \in \{1,2,3\}$.

Proof. For $n = 3,4$, results are easily verifiable:
$$\mathfrak{D} = \mathfrak{D}_r = \mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_1, v_3\} \text{ and } \gamma(DL_n) = \gamma_r(DL_n) = \gamma_t(DL_n) = \gamma_{tr}(DL_n) = 2.$$

For $n \ge 5$, following $\mathfrak{D}$ satisfies the conditions of suitable domination and followed by the results:

(i) $\mathfrak{D} = \mathfrak{D}_r = \{u_i, v_{i+2} \mid 1 \le i \le 1 \ (\mathrm{mod}\, 4) \le n \text{ and } (i+2) \le n\} \cup \{x\}$
where $x = \begin{cases} u_n, & \text{when } n \equiv 2 \ (\mathrm{mod}\, 4), \\ \phi, & \text{otherwise.} \end{cases}$

(ii) $\mathfrak{D}_t = \mathfrak{D}_{tr} = \{u_i, u_{i+1}, v_{i+3}, v_{i+4} \mid 1 \le i \le 1 \ (\mathrm{mod}\, 6) < n\} \cup \{x\}$
$x = \begin{cases} \phi, & \text{when } n \equiv 0,2,5 \ (\mathrm{mod}\, 6), \\ u_n, & \text{when } n \equiv 1 \ (\mathrm{mod}\, 6), \\ v_{n-1}, & \text{when } n \equiv 4 \ (\mathrm{mod}\, 6). \end{cases}$ □

Theorem 5.3. *For a mobius ladder graph ML_n $(n \ge 3)$,*

(i) $$\gamma(ML_n)=\gamma_r(ML_n)=\begin{cases} 2\left\lfloor \frac{n}{4}\right\rfloor+3, & \textit{for } n\equiv 0,1,2 \pmod 5,\\ 2\left\lceil \frac{n}{4}\right\rceil, & \textit{for } n\equiv 3 \pmod 5,\\ \frac{n}{4}+1, & \textit{for } n\equiv 3 \pmod 5). \end{cases}$$

(ii) $$\gamma_t(ML_n)=\gamma_{tr}(ML_n)=\begin{cases} n-1, & n=3,4,5,\\ n-2, & n=6,\\ 4\left\lceil \frac{n}{6}\right\rceil+x, & \textit{for } 10\le n\equiv 0,4,5 \pmod 6,\\ 4\left\lfloor \frac{n}{6}\right\rfloor+2, & \textit{for } 7\le n\equiv 1,2,3 \pmod 6). \end{cases}$$

$$\textit{where } x=\begin{cases} -1, & \textit{when } n\equiv 4 \pmod 6,\\ 0, & \textit{when } n\equiv 5 \pmod 6,\\ 1, & \textit{when } n\equiv 0 \pmod 6). \end{cases}$$

Proof. For $n=3,4$, results are easily verifiable.

When $n=3$:

$$\mathfrak{D}=\mathfrak{D}_r=\mathfrak{D}_t=\mathfrak{D}_{tr}=\{u_2,v_2\} \text{ and } \gamma(DL_2)=\gamma_r(DL_2)=\gamma_t(DL_2)=\gamma_{tr}(DL_2)=2.$$

When $n=4$:

$$\mathfrak{D}=\mathfrak{D}_r=\mathfrak{D}_t=\mathfrak{D}_{tr}=\{u_1,v_3,\ v_3\} \text{ and } \gamma(DL_3)=\gamma_r(DL_3)=\gamma_t(DL_3)=\gamma_{tr}(DL_3)=3.$$

(i) For $n\ge 5$, following $\mathfrak{D}$ satisfies the conditions of suitable domination and followed by the results,

$$\mathfrak{D}=\mathfrak{D}_r=\{u_i,v_{i+2}\mid 1\le i\le 1 \pmod 4\le n \text{ and } (i+2)\le n\}\cup\{x\},$$

$$\text{where } x=\begin{cases} u_n, & \text{when } n\equiv 2 \pmod 4,\\ \phi, & \text{otherwise.} \end{cases}$$

(ii) When $n=5,6$:

$$\mathfrak{D}_t=\mathfrak{D}_{tr}=\{u_2,v_2,\ u_5,v_5\} \text{ and } \gamma_t=\gamma_{tr}=4.$$

When $n\ge 7$:

$$\mathfrak{D}_t=\mathfrak{D}_{tr}=\{u_i,u_{i+1},v_{i+3},v_{i+4}\mid 1\le i\le 1 \pmod 6<n\}\cup\{x\},$$

$$\text{where } x=\begin{cases} \phi, & \text{when } n\equiv 0,2,5 \pmod 6,\\ u_n, & \text{when } n\equiv 1 \pmod 6,\\ v_{n-1}, & \text{when } n\equiv 4 \pmod 6). \end{cases}$$ □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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