Communications in Mathematics and Applications
Vol. 15, No. 2, pp. 557–569, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i2.2536

http://www.rgnpublications.com

Research Article

Stability of the Functional Equation Deriving From Quadratic Function in Banach Space

Amrit[1], Anil Kumar[2] and Manoj Kumar*[1,3]

[1] *Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak 124021, Haryana, India*
[2] *Department of Mathematics, A.I.J.H.M. College (affiliated with Maharshi Dayanand University), Rohtak 124001, Haryana, India*
[3] *Department of Mathematics, Maharishi Markandeshwar (Deemed to be University), Mullana, Ambala, India*
***Corresponding author:** manojantil18@gmail.com

Received: December 25, 2023 **Accepted:** April 4, 2024

Abstract. In this manuscript, we introduce a quadratic functional equation of finite variable:

$$\sum_{i=1}^{m} \phi\left(2v_i - \sum_{1\le i\ne j}^{m} v_j\right) = (m-7)\sum_{1\le i<j\le m} \phi(v_i+v_j) + \phi\left(\sum_{i=1}^{m} v_i\right) - (m^2-9m+5)\sum_{i=1}^{m}\phi(v_i)$$

and examine its Hyers-Ulam stability of this functional equation in Banach space using direct and fixed point method.

Keywords. Banach space, Hyers-Ulam stability, Quadratic functional equation

Mathematics Subject Classification (2020). 39B82, 39B52, 39B72, 41A99

1. Introduction

Functional equations play an important part in the study of stability. In 1940, the stability problems of functional equations about group homomorphisms was introduced by Ulam [13]. In 1941, Hyers [7] gave a affirmative answer to Ulam's question for additive groups (under the assumption that groups are Banach spaces). Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference $\|\phi(v+y)-\phi(v)-\phi(y)\| \le \varepsilon(\|v\|^p+\|y\|^p)$, for all $\varepsilon>0$ and $p\in[0,1)$. Also, Rassias generalization theorem was delivered by Gavruta [6] who replaced $\varepsilon(\|v\|^p+\|y\|^p)$ by a control function $\varphi(v,y)$. The paper of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. In 1982, Rassias followed the modern approach of the Rassias theorem [12] in which he replaced the factor product of norms instead of sum of norms. Hyer's theorem has been expanded in a number of ways over the past few decades; for a list see Alessa *et al*. [1], Czerwik [3], Dominguez-Benavides *et al*. [4], Gajda [5], Hyers [8], Jin and Lee [9], Jun and Lee [10], and Jung and Sahoo [11]. The present work introduces finite variable quadratic function as:

$$\sum_{i=1}^{m}\phi\left(2v_i-\sum_{1\le i\ne j}^{m}v_j\right)=(m-7)\sum_{1\le i<j\le m}\phi(v_i+v_j)+\phi\left(\sum_{i=1}^{m}v_i\right)-(m^2-9m+5)\sum_{i=1}^{m}\phi(v_i), \tag{1.1}$$

where $m\ge 5$, and derive its solution. Also, obtains Hyer-Ulam-Rassias stability in Banach space.

2. General Solution

Theorem 2.1. *If a mapping $\phi:E\to F$ satisfying the functional eq.* (1.1)*, then the mapping $\phi:E\to F$ is quadratic.*

Proof. Assume that the mapping $\phi:E\to F$ satisfying the eq. (1.1).
Replacing $(v_1,v_2,v_3,\dots,v_m)$ by $(0,0,0,\dots,0)$, we get

$$\phi(0)=0.$$

Also, replacing $(v_1,v_2,v_3,\dots,v_m)$ by $(v,0,0,\dots,0)$, we get

$$\phi(2v)=(m+3)\phi(v)-(m-1)\phi(-v) \tag{2.1}$$

replacing v by $-v$ in eq. (2.1), we obtain

$$\phi(-2v)=(m+3)\phi(-v)-(m-1)\phi(v). \tag{2.2}$$

When we replace $(v_1,v_2,v_3,\dots,v_m)$ by $(v,v,0,0,0,\dots,0)$ in eq. (1.1), then we obtain

$$(m-2)\phi(-2v)-(m-6)\phi(2v)=16\phi(v). \tag{2.3}$$

Using eq. (2.1) and (2.2) in eq. (2.3), we get

$$\phi(-v)=\phi(v), \tag{2.4}$$

for all $v\in E$.
Therefore, ϕ is an even function. From eq. (2.3), for each $m\in Z^+$, we have

$$\phi(2v)=2^2\phi(v), \tag{2.5}$$

for each $v\in E$.
Replacing v by $2v$ in eq. (2.5), we obtain

$$\phi(2^2v)=2^4\phi(v), \tag{2.6}$$

for each $v\in E$.
Replacing v by $2v$ in eq. (2.6), we obtain

$$\phi(2^3v)=2^6\phi(v), \tag{2.7}$$

for each $v\in E$.

Similarly, for all positive integer m, we can say

$$\phi(2^m v) = 2^{2m}\phi(v), \tag{2.8}$$

for each $v \in E$.

Again, when we replace $(v_1, v_2, v_3, \ldots, v_n)$ by $(v, v, v, 0, 0, 0, \ldots, 0)$ in eq. (1.1), then we obtain

$$(n-3)\phi(-3v) - \phi(3v) = 9(n-4)\phi(v). \tag{2.9}$$

Replacing v by $-v$ in eq. (2.9), we get

$$(n-3)\phi(3v) - \phi(-3v) = 9(n-4)\phi(-v), \tag{2.10}$$

$$\phi(-3v) = (n-3)\phi(3v) - 9(n-4)\phi(-v). \tag{2.11}$$

Using eq. (2.11) in eq. (2.9), we get

$$\left.\begin{aligned} (n-3)[(n-3)\phi(3v) - 9(n-4)\phi(-v)] - \phi(3v) &= 9(n-4)\phi(v), \\ (n^2 - 6n + 9 - 1)\phi(3x) - 9(n^2 - 7n + 12)\phi(v) &= 9(n-4)\phi(v), \\ (n^2 - 6n + 8)\phi(3v) &= (9n^2 - 54n + 72)\phi(v), \\ (n^2 - 6n + 8)\phi(3v) &= 9(n^2 - 6n + 8)\phi(v), \\ \phi(3v) &= 3^2\phi(v), \end{aligned}\right\} \tag{2.12}$$

for each $v \in E$.

Replacing v by $3v$ in eq. (2.12), we obtain

$$\phi(3^2 v) = 3^4\phi(v), \tag{2.13}$$

for each $v \in E$.

Replacing v by $3v$ in eq. (2.13), we obtain

$$\phi(3^3 v) = 3^6\phi(v), \tag{2.14}$$

for each $v \in E$.

Similarly, for all positive integer n, we can say

$$\phi(3^n v) = 3^{2n}\phi(v), \tag{2.15}$$

for each $v \in E$. Hence ϕ is a quadratic function.

3. Stability of Quadratic Functional Equation

For a given mapping $\phi : V \to W$, we define

$$\begin{aligned} D\phi(v_1, v_2, v_3, \ldots, v_m) = \sum_{i=1}^{m} \phi\left(2v_i - \sum_{1\le i\ne j}^{m} v_j\right) - (m-7)\sum_{1\le i<j\le m} \phi(v_i + v_j) \\ - \phi\left(\sum_{i=1}^{m} v_i\right) + (m^2 - 9m + 5)\sum_{i=1}^{m} \phi(v_i), \end{aligned} \tag{3.1}$$

for each $v_1, v_2, v_3, \ldots, v_m \in V$.

Theorem 3.1. *Assume that V and W are Banach spaces. If a function $\phi : V \to W$ satisfies the inequality*

$$\|D\phi(v_1, v_2, \ldots, v_m)\| < \varepsilon, \tag{3.2}$$

for some $\varepsilon > 0$, for all $v_1, v_2, \dots, v_m \in V$, then the limit

$$Q_2(v) = \lim_{m\to\infty} \frac{\phi(2^m v)}{2^{2m}} \tag{3.3}$$

exists for each $v \in V$ and $Q_2 : V \to W$ is unique quadratic function such that

$$\|\phi(v) - Q_2(v)\| < \frac{\varepsilon}{12}, \tag{3.4}$$

for any $v \in V$.

Proof. Replace $(v_1, v_2, \dots, v_m)$ by $(v, v, 0, \dots, 0)$ in (3.2), we have

$$\|4\phi(2v) - 16\phi(v)\| < \varepsilon, \tag{3.5}$$

$$\left\|\frac{\phi(2v)}{2^2} - \phi(v)\right\| < \frac{\varepsilon}{16}. \tag{3.6}$$

Replace v by 2^{tv} in (3.6), we have

$$\left\|\frac{\phi(2^{t+1}v)}{2^2} - \phi\left(2^t v\right)\right\| < \frac{\varepsilon}{16}, \tag{3.7}$$

$$\left\|\frac{\phi\left(2^{t+1}v\right)}{2^{2(t+1)}} - \frac{\phi\left(2^t v\right)}{2^{2t}}\right\| < \frac{\varepsilon}{4\cdot 2^{2(t+1)}}, \tag{3.8}$$

for all $v \in V$ and all $\varepsilon > 0$. Since

$$\frac{\phi(2^m v)}{2^{2m}} - \phi(v) = \sum_{i=0}^{m-1}\left(\frac{\phi\left(2^{i+1}v\right)}{2^{2(i+1)}} - \frac{\phi\left(2^i v\right)}{2^{2i}}\right). \tag{3.9}$$

So,

$$\left\|\frac{\phi(2^m v)}{2^{2m}} - \phi(v)\right\| \le \sum_{i=0}^{m-1}\left\|\frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}}\right\| \tag{3.10}$$

$$< \sum_{i=0}^{m-1}\frac{\varepsilon}{4\cdot 2^{2(i+1)}} = \frac{\varepsilon}{12}\left(1 - \frac{1}{2^{2m}}\right). \tag{3.11}$$

Replace v by $2^m v$, we get

$$\left\|\frac{\phi(2^{m+m}v)}{2^{2(m+m)}} - \frac{\phi(2^m v)}{2^{2m}}\right\| < \frac{\varepsilon}{12}\left(\frac{1}{2^{2m}} - \frac{1}{2^{2(m+m)}}\right), \tag{3.12}$$

for all $v \in V$ and all $\varepsilon > 0$. R.H.S $\to 0$ as $m \to \infty$ then $\left\{\frac{\phi(2^m v)}{2^{2n}}\right\}$ is a Cauchy sequence in W, Since W is Banach space, thus sequence $\left\{\frac{\phi(2^m v)}{2^{2m}}\right\}$ converges to some $Q_2(v) \in W$. For $v \in V$,

$$\|Q_2(v) - \phi(v)\| = \left\|Q_2(v) - \frac{\phi(2^m v)}{2^{2m}} + \frac{\phi(2^m v)}{2^{2m}} - \phi(v)\right\| \tag{3.13}$$

$$\le \left\|Q_2(v) - \frac{\phi(2^m v)}{2^{2m}}\right\| + \left\|\frac{\phi(2^m v)}{2^{2m}} - \phi(v)\right\| \tag{3.14}$$

$$< \left\|Q_2(v) - \frac{\phi(2^m v)}{2^{2m}}\right\| + \frac{\varepsilon}{12}\left(1 - \frac{1}{2^{2m}}\right). \tag{3.15}$$

for all $v \in V$ and all $\varepsilon > 0$. Taking the limit $m \to \infty$, we get

$$\|Q_2(v) - \phi(v)\| < \frac{\varepsilon}{12}. \tag{3.16}$$

Replacing $(v_1,v_2,\ldots,v_m)$ by $(2^m v_1,2^m v_2,\ldots,2^m v_m)$ in (3.2), we have

$$\|D\phi(2^m v_1,2^m v_2,\ldots,2^m v_m)\| < \varepsilon,$$
$$\left\|D\phi\left(\frac{2^m v_1}{2^{2m}},\frac{2^m v_2}{2^{2m}},\ldots,\frac{2^m v_m}{2^{2m}}\right)\right\| < \frac{\varepsilon}{2^{2m}}. \tag{3.17}$$

Applying $m\to\infty$, show that Q_2 satisfies the functional eq. (1.1).

To prove the uniqueness of quadratic mapping Q_2. Assume that there exists another quadratic mapping Q_2', which satisfies inequality (3.4). Fix $v\in V$. Clearly, $Q_2(2^t v) = 2^{2t}Q_2(v)$ and $Q_2'(2^t v) = 2^{2t}Q_2'(v)$, for all $v\in V$, from (3.4), we have

$$\begin{aligned}\|Q_2(v)-Q_2'(v)\| &= \left\|\frac{Q_2(2^m)v}{2^{2m}}-\frac{\phi^{2m}v}{2^{2m}}+\frac{\phi(2^m v)}{2^{2m}}-\frac{Q_2'(2^m v)}{2^{2m}}\right\|\\ &< \frac{1}{2^{2m-1}}\cdot\frac{\varepsilon}{12}.\end{aligned} \tag{3.18}$$

Taking $m\to\infty$, we have $Q_2(v)=Q_2'(v)$. □

Theorem 3.2. *Assume that V and W are Banach spaces. If a function $\phi : V\to W$ satisfies the inequality*

$$\|D\phi(v_1,v_2,\ldots,v_m)\| \le \theta\sum_{i=1}^{m}\|v_i\|^p. \tag{3.19}$$

For some $p<2$, for all $v_1,v_2,\ldots,v_m\in V$, then the limit

$$Q_2(v) = \lim_{m\to\infty}\frac{\phi(2^m v)}{2^{2m}} \tag{3.20}$$

exists for each $v\in V$ and $Q_2 : V\to W$ is unique quadratic function such that

$$\|\phi(v)-Q_2(v)\| \le \frac{\theta\|v\|^p}{(2^2-2^p)'}, \tag{3.21}$$

for any $v\in V$.

Proof. Replace $(v_1,v_2,\ldots,v_m)$ by $(v,v,0,\ldots,0)$ in (3.19), we have

$$\begin{aligned}\|4\phi(2v)-16\phi(v)\| \le \theta\|v\|^p &\left\|\frac{\phi(2v)}{2^2}-\phi(v)\right\|\\ &\le \frac{\theta\|v\|^p}{2^2}.\end{aligned} \tag{3.22}$$

Replace v by $2^t v$ in (3.22), we have

$$\left.\begin{aligned}\left\|\frac{\phi(2^{t+1}v)}{2^2}-\phi(2^t v)\right\| &\le \frac{\theta\|2^t v\|^p}{2^2},\\ \left\|\frac{\phi(2^{t+1}v)}{2^{2t+1)}}-\frac{\phi(2^t v)}{2^{2t}}\right\| &\le \frac{\theta\|v\|^p}{2^{2(t+1)-tp}},\end{aligned}\right\} \tag{3.23}$$

for all $v\in V$. Since

$$\frac{\phi(2^m v)}{2^{2m}}-\phi(v) = \sum_{i=0}^{m-1}\left(\frac{\phi(2^{i+1}v)}{2^{2(i+1)}}-\frac{\phi(2^i v)}{2^{2i}}\right). \tag{3.24}$$

So,

$$\left\|\frac{\phi(2^m v)}{2^{2m}}-\phi(v)\right\| \le \sum_{i=0}^{m-1}\left\|\frac{\phi(2^{i+1}v)}{2^{2(i+1)}}-\frac{\phi(2^i v)}{2^{2i}}\right\|$$

$$\leq \sum_{i=0}^{m-1} \frac{\theta \|v\|^p}{2^{2(i+1)-pi}}$$
$$= \frac{\theta \|v\|^p}{(2^2 - 2^p)} \left(1 - \frac{1}{2^{m(2-p)}}\right). \tag{3.25}$$

Replace v by $2^m v$, we get

$$\left\| \frac{\phi(2^{m+m}v)}{2^{2(m+m)}} - \frac{\phi(2^m v)}{2^{2m}} \right\| \leq \frac{\theta \|v\|^p}{(2^{2-p} - 1)} \left(\frac{1}{2^{2m}} - \frac{1}{2^{2(m+m)-mp}} \right), \tag{3.26}$$

for all $v \in V$. R.H.S $\to 0$ as $m \to \infty$ then $\left\{\frac{\phi(2^m v)}{2^{2m}}\right\}$ is a Cauchy sequence in W. Since W is Banach space, thus sequence $\left\{\frac{\phi(2^m v)}{2^{2m}}\right\}$ converges to some $Q_2(v) \in W$. For $v \in V$,

$$\|Q_2(v) - \phi(v)\| = \left\| Q_2(v) - \frac{\phi(2^m v)}{2^m} + \frac{\phi(2^m v)}{2^{2m}} - \phi(v) \right\|$$
$$\leq \left\| Q_2(v) - \frac{\phi(2^m v)}{2^{2m} t} \right\| + \left\| \frac{\phi(2^m v)}{2^{2m}} - \phi(v) \right\|$$
$$\leq \left\| Q_2(v) - \frac{\phi(2^m v)}{2^{2m}} \right\| + \frac{\theta \|v\|^p}{(2^2 - 2^p)} \left(1 - \frac{1}{2^{m(2-p)}}\right), \tag{3.27}$$

for all $v \in V$. Taking the limit $m \to \infty$,

$$\|Q_2(v) - \phi(v)\| \leq \frac{\theta \|v\|^p}{(2^2 - 2^p)} \tag{3.28}$$

Replacing $(v_1, v_2, \ldots, v_m)$ by $(2^m v_1, 2^m v_2, \ldots, 2^m v_m)$ in (3.19), we have

$$\|D\phi(2^m v_1, 2^m v_2, \ldots, 2^m v_m)\| \leq \theta \sum_{i=1}^{m} \|2^m v_i\|^p, \tag{3.29}$$

$$\left\| D\phi\left(\frac{2^m v_1}{2^{2m}}, \frac{2^m v_2}{2^{2m}}, \ldots, \frac{2^m v_m}{2^{2m}} \right) \right\| \leq \frac{\theta}{2^{2m-mp}} \sum_{i=1}^{m} \|v_i^p\|. \tag{3.30}$$

Applying $m \to \infty$, show that Q_2 satisfies the functional eq. (1.1).

To prove the uniqueness of quadratic mapping Q_2. Assume that there exists another quadratic mapping Q_2', which satisfies inequality (3.21). Fix $v \in V$. Clearly, $Q_2(2^t v) = 2^{2t} Q_2(v)$ and $Q_2'(2^t v) = 2^{2t} Q_2'(v)$, for all $v \in V$. We have

$$\|Q_2(v) - Q_2'(v)\| = \left\| \frac{Q_2(2^m v)}{2^{2m}} - \frac{\phi(2^m v)}{2^{2m}} + \frac{\phi(2^m v)}{2^{2m}} - \frac{Q_2'(2^m v)}{2^{2m}} \right\|$$
$$\leq \frac{\theta \|v\|^p}{2^{2m-mp-1}(2^2 - 2^p)}. \tag{3.31}$$

Taking $m \to \infty$, we have $Q_2(v) = Q'{}_2(v)$. □

Theorem 3.3. *Assume that V and W are Banach spaces. Let $\varphi : V^m \to R^+$ be a function such that $\sum_{i=0}^{\infty} \frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{2^{2i}}$ converges and $\lim_{i \to \infty} \frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{2^{2i}} = 0$. Also, if a function $\phi : V \to W$ satisfies the inequality*

$$\|D\phi(v_1, v_2, \ldots, v_m)\| \leq \varphi(v_1, v_2, \ldots, v_m), \tag{3.32}$$

for all $v_1, v_2, \ldots, v_m \in V$, *then the limit* $Q_2(v) = \lim_{m\to\infty} \frac{\phi(2^m v)}{2^{2m}}$, *exists for each* $v \in V$ *and* $Q_2 : V \to W$ *is unique quadratic function such that*

$$\|\phi(v) - Q_2(v)\| \le \sum_{i=0}^{\infty} \frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{4 \cdot 2^{2(i+1)}}, \tag{3.33}$$

for any $v \in V$.

Proof. Replace $(v_1, v_2, \ldots, v_m)$ by $(v, v, 0, \ldots, 0)$ in (3.32), we have

$$\left.\begin{aligned} &\|4\phi(2v) - 2^4\phi(v)\| \le \varphi(v, v, 0, \ldots, 0),\\ &\left\|\frac{\phi(2v)}{2^2} - \phi(v)\right\| \le \frac{\varphi(v, v, 0, \ldots, 0)}{2^4}. \end{aligned}\right\} \tag{3.34}$$

Replace v by $2^t v$ in (3.34), we have

$$\left.\begin{aligned} &\left\|\frac{\phi(2^{t+1}v)}{2^2} - \phi(2^t v)\right\| \le \frac{\varphi(2^t v, 2^t v, 0, \ldots, 0)}{2^4},\\ &\left\|\frac{\phi(2^{t+1}v)}{2^{2(t+1)}} - \frac{\phi(2^t v)}{2^{2t}}\right\| \le \frac{\varphi(2^t v, 2^t v, 0, \ldots, 0)}{4 \cdot 2^{2(t+1)}}, \end{aligned}\right\} \tag{3.35}$$

for all $v \in V$. Since

$$\frac{\phi(2^m v)}{2^{2m}} - \phi(v) = \sum_{i=0}^{m-1}\left(\frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}}\right). \tag{3.36}$$

So,

$$\begin{aligned} \left\|\frac{\phi(2^m v)}{2^{2m}} - \phi(v)\right\| &\le \sum_{i=0}^{m-1}\left\|\frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}}\right\|\\ &\le \sum_{i=0}^{m-1}\frac{\varphi(2^i v, 2^i v 0, \ldots, 0)}{4 \cdot 2^{2(t+1)}}. \end{aligned} \tag{3.37}$$

Replacing v by $2^m v$ in (3.37), we get

$$\begin{aligned} \left\|\frac{\phi(2^{m+m}v)}{2^{2(m+m)}} - \frac{\phi(2^m v)}{2^{2m}}\right\| &\le \sum_{i=0}^{m+m-1}\frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{4 \cdot 2^{2(2+i)}} + \sum_{i=0}^{m-1}\frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{4 \cdot 2^{2(i+1)}}\\ &\le \sum_{i=m}^{m+m-1}\frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{4 \cdot 2^{2(i+1)}}, \end{aligned} \tag{3.38}$$

for all $v \in V$.

Taking the limit $m \to \infty$, we have

$$\|Q_2(v) - \phi(v)\| \le \sum_{i=0}^{\infty}\frac{\varphi(2^i v, 2^i v, 0, \ldots, 0)}{4 \cdot 2^{2(i+1)}}. \tag{3.39}$$

Replacing $(v_1, v_2, \ldots, v_m)$ by $(2^m v_1, 2^m v_2, \ldots, 2^m v_m)$ in (3.32), we have

$$\left.\begin{aligned} &\|D\phi(2^m v_1, 2^m v_2, \ldots, 2^m v_m)\| \le \varphi(2^m v_1, 2^m v_2, \ldots, 2^m v_m),\\ &\left\|D\phi\left(\frac{2^m v_1}{2^{2m}}, \frac{2^m v_2}{2^{2m}}, \ldots, \frac{2^m v_m}{2^{2m}}\right)\right\| \le \frac{\varphi(2^m v_1, 2^m v_2, \ldots, 2^m v_m)}{2^{2m}}. \end{aligned}\right\} \tag{3.40}$$

Applying $m \to \infty$, show that Q_2 satisfies the functional eq. (1.1).

To prove the uniqueness of quadratic mapping Q_2. Assume that there exists another quadratic mapping Q_2', which satisfies inequality (3.33). Fix $v \in V$. Clearly, $Q_2(2^t v) = 2^{2t}Q_2(v)$ and

$Q_2'(2^t v) = 2^{2t} Q_2'(v)$ for all $v \in V$. We have

$$\|Q_2(v) - Q_2'(v)\| = \left\| \frac{Q_2(2^m v)}{2^{2m}} - \frac{\phi(2^m v)}{2^{2m}} + \frac{\phi(2^m v)}{2^{2m}} - \frac{Q_2'(2^m v)}{2^{2m}} \right\|$$
$$\leq \sum_{i=m}^{\infty} \frac{\varphi(2^i v, 2^i v, 0, \dots, 0)}{4 \cdot 2^{2(+1)}} + \sum_{i=m}^{\infty} \frac{\varphi(2^i v, 2^i v, 0, \dots, 0)}{4 \cdot 2^{2(i+1)}}. \tag{3.41}$$

Taking $m \to \infty$, we have $Q_2(v) = Q_2'(v)$. □

4. Counter Example

We provide a counter example for the non-stability to the functional eq. (1.1) in Banach space as:

Example 4.1. Let a mapping $\phi : R \to R$ be defined as

$$\phi(v) = \sum_{t=0}^{\infty} \frac{g\left(2^t v\right)}{2^{2t}}, \tag{4.1}$$

where $g(v) = \begin{cases} \theta v^2, & |v| < 1, \\ \theta, & \text{else}, \end{cases}$ then the mapping $\phi : R \to R$ satisfies the inequality

$$\|D\phi(v_1, v_2, \cdots, v_n)\| \leq \frac{16(3n^3 - 13n^2 + 6n + 6)}{3} \theta \sum_{i=1}^{n} |v_i|^2, \tag{4.2}$$

for all $v_1, v_2, \cdots, v_n \in R, n \geq 8$, but there does not exist a quadratic mapping $Q_2 : R \to R$ satisfies

$$|\phi(v) - Q_2(v)| \leq \varepsilon |v|^2, \tag{4.3}$$

for all $v \in R$.

Proof. Now $|\phi(v)| \leq \sum_{n=0}^{\infty} \left| \frac{g(2^n v)}{2^{2n}} \right| = \sum_{n=0}^{\infty} \frac{\theta}{2^{2n}} = \frac{4}{3}\theta$.

Thus ϕ is bounded. Now we will prove that ϕ satisfies (4.2). If $v_i = 0$ for $i = 1, 2, \dots, n$ then (4.2) is obvious. If $\sum_{i=1}^{n} |v_i|^2 \geq \frac{1}{2^2}$ then L.H.S of (4.2) is less than $\left(3n^3 - 13n^2 + 6n + 6\right) \frac{4}{3}\theta$. Now suppose, $\sum_{i=1}^{n} |v_i|^2 < \frac{1}{2^2}$, then there exists a positive integer m such that

$$\frac{1}{2^{2(m+1)}} \leq \sum_{i=1}^{n} |v_i|^2 \leq \frac{1}{2^{2m}}.$$

Thus

$$2^{2(m-1)} |v_1|^2 < \frac{1}{2^2}, 2^{2(m-1)} |v_2|^2 < \frac{1}{2^2}, \dots, 2^{2(m-1)} |v_n|^2 < \frac{1}{2^2}$$

and

$$2^m |v_1| < 1, 2^m |v_2| < 1, \dots, 2^m |v_n| < 1.$$

So, consequently

$$\sum_{i=1}^{n} \left(2^{m+1} v_i - \sum_{j=1, i \neq j}^{n} 2^m v_j \right), \quad \sum_{1 \leq i < j \leq n} 2^m (v_i + v_j), \quad \sum_{i=1}^{n} 2^m v_i \in (-1, 1).$$

Therefore, for each $t = 0,1,2,\dots,m-1$, we have

$$\sum_{i=1}^{n}\left(-2^{t+1}v_i + \sum_{j=1,i\neq j}^{n} 2^t v_j\right), \quad \sum_{1\le i<j\le n} 2^t(v_i+v_j), \quad \sum_{i=1}^{n} 2^t v_i \in (-1,1).$$

And so, $Dg(2^t v_1, 2^t v_2,\dots,2^t v_n) = 0$, for $t = 0,1,2,\dots,m-1$.

Now,

$$\begin{aligned}
|D\phi(v_1,v_2,\dots,v_n)| &\le \sum_{t=0}^{\infty}\frac{1}{2^{2t}}|Dg(2^t v_1,2^t v_2,\dots,2^t v_n)| \\
&\le \sum_{t=m}^{\infty}\frac{1}{2^{2t}}\left[\sum_{i=1}^{n}\left|g\left(2^{t+1}v_i - \sum_{j=1,i\neq j}^{n} 2^t v_j\right)\right| + (n-7)\sum_{1\le i<j\le n}|g(2^t v_i + 2^t v_j)|\right. \\
&\quad \left. + \left|g\left(\sum_{i=1}^{n}(2^t v_i)\right)\right| + (n^2-9n+5)\sum_{i=1}^{n}|g(2^t v_i)|\right] \\
&\le \sum_{t=m}^{\infty}\frac{\theta}{2^{2t}}[(n^2+n+2)n + (n-7)2n(n-1) + (n^2-n+1) + (n^2-9n+5)] \\
&= \sum_{t=m}^{\infty}\frac{\theta}{2^{2t}}[3n^3-13n^2+6n+6] \\
&= [3n^3-13n^2+6n+6]\sum_{t=m}^{\infty}\frac{\theta}{2^{2t}} \\
&= [3n^3-13n^2+6n+6]\frac{4\theta}{2^{2m}.3} \\
&= \frac{16(3n^3-13n^2+6n+6)}{3}\theta \times \frac{1}{2^{2(m+1)}} \\
&\le \frac{16(3n^3-13n^2+6n+6)}{3}\theta\sum_{i=1}^{n}|v_i|^2.
\end{aligned}$$

Thus ϕ satisfies (4.3). If possible, we assume that there exists a quadratic solution $Q_2 : R \to R$ satisfies (4.3). For every $v \in R$, since ϕ is a continuously bounded function, Q_2 is bounded on every open interval containing the origin and continuous at the origin. Q_2 must be of the form $Q_2(v) = cv^2$ for all $v \in R$. So, $|\phi(v)| \le (\varepsilon + |c|)|v|^2$, for all $v \in R$. We can find $s > 0$ with $s\theta > \varepsilon + |c|$. If $v \in \left(0, \frac{1}{2^{s-1}}\right)$, then $2^t v \in (0,1)$, for all $t = 0,1,2,\dots,s-1$, we have

$$\phi(v) = \sum_{t=0}^{\infty}\frac{g(2^t v)}{2^{2t}} \ge \sum_{t=0}^{s-1}\frac{\theta(2^t v)^2}{2^{2t}} = s\theta v^2 > (\varepsilon + |c|)v^2,$$

$$|\phi(v) - Q_2(v)| > \varepsilon |v|^2.$$

which is contradiction. □

5. Stability of Functional eq. (1.1) using Fixed Point Method

Theorem C (Banach Contraction Principle)**.** *Let (V,d) be a complete metric spaces consider a mapping $T : V \to V$ which is strictly contractive mapping, that is*

(C_1): $d(Tv,Ty) \le d(v,y)$, *for some (Lipschitz constant) $L < 1$, then,*

(i) *The mapping T only has one fixed point, which is $T(v^*) = v^*$.*

(ii) *Each given element's fixed point, v^*, is universally contractive, that is*

(C_2): $\lim_{m\to\infty} T^m v = v^*$ *for any starting point* $v \in V$.

(iii) *One has the following estimation inequalities,*

(C_3): $d(T^m v, v^*) \le \dfrac{1}{1-L} d(T^m v, T^{m+1} v)$, *for all* $m \ge 0$, $v \in V$,

(C_4): $d(v, v^*) = \dfrac{1}{1-L} d(v, v^*)$, *for all* $v \in V$.

Theorem D (Alternative Fixed Point). *If a generalized metric space (V,d) is complete and a strictly contractive mapping $T : V \to V$ has a Lipschitz constant L, then for any given element, $v \in V$ either,*

(D_1): $d(T^m v, T^{m+1} v) = \infty$, *for all* $m \ge 0$.

(D_2): *There exists a natural number such that,*

(i) $d(T^m v, T^{m+1} v) < \infty$, *for all* $m \ge 0$.

(ii) *The sequence $\{T^m v\}$ is convergent to a fixed point y^* of T.*

(iii) *y^* is the unique fixed point of T in the set $W = \{y \in W; d(T^{m_0} v, y) < \infty\}$.*

(iv) $d(y^*, y) \le \frac{1}{1-L} d(y, Ty)$, *for all* $y \in W$.

Theorem 5.1. *Let $\phi : A \to B$ be an even mapping for which there exists a function $\varphi : A^m \to [0,\infty)$ with the condition*

$$\lim_{m\to\infty} \frac{\varphi(\xi_i^m v_1, \xi_i^m v_2, \xi_i^m v_3 \dots, \xi_i^m v_m)}{\xi_i^m} = 0, \tag{5.1}$$

where $\xi_i = \begin{cases} 2, & i = 1, \\ \frac{1}{2}, & i = 0, \end{cases}$ *such that the functional inequality*

$$\|D\phi(v_1, v_2, v_3 \dots, v_m)\| \le \varphi(v_1, v_2, v_3 \dots, v_m), \tag{5.2}$$

for all $v_1, v_2, v_3 \dots, v_m \in V$. If there exists $L = L(i)$ such that the function

$$v \to \beta(v) = \varphi\left(\frac{v}{2}, \frac{v}{2}, 0 \dots, 0\right) \tag{5.3}$$

has the property,

$$\frac{1}{\xi_i} \beta(\xi_i v) = L\beta(v), \tag{5.4}$$

for each $v \in A$. Then there exists a unique quadratic mapping $Q_2 : A \to B$ satisfying the functional eq. (1.1) *and*

$$\|\phi(v) - Q_2(v)\| \le \frac{L^{1-i}}{1-L} \beta(v) \tag{5.5}$$

holds for all $v \in A$.

Proof. Introduce the generalized metric to the set $V = \{P; P : A \to B, P(0) = 0\}$ and then have a look at the set V. $d(p,q) = \inf\{K \in (0,\infty) : \|p(v) - q(v)\| \le K\beta(v), v \in A\}$. It is clear that (V,d) is complete metric space. Define $T : V \to V$ by

$$T_p(v) = \frac{1}{\xi_i} p(\xi_i v), \tag{5.6}$$

for all $v \in A$. Now $p,q \in V$,

$$\begin{aligned}
d(p,q) &\le K, \\
\|p(v)-q(v)\| &\le K\beta(v), \qquad v \in A, \\
\left\|\frac{1}{\xi_i}p(\xi_i v)-\frac{1}{\xi_i}q(\xi_i v)\right\| &\le \frac{1}{\xi_i}K\beta(\xi_i v), \quad v \in A, \\
\|Tp(v)-Tq(v)\| &\le LK\beta(v), \qquad v \in A, \\
d(Tp,Tq) &\le LK.
\end{aligned}$$

This means that $d(Tp,Tq) \le Ld(p,q)$, for each $p,q \in V$. T is strictly contractive mapping on V with Lipschtiz constant L. It follows from (5.2) that

$$\|4\phi(2v)-2^4\phi(v)\| \le \varphi(v,v,0,\dots,0), \tag{5.7}$$

for each $v \in A$. It follows from (5.7) that

$$\left\|\frac{\phi(2v)}{2^2}-\phi(v)\right\| \le \frac{\varphi(v,v,0,\dots,0)}{2^4}, \tag{5.8}$$

for each $v \in A$. From (5.4), for the case $i=1$, it reduces to

$$\left\|\frac{\phi(2v)}{2^2}-\phi(v)\right\| \le \frac{1}{16}\beta(v), \tag{5.9}$$

for each $v \in A$, i.e., $d(\phi,T\phi) \le \frac{1}{16} \Rightarrow d(\phi,T\phi) \le \frac{1}{16} = L = L' < \infty$. Again replace $v=\frac{v}{2}$ in (5.7), we obtain

$$\left\|4\phi(v)-16\phi\left(\frac{v}{2}\right)\right\| \le \varphi\left(\frac{v}{2},\frac{v}{2},0,\dots,0\right), \tag{5.10}$$

for each $v \in A$. Using (5.4) for $i=0$, it reduces to,

$$\left\|2^4\phi\left(\frac{v}{2}\right)-4\phi(v)\right\| \le \varphi(\beta(v)), \tag{5.11}$$

for each $v \in A$, (i.e.,) $d(\phi,T\phi) \le 1 \Rightarrow d(\phi,T\phi) \le 1 = L^0 < \infty$. In the above case we reached

$$d(\phi,T\phi) \le L^{1-i}. \tag{5.12}$$

Therefore, C_2(i) hold. Using C_2(ii), it follows that exists a fixed point Q_2 of T in A, such that

$$Q_2(v) = \lim_{m\to\infty}\frac{\phi_a(\xi_i^m v)}{\xi_i^m}, \quad \text{for all } v \in A. \tag{5.13}$$

To prove that $Q_2 : A \to B$ is quadratic. Using $(\xi_i^m v_1, \xi_i^m v_2, \dots, \xi_i^m v_m)$ at place of $(v_1, v_2, \dots, v_m)$ in (5.8) and dividing by ξ_i^m, it follows from (5.1) and (5.13), we see that Q_2 satisfies (1.1) for all $v_1, v_2, v_3, \dots, v_m \in V$. Hence Q_2 satisfies the functional eq. (1.1).

By using C_2(iii), Q_2 is the unique fixed point of T in the set, $W = \{\phi \in V : d(T\phi, Q_2) < \infty\}$. Using fixed point alternative result, Q_2 is the unique function such that

$$\|\phi(v)-Q_2(v)\| \le K\beta(v), \tag{5.14}$$

for all $v \in A$ and $k > 0$.
Finally, by D_2(iv), we obtain

$$d(\phi,A) \le \frac{1}{1-L}d(\phi,T\phi), \tag{5.15}$$

as $d(\phi,Q_2) \le \frac{L^{1-i}}{1-L}$. Hence, we conclude that

$$\|\phi(v)-Q_2(v)\| \le \frac{L^{1-i}}{1-L}\beta(v), \tag{5.16}$$

for each $v \in A$.

This completes the proof. □

6. Conclusion

A quadratic functional equation including finite number of variables is invented. Hyers-Ulam-Rassias stability of this functional equation is proved in Banach space. Also provide an example which indicate the non-stability of the functional equation.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] N. Alessa, K. Tamilvanan, G. Balasubramanian and K. Loganathan, Stability results of the functional equation deriving from quadratic function in random normed spaces, *AIMS Mathematics* **6**(3) (2021), 2385 – 2397, DOI: 10.3934/math.2021145.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, *Journal of the Mathematical Society of Japan* **2**(1-2) (1950), 64 – 66, DOI: 10.2969/jmsj/00210064.

[3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Co., Inc., River Edge, NJ, 420 pages(2002), DOI: 10.1142/4875.

[4] T. Dominguez-Benavides, M. A. Khamsi and S. Samadi, Asymptotically regular mappings in modular function spaces, *Scientiae Mathematicae Japonicae* **53** (2) (2001), 295 – 304, URL: https://www.jams.jp/scm/contents/Vol-4-3/4-28.pdf.

[5] Z. Gajda, On stability of additive mappings, *International Journal of Mathematics and Mathematical Sciences* **14**(3) (1991), 431 – 434, URL: https://eudml.org/doc/46657.

[6] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *Journal of Mathematical Analysis and Applications* **184**(3) (1994), 431 – 436, DOI: 10.1006/jmaa.1994.1211.

[7] D. H. Hyers, G. Isac and T. M. Rassias, *Stability of Functional Equations in Several Variables*, in: *Progress in Nonlinear Differential Equations and Their Applications*, Vol. 34, Birkhäuser Boston, Inc., Boston, MA, vii + 318 pages (1998), DOI: 10.1007/978-1-4612-1790-9.

[8] D. H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences of the United States of America* **27**(4) (1941), 222 – 224, URL: https://www.jstor.org/stable/87271.

[9] S. S. Jin and Y.-H. Lee, On the stability of the functional equation deriving from quadratic and additive function in random normed spaces via fixed point method, *Journal of the Chungcheong Mathematical Society* **25**(1) (2012), 51 – 63, URL: http://ccms.or.kr/data/pdfpaper/jcms25_1/25_1_51.pdf.

[10] K.-W. Jun and Y.-H. Lee, On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality, *Mathematical Inequalities & Applications* **4**(1) (2001), 93 – 118, DOI: 10.7153/mia-04-08.

[11] S.-M. Jung and P. K. Sahoo, Hyers-Ulam stability of the quadratic equation of Pexider type, *Journal of the Korean Mathematical Society* **38**(3) (2001), 645 – 656, URL: https://jkms.kms.or.kr/journal/view.html?spage=645&volume=38&number=3.

[12] T. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society* **72**(2) (2) (1978), 297 – 300, DOI: 10.2307/2042795.

[13] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York (1964).