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Research Article

The Solitary Wave Solutions for the Nonlinear Benjamin-Mahony Equation

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Abstract. The Benjamin-Bona-Mahony (BBM) equation is a nonlinear partial differential equation that describes the propagation of long waves in a shallow water channel. In this work, we present a comprehensive solution for the BBM equation using the Riccati-Bernoulli sub-ODE method. The method involves transforming the BBM equation into a Riccati equation, which is then further transformed into a Bernoulli equation. The Bernoulli equation is then solved analytically, and the solution is used to obtain the solution for the original BBM equation. Our results show that the Riccati-Bernoulli sub-ODE method provides an efficient and accurate solution for the BBM equation. The method can be extended to solve other nonlinear partial differential equations (NPDEs), making it a valuable tool for researchers in various fields.

Keywords. Benjamin-Bona-Mahony equation, Riccati-Bernoulli sub-ODE method, Water waves, Solitary wave

Mathematics Subject Classification (2020). 35C08, 35A20, 35A09, 35L05, 35Qxx

1. Introduction

The study of NLPDEs in mathematics and physics has resulted in the development of theories, applications, and concepts that characterize a wide range of physical systems (Agrawal [1], and Whitham [23]). Mathematicians have struggled to find the existence and uniqueness solutions to many complex issues employed in numerous fields of study, including applied science, engineering, chemistry, physics, astronomy, and biological (Hasegawa *et al*. [5]). Several studies have been conducted to investigate various techniques for solving NLPDEs (Ibrahim *et al*. [15], Inc *et al*. [16], and Sulaiman *et al*. [21]). NLPDE approaches and solutions are widely utilized in Continuum mechanics, fluid dynamics, Chaos theory for Dynamical systems, nonlinear optics, quantum theory, and other fields (Fang *et al*. [4], Ibrahim [8], Kudryashov [18], [1]). The NLPDE theories and concepts can be expanded to investigate the commutative of linear time varying systems (LTVSs) (Ibrahim [6, 7], Ibrahim and Rababah [11], Ibrahim and Köksal [12–14]).

A nonlinear partial differential equation that represents the propagation of long waves in shallow water is the (BBM) equation. This equation is widely used in fluid dynamics, oceanography, and other branches of applied mathematics. The BBM equation has a complex solution structure that can display phenomena like as soliton production, wave breaking, and turbulence. Finding accurate and effective analytic methods to solve the BBM equation is one of the obstacles in understanding it. To solve the BBM problem, several scholars used several approach to solve the BBM equation (Ali *et al*. [2], Elmandouh and Fadhal [3], Shakeel *et al*. [20], Wang [22], and Xie and Li [24]).

The goal of this work is to investigate the optical traveling wave solutions of the Benjamin-Bona-Mahony (BBM) equation using the Riccati-Bernoulli sub-ODE technique. Some figures are used to demonstrate and back up our findings.

The third-order $(1+1)$-dimensional BBM equations is provided by,

$$\vartheta_t = -\vartheta_x - \alpha\vartheta_x\vartheta^2 - \beta\vartheta_{xxt}, \tag{1}$$

where α and β are nonzero real parameters. The Riccati-Bernoulli sub-ODE method was a powerful and efficient approach for solving nonlinear differential equations. This method has since been used to solve a wide variety of nonlinear differential equations, including partial differential equations. The method has been demonstrated to be efficient and accurate in tackling a wide range of nonlinear problems. The Riccati-Bernoulli sub-ODE method has recently gained popularity for solving NLPDs. The original equation is transformed into a system of first-order differential equations, and then the Riccati-Bernoulli sub-ODE method is used to solve it. Using the RB-sub equation method, several ways have been used to obtain soliton solutions for various NLPDEs (Ibrahim [9, 10], Karaman [17], Ozdemir *et al*. [19], and Yang *et al*. [25]).

In this paper, we employ the approach in a similar way to examine the traveling wave solutions to the third-order BBM equation. So yet, the approach has not been applied to the proposed novel third-order BBM NPDE. The paper is planned as follows: The approach was introduced in Section 2. Section 3 has the application and figures, Section 4 describe the results and 5 concludes the paper.

2. Description of RB Sub-ODE Method

We present the RB sub-equation approach in this section. Assume we have an NLPDE,

$$P(\vartheta, \vartheta_t, \vartheta_x, \vartheta_{tt}, \vartheta_{xx}, \vartheta_{tx}, \dots) = 0, \tag{2}$$

P denotes a polynomial. The RB sub-equation approach is divided into three stages.

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Step 1: We take a look at the following traveling wave transformation,

$$\vartheta(\xi)=\vartheta(x,t), \quad \xi=K(x\pm vt), \tag{3}$$

which resulted in the following ODE

$$P(\vartheta,\vartheta',\vartheta'',\dots)=0, \tag{4}$$

where $\vartheta'(\xi)=\frac{d\vartheta}{d\xi}$.

Step 2: Let eq. (4) be the solution to the RB equation,

$$\vartheta'=b\vartheta+a\vartheta^{2-m}+c\vartheta^{m}, \tag{5}$$

where a,b,c, and m are random constants.

Differentiating eq. (5) results in

$$\vartheta''=\vartheta^{-1-2m}(a\vartheta^2+c\vartheta^{2m}+b\vartheta^{1+m})(-a(-2+m)\vartheta^2+cm\vartheta^{2m}+b\vartheta^{1+m}), \tag{6}$$

$$\begin{aligned}\vartheta'''=\vartheta^{-2(1+m)}(bu+a\vartheta^{2-m}+c\vartheta^{m})(a^2(-2+m)(-3+2m)\vartheta^4+c^2m(-1+2m)\vartheta^{4m}\\ +ab(-3+m)(-2+m)\vartheta^{3+m}+(b^2+2ac)\vartheta^{2+2m}+bcm(1+m)\vartheta^{1+3m}),\end{aligned} \tag{7}$$

and so on.

Take note that the solutions to eq. (5) results in:

Case 1. As $m=1$, eq. (5) become

$$\vartheta(\xi)=Je^{(b+a+c)\xi}. \tag{8}$$

Case 2. As $m\neq 1$, $b=0$ and $c=0$, eq. (5) become

$$\vartheta(\xi)=(a(m-1)(\xi+J))^{\frac{1}{m-1}}. \tag{9}$$

Case 3 . As $m\neq 1$, $b\neq 0$ and $c=0$, eq. (5) become

$$\vartheta(\xi)=\left(Je^{(b(m-1)\xi)}-\frac{a}{b}\right)^{\frac{1}{m-1}}. \tag{10}$$

Case 4. As $m\neq 1$, $a\neq 0$ and $b^2-4ac<0$, the results of eq. (5) becomes

$$\vartheta(\xi)=\left(-\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}} \tag{11}$$

and

$$\vartheta(\xi)=\left(-\frac{b}{2a}-\frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{(1-m)\sqrt{4ac-b^2}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}}. \tag{12}$$

Case 5. As $m\neq 1$,$a\neq 0$ and $b^2-4ac>0$, the results of eq. (5) becomes

$$\vartheta(\xi)=\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\tanh\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}} \tag{13}$$

and

$$\vartheta(\xi)=\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\coth\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi+J)\right]\right)^{\frac{1}{1-m}}. \tag{14}$$

Case 6. As $m \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, the results of eq. (5) become

$$\vartheta(\xi) = \left(\frac{1}{a(m-1)(\xi + J)} - \frac{b}{2a}\right)^{\frac{1}{1-m}}, \tag{15}$$

where J is a constant.

Step 3. When the derivatives of ϑ are entered into eq. (4), the equation in terms of ϑ is obtained. The solution is found by collecting terms that belong together and solving for the unknown constants of eq. (2) (see Ibrahim [10]).

2.1 Bäcklund Transformation

Supposed $\vartheta_n(\xi)$ and $\vartheta_{n-1}(\xi)$ are solutions of eq. (2), then

$$\frac{d\vartheta_n(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}(\xi)\xi}\frac{d\vartheta_{n-1}(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}\xi}(a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^m), \tag{16}$$

namely,

$$\frac{d\vartheta_n(\xi)}{a\vartheta_n^{2-m} + b\vartheta_n + c\vartheta_n^m} = \frac{d\vartheta_{n-1}(\xi)}{a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^m}. \tag{17}$$

Integrating eq. (17) with respect to ξ leads

$$\vartheta_n(\xi) = \left(\frac{-cA_1 + aA_2(\vartheta_{n-1}(\xi))^{1-m}}{bA_1 + aA_2 + aA_1(\vartheta_{n-1}(\xi))^{1-m}}\right)^{\frac{1}{1-m}}, \tag{18}$$

where A_1 and A_2 are random constants. The solution to eq. (2) can be derived using eq. (18), and the procedure is known as a Bäcklund transformation [25].

3. Results

To solve the third-order NLPDE provided in eq. (1), we must use the traveling wave transformation,

$$\vartheta(x,t) = \vartheta(\xi), \quad \xi = K(x + vt), \tag{19}$$

by plugging into eq. (1), We arrived at the following equation:

$$Kv\vartheta' + K\vartheta' + K\alpha\vartheta^2\vartheta' + K^3 v\beta\vartheta^{(3)} = 0. \tag{20}$$

Plugging eqs. (5), (6), (7) as well as its derivative into eq. (20), setting $m = 0$ and collecting all the coefficients of $U^i(\xi)$ (for $i = 0,1,2,3,4$), we get the following:

$$\left.\begin{aligned}
&\vartheta^0(\xi): cK + cKv + b^2cK^3v\beta + 2ac^2K^3v\beta - 0,\\
&\vartheta^1(\xi): bK + bKv + b^3K^3v\beta + 8abcK^3v\beta = 0,\\
&\vartheta^2(\xi): aK + aKv + cK\alpha + 7ab^2K^3v\beta + 8a^2cK^3v\beta = 0,\\
&\vartheta^3(\xi): bK\alpha + 12a^2bK^3v\beta = 0,\\
&\vartheta^4(\xi): aK\alpha + 6a^3K^3v\beta = 0.
\end{aligned}\right\} \tag{21}$$

Solving the algebraic equation system of eq. (21) results in

$$\left.\begin{aligned}
&a = \tfrac{cK^2\alpha\beta + \sqrt{6K^2\alpha\beta + c^2K^4\alpha^2\beta^2}}{6K^2\beta},\\
&b = 0,\\
&v = \tfrac{1}{3}(-3 + 6acK^2\beta - 2c^2K^2\alpha\beta).
\end{aligned}\right\} \tag{22}$$

We find the solutions of eq. (1) by combining the solutions of eq. (22), eqs. (8)-(15) and (19) as:
The periodic singular is as follows:

$$\vartheta_1^{\pm}(x,t)=\frac{\sqrt{6}c\tan\left[\frac{\sqrt{\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}(3J+K(3x+t(-3-c^2K^2\alpha\beta+c\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})))}{3\sqrt{6}}\right]}{\sqrt{\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}}, \tag{23}$$

$$\vartheta_2^{\pm}(x,t)=-\frac{\sqrt{6}c\cot\left[\frac{\sqrt{\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}(3J+K(3x+t(-3-c^2K^2\alpha\beta+c\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})))}{3\sqrt{6}}\right]}{\sqrt{\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}}. \tag{24}$$

The dark optical soliton is as follow:

$$\vartheta_3^{\pm}(x,t)=\frac{\sqrt{6}c\coth\left[\frac{\sqrt{-\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}(3J+K(3x+t(-3-c^2K^2\alpha\beta+c\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})))}{3\sqrt{6}}\right]}{\sqrt{-\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}}, \tag{25}$$

$$\vartheta_4^{\pm}(x,t)=\frac{\sqrt{6}c\tanh\left[\frac{\sqrt{-\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}(3J+K(3x+t(-3-c^2K^2\alpha\beta+c\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})))}{3\sqrt{6}}\right]}{\sqrt{-\frac{c(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})}{K^2\beta}}}, \tag{26}$$

and the singular soliton is as follow:

$$\vartheta_5^{\pm}(x,t)=\frac{18K^2\beta}{(cK^2\alpha\beta+\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})(3J+K(3x+t(-3-c^2K^2\alpha\beta+c\sqrt{K^2\alpha\beta(6+c^2K^2\alpha\beta)})))}. \tag{27}$$

4. Interpretation and Description of the Results

The purpose of this section is to discuss the behavior of some soliton and to understand some of the results gained from the previous section and simulations.

Figure 1 demonstrate the periodic singular wave solution for $\vartheta_1(x,t)$ of eq. (23). We consider the following parameters in a similar manner: $\beta=-1.35$; $\alpha=-2.3$; $c=-2.2$; $\gamma=-2.4$; $J=5.55$; $K=-3.8$.

Figure 2 demonstrate the periodic singular wave solution for $\vartheta_2(x,t)$ of eq. (22). We consider the following parameters in a similar manner: $\beta=-4.4$; $\alpha=2.8$; $c=8.3$; $\gamma=-0.8$; $J=-1.05$; $K=4.8$.

Figure 3 demonstrate the dark dark soliton solution for $\vartheta_3(x,t)$ of eq. (25). We consider the following parameters in a similar manner: $\beta=7.25$; $\alpha=1.85$; $c=5.85$; $\gamma=6.9$; $J=6.5$; $K=2.25$.

Figure 4 demonstrate the dark dark soliton solution for $\vartheta_4(x,t)$ of eq. (26). We consider the following parameters in a similar manner: $\beta=-3.25$; $\alpha=-3.25$; $c=0.75$; $\gamma=2.45$; $J=-3.25$; $K=-1.75$.

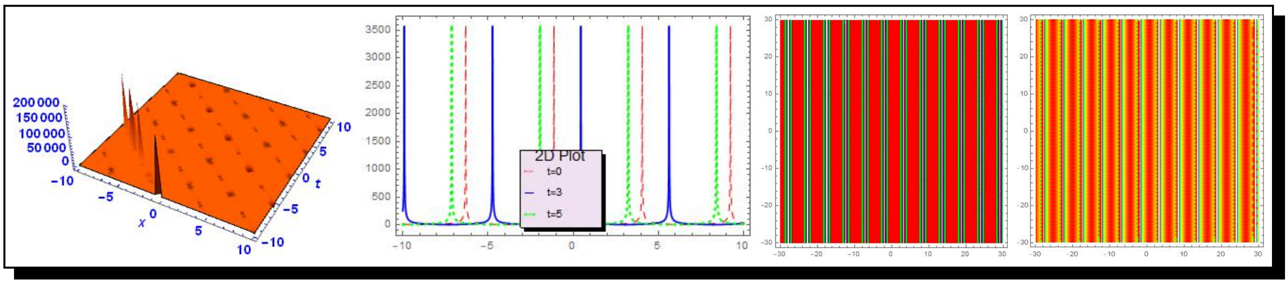

Figure 1. Plot of 3D, 2D, density and contour of (23)

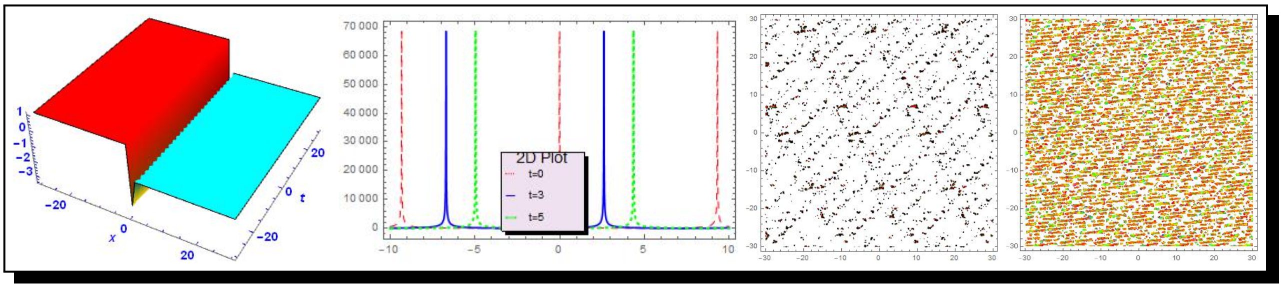

Figure 2. Plot of 3D, 2D, density and contour of (24)

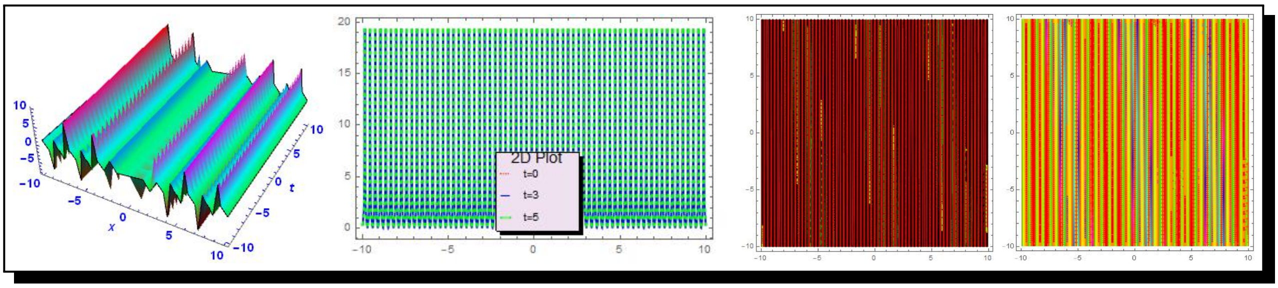

Figure 3. Plot of 3D, 2D, density and contour of (25)

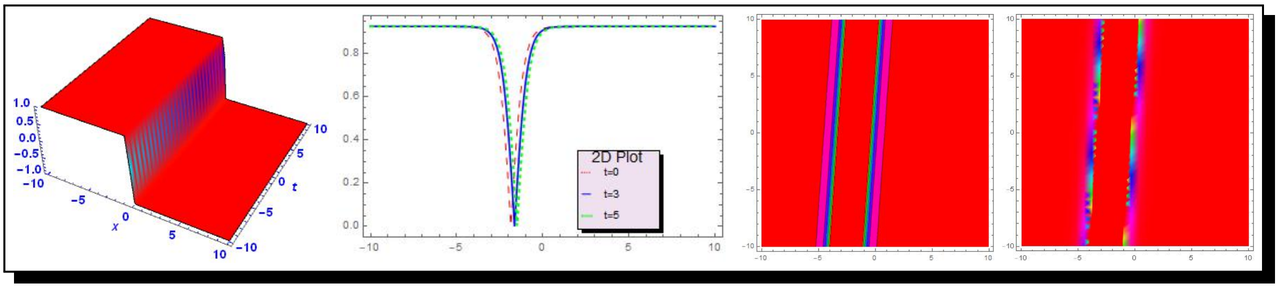

Figure 4. Plot of 3D, 2D, density and contour of (26)

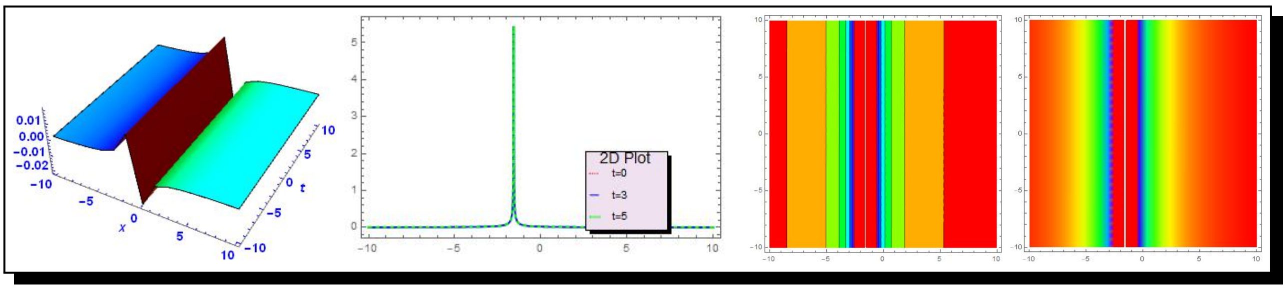

Figure 5. Plot of 3D, 2D, density and contour of (27)

Figure 5 demonstrate the singular wave solution for $\vartheta_5(x,t)$ of eq. (27). We consider the following parameters in a similar manner: $\beta = -2.8$; $\alpha = -4.4$; $c = 4.7$; $\gamma = 6.8$; $J = -6.65$; $K = -4.2$.

5. Concluding Remarks

The BBM equation is a NLPDE that describes the propagation of long waves in shallow water. The solution of the BBM equation using analytic and precise methods has been a focus of recent research. The tanh approach and the Hirota bilinear method are two of the most used methods for solving the BBM equation. Several academics have also investigated the behavior of the BBM equation under various boundary conditions. These investigations shed light on the physical features of the BBM equation and its behavior in dispersive media. The Riccati-Bernoulli sub-ODE method is an effective numerical method for solving NPDEs. The method has been demonstrated to be efficient fast and accurate in estimating the solutions of numerous NPDEs in diverse sectors of science and engineering. In this work, we applied the Riccati-Bernoulli sub-ODE method to solved the BBM equation and this lead to several solutions, such as soliton solutions, periodic solutions, and dark soliton solution, have also been used to find various sorts of solutions. Our results show that the Riccati-Bernoulli sub-ODE method solves the BBM problem in an efficient and accurate manner. The method can be used to solve additional NPDEs, making it a useful tool for researchers in a variety of domains.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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