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Research Article

Existence of Solutions of Mixed Variational-like Inequalities and Gap Functions

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Abstract. In this work, a gap function and an extended gap function are introduced, which gives rise to an optimization problem formulation to mixed variational-like inequalities. Furthermore, a convex lower bound to an extended gap function for Stampachhia and Minty mixed variational-like inequality problems is developed. The results presented in this paper generalize some well-known results of several authors, which play a significant role in the theory of variational inequalities.

Keywords. Gap function, Pseudomonotonicity, Strongly monotone, Lipschitz continuous

Mathematics Subject Classification (2020). 90C33, 49J40, 65K10

1. Introduction

The systematic study of variational inequality began in the early 1960's with the influential work of Hartman and Stampachhia [10], and Stampachhia [23,24] which has been used as an analytic tool for solving a wide variety of problems in mathematical programming, including system of nonlinear equations, optimization problems and fixed point theorems and many practical problems related to equilibrium. Gap functions studied by Ansari and Yao [1], Auchmuty [2], Gibali and Salahuddin [8], Hung *et al*. [11], Khaliq and Wu [13], and Wang *et al*. [26] play an important role in transforming variational inequality problems into optimization problems. The minimum gap function is a feasible approach for solving variational inequality problems. Further, different classes of gap functions for variational inequalities were applied which are known as primal (Larsson and Patriksson [14]), dual (Marcotte and Zhu [16]), regularized

(Ansari and Yao [1], Zhu and Marcotte [28]), difference (Scarf [19], Yamashita *et al*. [27]) and Giannessi's [7]. In recent years, this theory has been extended to variational-like inequalities and its many variants by numerous authors in different directions (see for example, Ansari and Yao [1], Ding [5], Khaliq and Wu [13], Lescarret [15], Parida *et al*. [17], Siddiqi *et al*. [20,21], Sonia and Sarma [22]). That is, the minimum of the non-differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities. The usual variational inequalities formulation admits various modifications and extensions which can also be in principle applied to economic equilibrium problems. Consider the *mixed variational inequality problem* of finding a point $u^* \in K$ such that

$$\langle F(u^*), u-u^*\rangle + \Psi(u) - \Psi(u^*) \geq 0, \quad \text{for all } u \in K, \tag{1}$$

where K is a nonempty convex set in the real Euclidean space $\mathbb{R}^n$, $F : K \to \mathbb{R}^n$ is a mapping, $\Psi : V \to \mathbb{R}$ is convex and V is a nonempty subset of $\mathbb{R}^n$ such that $K \subset V$. Problem (1) was originally considered by Lescarret [15] and Browder [4] in connection with its applications in mathematical physics and afterward studied by many authors (see for example, Baiocchi and Capelo [3], Duvaut and Lions [6], Grad and Lara [9], Ruiz-Garzón *et al*. [18], Ullah and Noor [25]). Motivated and inspired by the aforesaid work, the paper is devoted to study gap functions and extended gap functions for Stampachhia and Minty mixed variational-like inequalities.

The rest of the paper is organized as follows: In Section 2, we introduced the concept of Stampachhia and Minty variational-like inequality problem and Stampachhia and Minty mixed variational-like inequality problem. In Section 3, we establish a gap function for mixed variational-like inequalities. Extended gap function for mixed variational-like inequalities is discussed in Section 4.

2. Formulation and Preliminaries

Let K be a nonempty closed and convex subset of $\mathbb{R}^n$ and $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}^n$ be a bifunction. Let $F : K \to \mathbb{R}^n$ be the mapping, then *Stampachhia Variational Inequality Problem* (for short, SVIP) is the problem of finding $u^* \in K$ such that

$$\langle F(u^*), u-u^*\rangle \geq 0, \quad \text{for all } u \in K.$$

Minty Variational Inequality Problem (for short, MVIP) is the problem of finding $u^* \in K$ such that

$$\langle F(u), u-u^*\rangle \geq 0, \quad \text{for all } u \in K.$$

Let $\Psi : K \to \mathbb{R}$ be a mapping, then *Stampachhia Mixed Variational-like Inequality Problem* (for short, SMVLIP) is the problem of finding $u^* \in K$ such that

$$\langle F(u^*), \eta(u,u^*)\rangle + \Psi(u) - \Psi(u^*) \geq 0, \quad \text{for all } u \in K.$$

Minty Mixed Variational-like Inequality Problem (for short, MMVLIP) is the problem of finding $u^* \in K$ such that

$$\langle F(u), \eta(u,u^*)\rangle + \Psi(u) - \Psi(u^*) \geq 0, \quad \text{for all } u \in K.$$

Next we define two mappings $e_1, e_2 : K \to \mathbb{R}$ as follows:

For $u \in K$,

$$e_1(u) = \max_{v \in K}\{\langle F(u), \eta(v,u)\rangle + \Psi(v) - \Psi(u)\}$$

and

$$e_2(u) = \max_{v \in K}\{\langle F(u), \eta(u,v)\rangle + \Psi(u) - \Psi(v)\}.$$

We also write

$$\Gamma(u) = \arg\max_{v \in K}\langle F(v), \eta(u,v)\rangle + \Psi(u) - \Psi(u^*). \tag{2}$$

Further, we denote by sol(SMVLIP) (respectively, sol(MMVLIP)), the solution set of (SMVLIP) (respectively, (MMVLIP)). If for all $u,v \in K$, $\eta(u,v) + \eta(v,u) = 0$ and $\eta(u,u) = 0$, then $u^* \in$ sol(SMVLIP) (respectively, $u^* \in$ sol(MMVLIP)) if and only if, it is a global minimizer with objective values zero of the function $e_1(u^*)$ (respectively, $e_2(u^*)$).

Definition 1. The function $F : K \to \mathbb{R}^n$ is said to be:

(i) Ψ-η-monotone on K if

$$\langle F(u) - F(v), \eta(u,v)\rangle + \Psi(u) - \Psi(v) \geq 0, \quad \text{for all } u,v \in K,$$

(ii) Ψ-η-pseudomonotone on K if

$$\langle F(u), \eta(v,u)\rangle + \Psi(v) - \Psi(u) \geq 0$$
$$\Longrightarrow \quad \langle F(v), \eta(v,u)\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \text{for all } u,v \in K,$$

(iii) Ψ-η-strongly pseudomonotone on K with modulus α if there exists a positive constant α such that

$$\langle F(u), \eta(v,u)\rangle + \Psi(v) - \Psi(u) \geq 0$$
$$\Longrightarrow \quad \langle F(v), \eta(v,u)\rangle + \Psi(v) - \Psi(u) \geq \alpha\|u - v\|^2, \quad \text{for all } u,v \in K,$$

(iv) Ψ-η-pseudomonotone $^+$ on K if, it is Ψ-η-pseudomonotone on K and

$$\langle F(u), \eta(v,u)\rangle + \Psi(v) - \Psi(u) \geq 0,\ \langle F(v), \eta(v,u)\rangle + \Psi(v) - \Psi(u) = 0$$
$$\Longrightarrow \quad F(u) = F(v), \quad \text{for all } u,v \in K.$$

Remark 2.1. Ψ-η-monotone, Ψ-η-pseudomonotone, Ψ-η-strongly pseudomonotone, and Ψ-η-pseudomonotone$^+$ maps reduces to monotone, pseudomonotone, strongly pseudomonotone, and pseudomonotone$^+$ provided $\Psi = 0$ and $\eta(u,v) = u - v$, for all $u,v \in K$, respectively.

3. Gap Function for Mixed Variational-Like Inequalities

In this section, gap functions for (SMVLIP) and (MMVLIP) are introduced.

Definition 2. A function $f : K \to \mathbb{R}$ is known as gap function for (SMVLIP) (respectively, (MMVLIP)) if and only if

(i) $f(u) \geq 0$, for all $u \in K$,

(ii) $f(u^*) = 0 \Longleftrightarrow u^* \in$ sol(SMVLIP) (respectively, $u^* \in$ sol(MMVLIP)).

We note that if $\eta(u,v) = u - v$, for all $u,v \in K$, then e_2, defined above is called dual gap function associated with the (SMVLIP) and $e_2(u) = \langle F(\bar{v}), u - \bar{v}\rangle$, where $\bar{u}$ is any point in the set $\Gamma(u)$ (see Marcotte and Zhu [16]).

Proposition 3.1. *Let K be a nonempty convex subset of $\mathbb{R}^n$, $F : K \to \mathbb{R}^n$ be a function and $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}^n$ be a bifunction,*

(i) *If $\eta(v,u)$ is skew symmetric, for all $u,v \in K$, that is, $\eta(v,u) = -\eta(u,v)$, and $\eta(u,u) = 0$, then e_1 and e_2, are gap functions for* (SMVLIP) *and* (MMVLIP)*, respectively.*

(ii) *If F be an Ψ-η-pseudomonotone and hemicontinuous mapping, $u \mapsto \langle F(\cdot), \eta(u,\cdot)\rangle$ be a convex mapping and $\eta(u,u) = 0$, for all $u \in K$, then* sol(SMVLIP) = sol(MMVLIP)*. If, in addition, we assume that $\eta(v,u)$ is skew symmetric, for all $u,v \in K$, that is, $\eta(v,u) = -\eta(u,v)$ and F is Ψ-η-pseudomonotone$^+$, then F is constant over* sol(SMVLIP).

Proof. (i): This follows from the definitions.

(ii): sol(SMVLIP) $\subseteq$ sol(MMVLIP), follows from the Ψ-η-pseudomonotonicity of F. We show that sol(MMVLIP) $\subseteq$ sol(SMVLIP).

Let $u^* \in$ sol(MMVLIP). Since K is convex, we have for $u \in K$, $u_\lambda = \lambda u + (1-\lambda)u^*$, $\lambda \in [0,1]$,

$$\langle F(u_\lambda), \eta(u_\lambda, u^*)\rangle + \Psi(u_\lambda) - \Psi(u^*) \geq 0. \tag{3}$$

Since $u \mapsto \langle F(\cdot), \eta(u,\cdot)\rangle$ is convex and $\eta(u,u) = 0$, for all $u \in K$, we have

$$\langle F(u_\lambda), \eta(u_\lambda, u^*)\rangle + \Psi(u_\lambda) - \Psi(u^*) \leq \lambda \langle F(u_\lambda), \eta(u, u^*)\rangle + \Psi(u) - \Psi(u^*). \tag{4}$$

From (3) and (4), we have

$$\langle F(u^* + \lambda(u - u^*)), \eta(u, u^*)\rangle + \Psi(u) - \Psi(u^*) \geq 0, \quad \text{for all } u \in K.$$

Since F is hemicontinuous, we can allow $\lambda \to 0^+$, to obtain $u^* \in$ sol(SMVLIP).

Now, let $u_1^*, u_2^* \in$ sol(SMVLIP). Since sol(SMVLIP) = sol(MMVLIP), we have

$$\langle F(u_1^*, \eta(u_2^*, u_1^*)\rangle + \Psi(u_2^*) - \Psi(u_1^*) \Longleftrightarrow \langle F(u_2^*, \eta(u_2^*, u_1^*)\rangle + \Psi(u_2{}^*) - \Psi(u_1{}^*) \geq 0. \tag{5}$$

and

$$\langle F(u_2^*, \eta(u_1^*, u_2^*)\rangle + \Psi(u_1^*) - \Psi(u_2^*) \Longleftrightarrow \langle F(u_1^*, \eta(u_1^*, u_2^*)\rangle + \Psi(u_1^*) - \Psi(u_2^*) \geq 0. \tag{6}$$

From (5) and (6), and the given condition, we have

$$\langle F(u_2{}^*), \eta(u_2{}^*, u_1^*)\rangle + \Psi(u_2{}^*) - \Psi(u_1{}^*) = 0.$$

Since F is Ψ-η-pseudomonotone$^+$ on K, we have

$$F(u_1{}^*) = F(u_2{}^*).$$

□

Proposition 3.2. *Let $u^* \in$* sol(SMVLIP)*. If F is Ψ-η-pseudomonotone on K and assume that $\eta(v,u)$ is skew symmetric, for all $u,v \in K$, that is, $\eta(v,u) = -\eta(u,v)$. Then, every $u^* \in$* sol(SMVLIP) *lies on the set*

$$G^* = \{u \in K : \langle F(u^*), \eta(u,u^*)\rangle + \Psi(u) - \Psi(u^*) = 0\}.$$

Proof. Since $u^*, \bar{u} \in$ sol(SMVLIP), we have

$$\langle F(u^*), \eta(\bar{u}, u^*)\rangle + \Psi(\bar{u}) - \Psi(u^*) \geq 0,$$

$$\langle F(\bar{u}), \eta(u^*, \bar{u})\rangle + \Psi(u^*) - \Psi(\bar{u}) \geq 0.$$

From Ψ-η-pseudomonotonicity of F, we have

$$\langle F(\bar{u}), \eta(\bar{u}, u^*)\rangle + \Psi(\bar{u}) - \Psi(u^*) \geq 0,$$

$$\langle F(u^*), \eta(u^*, \bar{u})\rangle + \Psi(u^*) - \Psi(\bar{u}) \geq 0.$$

From the given condition, we have

$$\langle F(u^*), \eta(\bar{u}, u^*)\rangle + \Psi(\bar{u}) - \Psi(u^*) \le 0.$$

Hence we conclude that $\bar{u} \in G^*$. □

Proposition 3.3. *If F is Ψ-η-pseudomonotone$^+$ and assume that $\eta(v,u)$ is skew symmetric, for all $u,v \in K$, that is, $\eta(v,u) = -\eta(u,v)$, then for any $u^* \in$ sol(SMVLIP), F is constant and equal to $F(u^*)$ over $\Gamma(u^*)$.*

Proof. For every $u^* \in$ sol(SMVLIP) and $\bar{u} \in \Gamma(u^*)$, we have

$$\langle F(\bar{u}, \eta(u^*, \bar{u})\rangle = 0.$$

Also, the given condition implies that

$$\langle F(\bar{u}), \eta(\bar{u}, u^*)\rangle + \Psi(\bar{u}) - \Psi(u^*) = 0. \tag{7}$$

Since $u^* \in$ sol(SMVLIP), we have

$$\langle F(u^*), \eta(\bar{u}, u^*)\rangle + \Psi(\bar{u}) - \Psi(u^*) \ge 0. \tag{8}$$

Since F is Ψ-η-pseudomonotone$^+$ on K, from (7) and (8), it follows that

$$F(\bar{u}) = F(u^*).$$ □

Proposition 3.4. *If F is Ψ-η-pseudomonotone on K, and for all $u,v,w \in K$, assume that $\eta(v,u)$ is skew symmetric, that is, $\eta(v,u) = -\eta(u,v)$ and $\eta(u,v) = \eta(u,w) + \eta(w,v)$. Then,* sol(SMVLIP) = $\Gamma(u^*)$*, for every $u^* \in$* sol(SMVLIP).

Proof. Let $\bar{u} \in$ sol(SMVLIP). Then by Proposition 3.2, we have

$$\langle F(\bar{u}), \eta(u^*, \bar{u})\rangle + \Psi(u^*) - \Psi(\bar{u}) = 0, \quad \text{for any } u^* \in \text{sol(SMVLIP)},$$

which implies that $\bar{u} \in \Gamma(u^*)$.

Conversely, let $\bar{u} \in \Gamma(u^*)$, then we have

$$\langle F(\bar{u}), \eta(u^*, \bar{u})\rangle + \Psi(u^*) - \Psi(\bar{u}) = 0. \tag{9}$$

Also from Proposition 3.3, we have

$$F(\bar{u}) = F(u^*). \tag{10}$$

From (9) and (10), and the given condition, we have, for all $u \in K$,

$$\begin{aligned} \langle F(\bar{u}), \eta(u, \bar{u})\rangle + \Psi(u) - \Psi(\bar{u}) &= \langle F(\bar{u}), \eta(u, u^*)\rangle + \Psi(u) - \Psi(u^*) \\ &\quad + \langle F(\bar{u}), \eta(u^*, \bar{u})\rangle + \Psi(u^*) - \Psi(\bar{u}) \\ &= \langle F(u^*), \eta(u, u^*)\rangle + \Psi(u) - \Psi(u^*) \\ &\ge 0. \end{aligned}$$

Hence $\bar{u} \in$ sol(SMVLIP). □

Remark 3.5. If $\eta(u,v) = u - v$ and $\Psi = 0$, then the gap functions e_1 and e_2 collapses into the gap function in [16].

4. Extended Gap Function for Mixed Variational-Like Inequalities

In this section, a more general gap function is introduced by a function $\Omega(u,v): K\times K \to \mathbb{R}$ which satisfies the following assumptions:

(i) $\Omega(u,v)\ge 0$, for all $u,v\in K$,

(ii) $\Omega(u,v)$ is continuously differentiable on $K\times K$,

(iii) $\Omega(u,u)=0$ and $\nabla_2\Omega(u,u)=0$, for all $u\in K$,

(iv) $\Omega(u,v)$ is strongly convex on K with respect to v, for all $u\in K$.

We define

$$e(u)=\max_{v\in K}\{\langle F(u),\eta(u,v)\rangle+\Psi(u)-\Psi(v)-\Omega(u,v)\}.$$

The inclusion of the term $\Omega(\cdot,\cdot)$ was also discussed by Marcotte and Zhu [16] for (SMVLIP). Next, we prove that e is a gap function for (SMVLIP).

Theorem 4.1. *Suppose that* $\Omega(\cdot,\cdot)$ *satisfies the assumption* (i)-(iv). *If* $\eta(\cdot,\cdot)$ *is affine in the first argument and assume that* $\eta(v,u)$ *is skew symmetric, for all* $u,v\in K$*, that is,* $\eta(v,u)=-\eta(u,v)$*, and* $\eta(u,u)=0$*, then* e *is a gap function for* (SMVLIP).

Proof. Assume that

$$h(u,v)=\langle F(u),\eta(v,u)\rangle+\Psi(v)-\Psi(u)+\Omega(u,v). \tag{11}$$

Since for all $u,v\in K$, $\eta(u,v)+\eta(v,u)=0$, we have

$$e(u)=\max_{v\in K}\{-h(u,v)\}=-\min_{v\in K}h(u,v).$$

Since $h(u,u)=0$, for all $u\in K$, we have $\min_{v\in K}h(u,v)\le 0$, for all $u\in K$. Thus, we know that $e(u)\ge 0$, for all $u\in K$. Assume that $u\in \text{sol(SMVLIP)}$. By assumption (i), we conclude that

$$\langle F(u),\eta(v,u)\rangle+\Psi(v)-\Psi(u)+\Omega(u,v)\ge 0, \quad \text{for all } v\in K,$$

which implies that $\min_{v\in K}h(u,v)\ge 0$ or $e(u)=-\min_{v\in K}h(u,v)\le 0$. Hence $e(u)=0$.

Let us assume further that $e(u)=0$. Therefore, $\min_{v\in K}h(u,v)=0$, for all $v\in K$. Also, $h(u,u)=0$, u is a solution of the following optimization problem:

$$\min_{v\in K}h(u,v). \tag{12}$$

Define a function

$$v\mapsto f(v)=h(u,v), \quad \text{for all } v\in K. \tag{13}$$

Then from (11) and (12), we have that f is a convex and attains its minimum at u. From [12, Theorem 3.4 and Theorem 3.8], the directional derivative of f exists and

$$f'(u)(v-u)\ge 0, \quad \text{for all } v\in K. \tag{14}$$

As $\eta(\cdot,\cdot)$ is affine in the first argument, we have from (12) and (13),

$$f'(u)(v-u)=\langle F(u),\eta(v,u)\rangle+\Psi(v)-\Psi(u)+\langle\nabla_2\Omega(u,u),v-u\rangle. \tag{15}$$

Therefore from (14) and (15), and assumption (iii), we have

$$\langle F(u),\eta(v,u)+\Psi(u)-\Psi(v)\ge 0, \quad \text{for all } v\in K.$$

Hence $u\in \text{sol(SMVLIP)}$. □

In the following result, we establish a convex lower bound to the gap function e, which is an extension of [16, Theorem 3.3] to the (SMVLIP).

Theorem 4.2. *Suppose that $u^* \in$ sol(SMVLIP) and assume that $\eta(v,u)$ is skew symmetric, for all $u,v \in K$, and assumption (i)-(ii) are satisfied. We further suppose that the following conditions are satisfied:*

(i) *F is η-strongly pseudomonotone with a modulus μ,*

(ii) *$\eta(\cdot,\cdot)$ is affine in the second argument, and $\eta(u,u) = 0$, for all $u \in K$,*

(iii) *$\Omega(\cdot,\cdot)$ is convex in the second argument and gradient of $\Omega(\cdot,\cdot)$ with respect to the second argument is Lipschitz continuous with a constant L_Ω.*

Then, there exists a positive constant α such that

$$e(u) \geq \alpha\|u-u^*\|^2, \quad \textit{for all } u \in K.$$

Proof. Since $u^* \in$ sol(SMVLIP), we have

$$\langle F(u^*), \eta(u,u^*) + \Psi(u) - \Psi(u^*) \geq 0, \quad \text{for all } v \in K. \tag{16}$$

As F is Ψ-η-pseudomonotone with modulus μ. Hence (16) implies that

$$\langle F(u), \eta(u,u^*)\rangle + \Psi(u) - \Psi(u^*) \geq \mu\|u-u^*\|^2, \quad \text{for all } u \in K.$$

Let $v = \lambda u^* + (1-\lambda)u$, for $0 \leq \lambda \leq 1$. Since $\Omega(\cdot,\cdot)$ is convex in the second argument and its gradient with respect to the second argument is Lipschitz continuous, we have

$$\begin{aligned}\Omega(u,v) - \Omega(u,u) &\leq \langle \nabla_2\Omega(u,v), v-u\rangle \\ &= \langle \nabla_2\Omega(u,v) - \nabla_2\Omega(u,u), v-u\rangle \\ &\leq L_\Omega\|v-u\|^2.\end{aligned}$$

Since $\eta(\cdot,\cdot)$ is affine in the second argument, $\eta(u,u) = 0$, for all $u \in K, F$ is Ψ-η-strongly pseudomonotone and $\Omega(\cdot,\cdot)$ is Lipschitz continuous in the second argument, we have

$$\begin{aligned}e(u) &\geq \langle F(u), \eta(u,v)\rangle + \Psi(u) - \Psi(v) - \Omega(u,v) \\ &= \lambda\langle F(u), \eta(u,u^*)\rangle + \Psi(u) - \Psi(u^*) - (\Omega(u,v) - \Omega(u,u)) \\ &\geq \lambda\langle F(u), \eta(u,u^*)\rangle + \Psi(u) - \Psi(u^*) - L_\Omega\|v-u^*\|^2 \\ &\geq \lambda\mu\|u-u^*\|^2 - L_\Omega\lambda^2\|u-u^*\|^2 \\ &= (\lambda\mu - L_\Omega\lambda^2)\|u-u^*\|^2.\end{aligned}$$

Since the unconstrained maximum of $(\lambda\mu - \lambda^2 L_\Omega)$ occurs at $\mu/2L_\Omega$, we choose

$$\lambda = \min\{1, \mu/2L_\Omega\},$$

to obtain

$$e(u) \geq \alpha\|u-u^*\|^2,$$

where

$$\alpha = \begin{cases}\mu - L_\Omega, & \text{if } \mu \geq 2L_\Omega, \\ \mu^2/4L_\Omega, & \text{otherwise.}\end{cases}$$

Hence the result is proved. □

5. Conclusion

The article introduces two types of functions, namely gap function and an extended gap function to find the existence of solutions of a class of mixed variational-like inequalities in the sense of Stampachhia and Minty. Further efforts are required to explore several applications of this kind of gap function to study these inequalities for the case of interval valued mapping and convexificators in vector optimization and nonsmooth analysis in particular, which may stimulate further research in this area.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] Q. H. Ansari and J. C. Yao, Generalised variational-like inequalities and a gap function, *Bulletin of the Australian Mathematical Society* **59**(1) (1999), 33 – 44, DOI: 10.1017/S0004972700032573.

[2] G. Auchmuty, Variational principles for variational inequalities, *Numerical Functional Analysis and Optimization* **10**(9-10) (1989), 863 –874, DOI: 10.1080/01630568908816335.

[3] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free Boundary Problems*, Wiley-Blackwel, New York, 462 pages (1984).

[4] F. E. Browder, On the unification of the calculus of variations and the theory of monotone nonlinear operators in Banach spaces, *The Proceedings of the National Academy of Sciences (PNAS)* **56**(2) (1966), 419 – 425, DOI: 10.1073/pnas.56.2.419.

[5] X. P. Ding, Generalized variational-like inequalities with nonmonotone set-valued mappings, *Journal of Optimization Theory and Applications* **95** (1997), 601 – 613, DOI: 10.1023/A:1022673922484.

[6] G. Duvaut and J.-L. Lions, *Les Inéquations en Mécanique et en Physiqe*, Dunod, Paris, xx + 387 pages (1972).

[7] F. Giannessi, Separation of sets and gap functions for quasi-variational inequalities, in: *Proceedings of the International School of Mathematics "G. Stampacchia", 19th Course on Variational Inequalities and Network Equilibrium Problem* (June 19-15, 1994, Erice, Italy), F. Giannassi and A. Maugeri (editors), Plennum Press, New York, pp. 101 – 121 (1995).

[8] A. Gibali and Salahuddin, Error bounds and gap functions for various variational type problems, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* **115** (2021), article number 123, DOI: 10.1007/s13398-021-01066-8.

[9] S.-M. Grad and F. Lara, Solving mixed variational inequalities beyond convexity, *Journal of Optimization Theory and Applications* **190** (2021), 565 – 580, DOI: 10.1007/s10957-021-01860-9.

[10] P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Mathematica* **115** (1966), 271 – 310, DOI: 10.1007/BF02392210.

[11] N. V. Hung, S. Migórski, V. M. Tam and S. Zeng, Gap functions and error bounds for variational–hemivariational inequalities, *Acta Applicandae Mathematicae* **169** (2020), 691 – 709, DOI: 10.1007/s10440-020-00319-9.

[12] J. Jahn, *Introduction to the Theory of Nonlinear Optimization*, 3rd edition, Springer, Berlin — Heidelberg, x + 292 pages (1994), DOI: 10.1007/978-3-540-49379-2.

[13] A. Khaliq and S. Y. Wu, Variational like inequalities and gap functions, *Nonlinear Analysis Forum* **10**(1) (2005), 1 – 11.

[14] T. Larsson and M. Patriksson, A class of gap functions for variational inequalities, *Mathematical Programming* **64** (1994), 53 – 79, DOI: 10.1007/BF01582565.

[15] C. Lescarret, Cas d'addition des applications monotones maximales dans un espace de Hilbert, *Comptes Rendus de l'Académie des Sciences, Paris* **261** (1965), 1160 – 1163 (French).

[16] P. Marcotte and D. L. Zhu, Weak sharp solutions of variational inequalities, *SIAM Journal on Optimization* **9**(1) (1998), 179 – 189, DOI: 10.1137/S1052623496309867.

[17] J. Parida, M. Sahoo and A. Kumar, A variational like inequality problem, *Bulletin of the Australian Mathematical Society* **39**(2) (1989), 225 – 231, DOI: 10.1017/S0004972700002690.

[18] G. Ruiz-Garzón, R. Osuna-Gómez and J. Ruiz-Zapatero, Mixed variational inequality interval-valued problem: Theorems of existence of solutions, *Taiwanese Journal of Mathematics* **26**(6) (2022), 1257 – 1280, DOI: 10.11650/tjm/220503.

[19] H. Scarf, The approximation of fixed points of a continuous mapping, *SIAM Journal on Applied Mathematics* **15**(5) (1967), 1328 – 1342, DOI: 10.1137/0115116.

[20] A. H. Siddiqi, Q. H. Ansari and R. Ahmed, On generalized variational-like inequalities, *Indian Journal of Pure and Applied Mathematics* **26** (1995), 1135 – 1141.

[21] A. H. Siddiqi, A. Khaliq and Q. H. Ansari, On variational-like inequalities, *Annales des sciences mathématiques du Québec* **18** (1994), 95 – 104.

[22] Sonia and R. D. Sarma, Generalized vector quasi-variational-like inequality problem: A new approach, *Rendiconti del Circolo Matematico di Palermo Series 2* **72** (2023), 3681 – 3694, DOI: 10.1007/s12215-022-00854-4.

[23] G. Stampacchia, Formes bilinearies coercivities sur les ensembles convexes, *Comptes Rendus de l'Académie des Sciences, Paris* **258** (1964), 4413 – 4416.

[24] G. Stampacchia, Variational inequalities: Theory and applications of monotone operators, *Proceedings of NATO Advanced Study Institute, Venice*, (1968), 101 – 192.

[25] S. Ullah and M. A. Noor, An efficient method for solving new general mixed variational inequalities, *Journal of Inequalities and Special Functions* **11**(3) (2020), 1 – 9.

[26] Z.-B. Wang, Z.-Y. Chen and Z. Chen, Gap functions and error bounds for vector inverse mixed quasi-variational inequality problems, *Fixed Point Theory and Applications* **2019** (2019), Article number: 14, DOI: 10.1186/s13663-019-0664-5.

[27] N. Yamashita, K. Taji and M. Fukushima, Unconstrained optimization reformulations of variational inequality problems, *Journal of Optimization Theory and Applications* **92** (1997), 439 – 456, DOI: 10.1023/A:1022660704427.

[28] D. L. Zhu and P. Marcotte, An extended descent framework for variational inequalities, *Journal of Optimization Theory and Applications* **80** (1994), 349 – 366, DOI: 10.1007/BF02192941.