Communications in Mathematics and Applications
Vol. 15, No. 1, pp. 81–94, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i1.2347

http://www.rgnpublications.com

Research Article

On Fixed Point of Monotone (α,β)-Nonexpansive Mappings in Ordered Hyperbolic Metric Spaces

Kiran Dewangan

Department of Mathematics, Government Dudhadhari Bajrang Girls' Postgraduate Autonomous College (affiliated to Pt. Ravishakar Shukla University), Raipur 492001, Chhattisgarh, India
***Corresponding author:** dewangan.kiran@gmail.com

Received: July 14, 2023 **Accepted:** December 28, 2023

Abstract. This paper is concerned with convergence and Δ-convergence of a sequence generated by Picard-Mann hybrid iteration scheme for monotone (α,β)-nonexpansive type mappings in ordered hyperbolic metric spaces along with some fixed point results. An application in $L_1([0,1])$ space is discussed here.

Keywords. Fixed point, (α,β)-nonexpansive mapping, Hyperbolic space

Mathematics Subject Classification (2020). 47H09, 47H10

1. Introduction

Study of monotone nonexpansive mappings increase rapidly in past few years. The concept of monotone nonexpansive mapping was introduced by Bachar and Khamsi [2] in 2015. They proved existence of fixed points for semigroups of nonlinear monotone mappings acting in a Banach vector space endowed with a partial order.

In 1953, Mann [12] introduced an iteration scheme known as Mann iteration, to approximate fixed point of a mapping in the framework of Hilbert space as follows:
Let X be a Hilbert space and K be a non-empty closed convex subset of X. Let $T: K \to K$ be a mapping, and $\{x_k\}$ be a sequence in K defined by

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T x_k, \quad \text{for any } k \geq 1, \tag{1.1}$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$. Dehaish and Khamsi [4] established some convergence results of Mann iteration in Banach space endowed with partial order relation.

In the same year, Khamsi and Khan [6] introduced Krasnoselskii-Ishikawa iteration scheme as follows:

Let $K \subset L_1([0,1])$ be a non-empty closed and compact set, and $T : K \to K$ be a monotone nonexpansive mapping. Let $\{x_k\}$ be a sequence in K defined by

$$x_{k+1} = \alpha_k x_k + (1-\alpha_k)Tx_k, \quad \text{for any } k \ge 1,$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$. They proved some convergence results of Krasnoselskii-Ishikawa iteration scheme for monotone noexpansive mappings in $L_1([0,1])$ space.

In 2016, Song *et al*. [14] proved weak convergence of Mann iteration scheme for monotone nonexpansive mappings in Banach spaces under some considerations. In 2018, Udiin *et al*. [17] proved convergence and Δ-convergence of Mann iteration for monotone nonexpansive mappings in ordred CAT(0) space. In 2021, Kalsoom *et al*. [8] proved some fixed point results for monotone α-nonexpansive mapping and generalized β-nonexpansive mappings by using iteration scheme (Kalsoom *et al*. [9]) in hyperbolic spaces.

Recently, Picard-Mann hybrid iteration scheme was introduced by Khan [7] in 2013, in the framework of Banach spaces as follows:

$$\begin{cases} x_{k+1} = Ty_k, \\ y_k = (1-a_k)x_k + a_k Tx_k, \quad \text{for any } k \ge 1, \end{cases} \tag{1.2}$$

where $\{a_k\}$ is sequence in $(0,1)$. The author proved that the iteration scheme (1.2) is faster than some well-known iteration schemes and also proved some convergence results for nonexpansive type mappings.

Several generalizations of nonexpansive mappings were developed by many researchers and it is still ongoing. The convergence results of iteration scheme (1.2) in the framework of ordered hyperbolic metric spaces for monotone (α,β)-nonexpansive mappings are discussed here and also discussed some examples in support of main results.

2. Preliminaries

Let X be a non-empty set. A point $x \in X$ is called a fixed point of a mapping $T : X \to X$ if $T(x) = x$. Throughout the literature $F(T)$ denotes set of fixed points of T, i.e., $F(T) = \{x \in X : Tx = x\}$. Note that a mapping $T : X \to X$ is called

(i) *Lipschitz* if $d(Tx,Ty) \le Ld(x,y)$, for all $x,y \in X$, where $L > 0$.

(ii) *Contraction* if $d(Tx,Ty) \le Ld(x,y)$, for all $x,y \in X$, where $0 < L < 1$.

(iii) *Nonexpansive* if $d(Tx,Ty) \le d(x,y)$, for all $x,y \in X$ and $L = 1$.

In 2008, Suzuki [16] introduced a generalization of nonexpansive mappings which is called condition (C) as follows:

A mapping T defined on a subset K of a Banach space X is said to satisfy condition (C), if

$$\frac{1}{2}\|x - Tx\| \le \|x-y\| \Rightarrow \|Tx - Ty\| \le \|x-y\|, \quad \text{for } x,y \in K.$$

We can say that T is generalization of nonexpansive mapping in the sense of Suzuki. It is obvious that every nonexpansive mapping satisfies condition (C), but the converse is not true. Consider the following examples:

Example 2.1. Let $T:[0,2]\to[0,2]$ defined by

$$Tx=\begin{cases}0, & x\neq 2,\\ 2, & x=2.\end{cases} \tag{2.1}$$

It is clear that T is a Suzuki nonexpansive mapping and also nonexpansive.

Example 2.2. Let $X=\mathbb{R}$ and $K=\left[0,\frac{5}{2}\right]$ is subset of X. Let $d:X\times X\to\mathbb{R}$ such that $d(x,y)=|x-y|$. Clearly, (X,d) is metric space. Let T be a mapping defined on K such that

$$Tx=\begin{cases}0, & x\in[0,2],\\ 4x-12, & x\in\left[0,\frac{5}{2}\right].\end{cases} \tag{2.2}$$

Then T is a Suzuki nonexpansive mapping. However, it is not a nonexpansive mapping.

In 2011, Aoyama and Kohsaka [1] introduced a new type of nonexpansive mapping that satisfies condition (C), known as α-nonexpansive mapping as follows:

Definition 2.1 ([1])**.** A mapping $T:D(T)\to R(T)$ is called α-nonexpansive if $\alpha<1$ and

$$\|Tx-Ty\|^2\le\alpha\|Tx-y\|^2+\alpha\|x-Ty\|^2+(1-2\alpha)\|x-y\|^2.$$

Definition 2.2 ([17])**.** Let K be a non-empty subset of an ordered metric space X. A mapping $T:K\to K$ is said to be:

(i) monotone if $Tx\preceq Ty$, for all $x,y,\in K$ with $x\preceq y$,

(ii) monotone nonexpansive if T is monotone and

$$d(Tx,Ty)\le d(x,y),\quad \text{for all } x,y\in K \text{ with } x\preceq y,$$

(iii) monotone quasi-nonexpansive if T is monotone and

$$d(Tx,p)\le d(x,p),\quad \text{for all } x\in K,\ p\in F(T).$$

Definition 2.3 ([13])**.** Let K be a non-empty closed subset of an ordered Banach space $(X,\preceq)$. A mapping $T:K\to X$ is

(i) monotone α-nonexpansive if T is monotone and for some $\alpha<1$,

$$\|Tx-Ty\|^2\le\alpha\|Tx-y\|^2+\alpha\|x-Ty\|^2+(1-2\alpha)\|x-y\|^2, \text{ for all } x,y\in K \text{ with } x\preceq y,$$

(ii) monotone quasi-nonexpansive if T is monotone, $F(T)\neq\emptyset$ and $\|Tx-p\|\le\|x-p\|$ for all $p\in F(T)$ and $x\in K$ with $x\preceq p$ or $p\preceq x$.

In 2017, Muangchoo-in *et al.* [13] introduced a new type of nonexpansive mappings known as monotone (α,β)-nonexpansive mapping as follows:

Definition 2.4 ([13])**.** Let K be a non-empty closed subset of an ordered Banach space $(X,\preceq)$. A mapping $T:K\to K$ is said to be monotone (α,β)-nonexpansive if T is monotone and

$$\|Tx-Ty\|^2\le\alpha\|Tx-y\|^2+\beta\|x-Ty\|^2+(1-(\alpha+\beta))\|x-y\|^2,$$

for all $x,y\in K$ with $x\preceq y$, and $\alpha,\beta<1$.

Remark 2.1 ([13])**.** (i) If $\alpha=\beta$, then α-nonexpansive implies (α,β)-nonexpansive mapping and converse is true.

(ii) Every nonexpansive mapping is a 0-nonexpansive mapping and $(0,0)$-nonexpansive mapping.

(iii) Every α-nonexpansive and (α,β)-nonexpansive mappings with $F(T)\neq\emptyset$ are quasi-nonexpansive mappings.

Definition 2.5 ([10]). A hyperbolic space (X,d,W) is a metric space (X,d) together with a convexity mapping $W: X\times X\times[0,1]\to X$ such that for all $x,y,z\in X$ and $\theta,\zeta\in[0,1]$, we have

(i) $d(u,W(x,y,\theta))\le(1-\theta)d(u,x)+\theta d(u,y)$,

(ii) $d(W(x,y,\theta),W(x,y,\zeta))=|\theta-\zeta|d(x,y)$,

(iii) $W(x,y,\theta)=W(y,x,1-\theta)$,

(iv) $d(W(x,z,\theta),W(y,w,\theta))\le(1-\theta)d(x,y)+\theta d(z,w)$.

Example 2.3. Let $X=\mathbb{R}$ be a Banach space. Let $d: X\times X\to[0,\infty)$ be a mapping defined by

$$d(x,y)=\|x-y\|.$$

It is clear that d is metric on X. Let $K=[0,1]$ be a subset of X. Further, define a mapping $W: X\times X\times[0,1]$ by

$$W(x,y,\alpha)=\alpha x+(1-\alpha)y,$$

for all $x,y\in X$ and $\alpha\in[0,1]$. Then (X,d,W) is a hyperbolic space.

Definition 2.6 ([3]). A non-empty subset K of a hyperbolic space (X,d,W) is said to be convex, if $W(x,y,\alpha)\in K$ for all $x,y\in K$ and $\alpha\in[0,1]$.

Definition 2.7 ([15]). A hyperbolic space X is said to be uniformly convex if for any $r>0$ and $\varepsilon\in(0,2]$, there exists a $\delta\in(0,1]$ such that for all $x,y,z\in X$,

$$d(W(x,y,\frac{1}{2}),z)\le(1-\delta)r,$$

provided $d(x,z)\le r$, $d(y,z)\le r$ and $d(x,y)\ge\varepsilon r$.

Definition 2.8 ([8]). Let K be a non-empty subset of a hyperbolic space (X,d,W) and $\{x_k\}$ be a bounded sequence in X. For $x\in X$, there is a continuous functional $r(\cdot,\{x_k\}): X\to[0,\infty)$ defined by

$$r(x,\{x_k\})=\limsup_{k\to\infty} d(x_k,x).$$

The asymptotic radius $r(K,\{x_k\})$ of $\{x_k\}$ with respect to K is given by

$$r(K,\{x_k\})=\inf\{r(x,\{x_k\}): x\in K\}.$$

A point $x\in K$ is said to be an asymptotic center of the sequence $\{x_k\}$ with respect to K, if

$$r(x,\{x_k\})=\inf\{r(y,\{x_k\}): y\in K\}.$$

The set of all asymptotic centres of $\{x_k\}$ with respect to K is denoted by $A(K,\{x_k\})$.

Remark 2.2 ([8]). $A(K,\{x_k\})\neq\emptyset$ and $A(K,\{x_k\})$ has exactly one point if X is uniformly convex.

Definition 2.9 ([8]). In hyperbolic space (X,d,W) an order interval is any of the subsets

$$[a,\to)=\{x\in X: a\preceq x\} \quad\text{or}\quad (\leftarrow,a]=\{x\in X: x\preceq a\},$$

for any $a \in X$. So an order interval $[x,y]$ for all $x,y \in X$ is given by

$$[x,y] = \{z \in X : x \preceq z \preceq y\}.$$

Remark 2.3 ([8])**.** The order intervals are closed and convex.

Definition 2.10 ([15])**.** Let X be a hyperbolic space. A map $\eta : (0,\infty)\times(0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0,2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r.

Lemma 2.1 ([11])**.** *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequences $\{x_k\}$ in X has a unique asymptotic center with respect to any non-empty closed convex subset K of X.*

Lemma 2.2 ([5])**.** *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_k\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_k\}$ and $\{y_k\}$ are sequences in X such that $\limsup_{k\to\infty} d(x_k,x^*) \le c$, $\limsup_{k\to\infty} d(y_k,x^*) \le c$, and $\lim_{k\to\infty} d(t_k x_k + (1-t_k)y_k, x^*) = c$, for some $c > 0$. Then $\lim_{k\to\infty} d(x_k,y_k) = 0$.*

3. Main Results

Lemma 3.1. *Let K be non-empty closed convex subset of an ordered hyperbolic metric space (X,d,W). Let $T : K \to K$ be a monotone (α,β)-nonexpansive mapping with $F(T) \neq \emptyset$. Then*

(i) *T is monotone quasi-nonexpansive,*

(ii) *for all $x,y \in K$, $\alpha,\beta < 1$ with $x \preceq y$,*

$$d^2(Tx,Ty) \le d^2(x,y) + \frac{\alpha+\beta}{1-\beta} d^2(Tx,x) + \frac{2}{1-\beta} d(Tx,x)[|\alpha| d(x,y) + |\beta| d(Tx,Ty)].$$

Proof. From Remark 2.1, (α,β)-nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping, hence for all $p \in F(T)$ and from definition of (α,β)-nonexpansive mapping, we have

$$\begin{aligned} d^2(Tx,p) &= d^2(Tx,Tp) \\ &\le \alpha d^2(Tx,p) + \beta d^2(x,Tp) + (1-(\alpha+\beta)) d^2(x,p) \\ &\le \alpha d^2(x,p) + \beta d^2(x,p) + (1-(\alpha+\beta)) d^2(x,p) \\ &\le d^2(x,p). \end{aligned}$$

This implies that T is monotone quasi-nonexpansive mapping with $x \preceq y$.

We consider the following cases:

Case I: When $0 \le \alpha$, $\beta < 1$, then

$$\begin{aligned} d^2(Tx,Ty) &\le \alpha d^2(Tx,y) + \beta d^2(x,Ty) + (1-(\alpha+\beta)) d^2(x,y) \\ &\le \alpha[d(Tx,x) + d(x,y)]^2 + \beta[d(Tx,x) + d(Tx,Ty)]^2 + (1-(\alpha+\beta)) d^2(x,y) \\ &\le \alpha[d^2(Tx,x) + d^2(x,y) + 2d(x,y)d(Tx,x)] + \beta[d^2(Tx,x) + d^2(Tx,Ty) \\ &\quad + 2d(Tx,x)d(Tx,Ty)] + (1-(\alpha+\beta)) d^2(x,y) \\ &\le (\alpha+\beta) d^2(Tx,x) + (1-\beta) d^2(x,y) + \beta d^2(Tx,Ty) \end{aligned}$$

$$\begin{aligned}
&\quad + 2d(Tx,x)[\alpha d(x,y)+\beta d(Tx,Ty)] \\
&\le d^2(x,y)+\frac{\alpha+\beta}{1-\beta}d^2(Tx,x)+\frac{2}{1-\beta}d(Tx,x)[\alpha d(x,y)+\beta d(Tx,Ty)].
\end{aligned}$$

Case II: When $0 \le \alpha < 1$, $\beta < 0$. Then

$$\begin{aligned}
d^2(Tx,Ty) &\le \alpha d^2(Tx,y)+\beta d^2(x,Ty)+(1-(\alpha+\beta))d^2(x,y) \\
&\le \alpha[d(Tx,x)+d(x,y)]^2+\beta[d(Tx,x)-d(Tx,Ty)]^2+((1-(\alpha+\beta))d^2(x,y) \\
&\le \alpha[d^2(Tx,x)+d^2(x,y)+2d(x,y)d(Tx,x)]+\beta[d^2(Tx,x)+d^2(Tx,Ty) \\
&\quad - 2d(Tx,x)d(Tx,Ty)]+(1-(\alpha+\beta))d^2(x,y) \\
&\le (\alpha+\beta)d^2(Tx,x)+(1-\beta)d^2(x,y)+\beta d^2(Tx,Ty) \\
&\quad + 2d(Tx,x)[\alpha d(x,y)-\beta d(Tx,Ty)] \\
&\le d^2(x,y)+\frac{\alpha+\beta}{1-\beta}d^2(Tx,x)+\frac{2}{1-\beta}d(Tx,x)[\alpha d(x,y)-\beta d(Tx,Ty)].
\end{aligned}$$

Case III: When $\alpha < 0$, $0 \le \beta < 1$. Then

$$\begin{aligned}
d^2(Tx,Ty) &\le \alpha d^2(Tx,y)+\beta d^2(x,Ty)+(1-(\alpha+\beta))d^2(x,y) \\
&\le \alpha[d(Tx,x)-d(x,y)]^2+\beta[d(Tx,x)+d(Tx,Ty)]^2+(1-(\alpha+\beta))d^2(x,y) \\
&\le \alpha[d^2(Tx,x)+d^2(x,y)-2d(x,y)d(Tx,x)]+\beta[d^2(Tx,x)+d^2(Tx,Ty) \\
&\quad - 2d(Tx,x)d(Tx,Ty)]+(1-(\alpha+\beta))d^2(x,y) \\
&\le (\alpha+\beta)d^2(Tx,x)+(1-\beta)d^2(x,y)+\beta d^2(Tx,Ty) \\
&\quad + 2d(Tx,x)[-\alpha d(x,y)+\beta d(Tx,Ty)] \\
&\le d^2(x,y)+\frac{\alpha+\beta}{1-\beta}d^2(Tx,x)+\frac{2}{1-\beta}d(Tx,x)[-\alpha d(x,y)+\beta d(Tx,Ty)].
\end{aligned}$$

Case IV: When $\alpha,\beta < 0$. Then

$$\begin{aligned}
d^2(Tx,Ty) &\le \alpha d^2(Tx,y)+\beta d^2(x,Ty)+(1-(\alpha+\beta))d^2(x,y) \\
&\le \alpha[d(Tx,x)-d(x,y)]^2+\beta[d(Tx,x)-d(Tx,Ty)]^2+(1-(\alpha+\beta))d^2(x,y) \\
&\le \alpha[d^2(Tx,x)+d^2(x,y)-2d(x,y)d(Tx,x)]+\beta[d^2(Tx,x)+d^2(Tx,Ty) \\
&\quad - 2d(Tx,x)d(Tx,Ty)]+(1-(\alpha+\beta))d^2(x,y) \\
&\le (\alpha+\beta)d^2(Tx,x)+(1-\beta)d^2(x,y)+\beta d^2(Tx,Ty) \\
&\quad + 2d(Tx,x)[-\alpha d(x,y)-\beta d(Tx,Ty)] \\
&\le d^2(x,y)+\frac{\alpha+\beta}{1-\beta}d^2(Tx,x)+\frac{2}{1-\beta}d(Tx,x)[-\alpha d(x,y)-\beta d(Tx,Ty)].
\end{aligned}$$

Hence, for all $x,y \in K$, $\alpha,\beta < 1$ with $x \preceq y$,

$$d^2(Tx,Ty) \le d^2(x,y)+\frac{\alpha+\beta}{1-\beta}d^2(Tx,x)+\frac{2}{1-\beta}d(Tx,x)[|\alpha|d(x,y)+|\beta|d(Tx,Ty)]. \qquad \square$$

Lemma 3.2. *Let K be a non-empty closed convex subset of an ordered complete hyperbolic metric space (X,d,W). Let $T : K \to K$ be a monotone (α,β)-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_k\}$ be a bounded sequence in K such that $\lim_{k\to\infty} d(x_k,Tx_k) = 0$. Then $F(T)$ is closed.*

Proof. Let $\{x_k\}$ be a bounded sequence in $F(T)$ such that $\{x_k\}$ converges to some $y \in K$. For $p \in F(T)$ and using Lemma 3.1, we have

$$\begin{aligned} d(x_k,Ty) &\le d(x_k,Tx_k)+d(Tx_k,p)+d(p,Ty) \\ &\le d(x_k,Tx_k)+d(x_k,p)+d(p,y) \\ &\le d(x_k,Tx_k)+d(x_k,y). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k,Ty)=0.$$

Hence $F(T)$ is closed. □

Lemma 3.3. *Let K be a non-empty closed convex subset of an ordered complete uniformly convex hyperbolic metric space (X,d,W). Let $T:K\to K$ be a monotone nonexpansive mapping. Let $\{x_k\}$ be a sequence in K defined by (1.2) with $x_1 \in K$ such that $x_1 \preceq Tx_1$. Then*

(i) *$x_k \preceq y_k \preceq Tx_k$, for any $k \ge 2$,*

(ii) *$x_k \preceq x_{k+1}$, foe any $k \ge 1$,*

(iii) *$x_k \preceq p$ provided that $\{x_k\}$ Δ-converges to p and $p \in K$.*

Proof. (i): We proceed by induction on K. Since $x_1 \preceq Tx_1$ and ordered intervals are convex, we have

$$x_1 \preceq (1-a_1)x_1 + a_1Tx_1 \preceq Tx_1 \Rightarrow x_1 \preceq y_1 \preceq Tx_1.$$

Hence induction is true for $k=1$. Now suppose that induction is true for $k \ge 2$. Since $x_k \preceq Tx_k$, by convexity of ordered intervals, we have

$$x_k \preceq (1-a_k)x_k + a_kTx_k \preceq Tx_k \Rightarrow x_k \preceq y_k \preceq Tx_k.$$

Now $x_k \preceq y_k \Rightarrow Tx_k \preceq Ty_k = x_{k+1}$, i.e., $Tx_k \preceq x_{k+1}$. Since $y_k \preceq Tx_k \preceq x_{k+1}$, we have $x_k \preceq y_k \preceq x_{k+1}$. Since T is monotone, $Ty_k \preceq Tx_{k+1}$, i.e., $x_{k+1} \preceq Tx_{k+1}$. Using the fact that ordered intervals are convex, we have $x_{k+1} \preceq (1-a_{k+1})x_{k+1} + a_{k+1}Tx_{k+1} \preceq Tx_{k+1} \Rightarrow x_{k+1} \preceq y_{k+1} \preceq Tx_{k+1}$. Hence, $x_k \preceq y_k \preceq Tx_k$, for any $k \ge 2$.

(ii): Since $x_k \preceq y_k$ and T is monotone, we have

$$Tx_k \preceq Ty_k, \text{ i.e., } Tx_k \preceq x_{k+1}.$$

Since $x_k \preceq Tx_k$, $Tx_k \preceq x_{k+1}$, by transitivity of order, we have $x_k \preceq x_{k+1}$.

(iii): Suppose that $\{x_k\}$ Δ-converges to a point $p \in K$. Since by (ii) $x_k \preceq x_{k+1}$ for any $k \ge 1$, $\{x_k\}$ is monotonically increasing and the order interval $[x_k,\to)$ is closed, therefore $p \in [x_k,\to)$. If $p \notin [x_k,\to)$, then we can construct a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ by leaving first $m-1$ terms of $\{x_k\}$, and then the asymptotic centre of $\{x_{k_n}\}$ would not be p, which contradict that p is Δ-limit of $\{x_k\}$. □

Lemma 3.4. *Let K be a non-empty closed convex subset of an ordered complete uniformly convex hyperbolic metric space (X,d,W). Let $T:K\to K$ be a monotone nonexpansive mapping. Let $\{x_k\}$ be a sequence in K defined by (1.2) with $x_1 \in K$ such that $x_1 \preceq Tx_1$. Let $F(T) \neq \emptyset$ with $p \in F(T)$ such that $x_1 \preceq p$. Then*

(i) *$x_k \preceq p$, for all $p \in F(T)$,*

(ii) $\{x_k\}$ *is bounded,*

(iii) $\lim_{k\to\infty} d(x_k,p)$ *exists, for all* $p \in F(T)$.

Proof. (i): We proceed by induction on k. By assumption, $x_1 \preceq p$, hence induction is true for $k = 1$. Now suppose that $x_k \preceq p$ for $k \geq 2$. Since T is monotone, we have $Tx_k \preceq Tp = p$. From Lemma 3.3, $y_k \preceq Tx_k$, by convexity of ordered intervals, $y_k \preceq p \Rightarrow Ty_k \preceq Tp$, i.e., $x_{k+1} \preceq p$. Hence $x_k \preceq p$ for all $p \in F(T)$.

(ii): Since $x_k \preceq p$ for all $p \in F(T)$, hence $\{x_k\}$ is bounded sequence in K.

(iii): Using Lemma 3.1, we have

$$\begin{aligned} d^2(x_{k+1},p) &= d^2(Ty_k,Tp) \\ &\leq d^2(y_k,p) + \frac{\alpha+\beta}{1-\beta} d^2(Tp,p) + \frac{2}{1-\beta} d(Tp,p)[|\alpha| d(y_k,p) + |\beta| d(Ty_k,Tp)] \\ &\leq d^2(y_k,p) \\ &\leq d^2(x_k,p). \end{aligned}$$

It conclude that $\{d(x_k,p)\}$ is monotonically decreasing sequence bounded below by zero, hence $\lim_{k\to\infty} d(x_k,p)$ exists, for all $p \in F(T)$. □

Theorem 3.5. *Let K be a non-empty closed convex subset of an ordered complete uniformly convex hyperbolic metric space (X,d,W) with monotone modulus of uniform convexity η. Let $T : K \to K$ be a monotone (α,β)-nonexpansive mapping. Let $\{x_k\}$ be a sequence in K defined by* (1.2) *with $x_1 \in K$ such that $x_1 \preceq Tx_1$ and $x_k \preceq u$, for some $u \in K$. Then $F(T) \neq \emptyset$ if and only if* $\lim_{k\to\infty} d(x_k,Tx_k) = 0$.

Proof. Let $F(T) \neq \emptyset$ with $p \in F(T)$. As $\lim_{k\to\infty} d(x_k,p)$ exists for all $p \in F(T)$, so suppose that $\lim_{k\to\infty} d(x_k,p) = q$.

If $q = 0$, by using Lemma 3.1, we have

$$\begin{aligned} d^2(Tx_k,x_k) &\leq [d(Tx_k,p) + d(p,x_k)]^2 \\ &\leq d^2(Tx_k,p) + d^2(p,x_k) + 2d(p,x_k)d(Tx_k,p) \\ &\leq 4d^2(p,x_k). \end{aligned}$$

Therefore, the result follows.

Suppose that $q > 0$. As $\lim_{k\to\infty} d(x_k,p) = q$, it follows that $\limsup_{k\to\infty} d(x_k,p) \leq q$. Also $d(Tx_k,p) \leq d(x_k,p)$, this implies that $\limsup_{k\to\infty} d(Tx_k,p) \leq q$. Now, for any sequence $\{t_k\}$ in $[\delta,1-\delta]$ for $\delta \in (0,1)$, we have

$$\begin{aligned} \limsup_{k\to\infty} d(t_k x_k + (1-t_k)Tx_k,p) &\leq t_k \limsup_{k\to\infty} d(x_k,p) + (1-t_k)d(Tx_k,p) \\ &\leq q. \end{aligned}$$

Hence, by Lemma 2.2, we have $\lim_{k\to\infty} d(x_k,Tx_k) = 0$.

Conversely suppose that $\lim_{k\to\infty} d(x_k, Tx_k) = 0$. Then there exists a subsequence say $\{x_{k_n}\}$ of the sequence $\{x_k\}$ such that

$$\lim_{k\to\infty} d(x_{k_n}, Tx_{k_n}) = 0.$$

Let $B_k = \{u \in K : x_k \preceq u\}$ for all $k \in \mathbb{N}$. Clearly, B_k is non-empty closed and convex. Define

$$B^* = \bigcap_{k=1}^{\infty} B_k.$$

Then B^* is non-empty closed convex subset of K. Let $u \in B^*$, then $x_{k_n} \preceq u$ for all k. By monotonicity of T, we have $Tx_{k_n} \preceq Tu$ and by transitivity of order, we have $x_{k_n} \preceq Tu$, which implies that $T(B^*) \subset B^*$.

Now consider a function $r(,\{x_{k_n}\}) : B^* \to [0,\infty)$ generated by sequence $\{x_{k_n}\}$ such that

$$r(u,\{x_{k_n}\}) = \limsup_{k\to\infty} d(x_{k_n}, u).$$

Then, there exists a unique point $v \in B^*$ such that

$$r(v,\{x_{k_n}\}) = \inf\{r(u,\{x_{k_n}\}) : u \in B^*\}.$$

By monotonicity of T we have

$$r(Tu,\{x_{k_n}\}) = \limsup_{k\to\infty} d(x_{k_n}, Tu).$$

Using Lemma 3.1, we have

$$d^2(Tx_{k_n}, Tv) \le d^2(x_{k_n}, v) + \frac{\alpha+\beta}{1-\beta} d^2(Tx_{k_n}, x_{k_n}) + \frac{2}{1-\beta} d(Tx_{k_n}, x_{k_n})[|\alpha| d(x_{k_n}, v) + |\beta| d(Tv, Tx_{k_n})],$$

$$\limsup_{k\to\infty} d^2(Tx_{k_n}, Tv) \le \limsup_{k\to\infty} d^2(x_{k_n}, v).$$

This implies that $r(Tv,\{Tx_{k_n}\}) \le r(v,\{x_{k_n}\})$.

Now

$$\begin{aligned}
& r(Tv,\{Tx_{k_n}\}) \le r(v,\{x_{k_n}\}) \\
\Rightarrow\quad & \limsup_{k\to\infty} d(Tx_{k_n}, Tv) \le \limsup_{k\to\infty} d(x_{k_n}, v) \\
& \limsup_{k\to\infty} d(Tx_{k_n}, x_{k_n}) + \limsup_{k\to\infty} d(x_{k_n}, Tv) \le \limsup_{k\to\infty} d(x_{k_n}, v) \\
& \limsup_{k\to\infty} d(x_{k_n}, Tv) \le \limsup_{k\to\infty} d(x_{k_n}, v) \\
\Rightarrow\quad & r(Tv,\{x_{k_n}\}) \le r(v,\{x_{k_n}\}).
\end{aligned}$$

Hence $Tv = v$, which implies that $F(T) \ne \emptyset$. □

Theorem 3.6. *Let K be a non-empty closed convex subset of an ordered complete uniformly convex hyperbolic metric space (X,d,W) with monotone modulus of uniform convexity η. Let $T : K \to K$ be a monotone (α,β)-nonexpansive mapping. Let $\{x_k\}$ be a sequence in K defined by* (1.2) *with $x_1 \in K$ such that $x_1 \preceq Tx_1$. Let $F(T) \ne \emptyset$. Then $\{x_k\}$ Δ-converges to a unique fixed point of T.*

Proof. Let $p \in F(T)$. From Lemma 3.4 the sequence $\{x_k\}$ is bounded and from Lemma 2.1 the sequence $\{x_k\}$ has unique asymptotic center, so suppose that $A(\{x_k\}) = \{x\}$. Since $\{x_k\}$ is bounded sequence, so suppose that there are subsequences $\{x_{k_n}\}$ and $\{x_{k_m}\}$ of $\{x_k\}$ Δ-converges to some $l \in K$ and $m \in K$ (refer Lemma 3.4) with $x_{k_n} \preceq l$ and $x_{k_m} \preceq m$. From Lemma 3.4, $\lim_{k\to\infty} d(x_{k_n}, l)$ exists, and $\lim_{k\to\infty} d(x_{k_m}, m)$ exists. From the Theorem 3.5, $\lim_{k\to\infty} d(x_{k_n}, Tx_{k_n}) = 0$ and $\lim_{k\to\infty} d(x_{k_m}, Tx_{k_m}) = 0$. We claim that l is a fixed point of T. Using Lemma 3.1, we have

$$\begin{aligned} d^2(Tx_{k_n}, Tl) &\le d^2(x_{k_n}, l) + \frac{\alpha+\beta}{1-\beta} d^2 d(x_{k_n}, Tx_{k_n}) \\ &\quad + \frac{2}{1-\beta} d^2(x_{k_n}, Tx_{k_n})[|\alpha| d(x_{k_n}, l) + |\beta| d(Tx_{k_n}, Tl)] \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Since

$$\begin{aligned} r(Tl, \{x_{k_n}\}) &= \limsup_{k\to\infty} d(x_{k_n}, Tl) \\ &\le \limsup_{k\to\infty} d(x_{k_n}, Tx_{k_n}) + d(Tx_{k_n}, Tl) \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence $r(Tl, \{x_{k_n}\}) = 0$. Now

$$\begin{aligned} d(l, Tl) &\le d(l, x_{k_n}) + d(x_{k_n}, Tx_{k_n}) + d(Tx_{k_n}, Tl) \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Again

$$\begin{aligned} r(l, \{x_{k_n}\}) &= \limsup_{k\to\infty} d(x_{k_n}, l) \\ &\to 0 \text{ as } k \to \infty \\ \Rightarrow \quad r(l, \{x_{k_n}\}) &= 0. \end{aligned}$$

Hence, we conclude that $r(Tl, \{x_{k_n}\}) = r(l, \{x_{k_n}\}) \Rightarrow Tl = l$.

Similarly, we can prove that $Tm = m$. Therefore, $l = m$. □

4. Numerical Example

Example 4.1. Let $M = \mathbb{R}$ with metric d defined by $d(x,y) = |x-y|$ endowed with usual order and $B = [-3,3]$ be subset of M. Define $F : [-3,3] \to [-3,3]$ by

$$Tx = \frac{x}{7}, \quad \text{for any } x \in [-3,3].$$

To prove that T is a monotone mapping. Consider the following two cases:

Case I: When $x \preceq y$, let $x = -3$, and $y \in [-3,3]$, then $Tx = \frac{-3}{7} \le \frac{y}{7} = Ty$, for all $y \in [-3,3]$.

Case II: When $y \preceq x$, let $y = -3$, and $x \in [-3,3]$, then $Ty = \frac{-3}{7} \le \frac{x}{7} = Tx$, for all $x \in [-3,3]$.

Second, we prove that F is $(0.5, 0.5)$-nonexpansive mapping. Since

$$\begin{aligned} d^2(Tx, Ty) &= |Tx - Ty|^2 \\ &= \left|\frac{x}{7} - \frac{y}{7}\right|^2 \\ &= \frac{1}{49}|x-y|^2 \end{aligned}$$

$$\leq \frac{1}{49}d^2(x,y),$$

$$\alpha d^2(Tx,y)+\beta d^2(x,Ty)+(1-\alpha-\beta)d^2(x,y) \leq |x|^2\left(\frac{49-48\alpha}{49}\right)+|y|^2\left(\frac{49-48\beta}{49}\right) + 2|x|\,|y|\left(\frac{7-6\alpha-6\beta}{7}\right) \leq \frac{1}{49}[25|x|^2+25|y|^2+14|x\|y|].$$

Now suppose that T is $(0.5,0.5)$-nonexpansive mapping, i.e.,

$$d^2(Tx,Ty) \leq \alpha d^2(Tx,y)+\beta d^2(x,Ty)+(1-\alpha-\beta)d^2(x,y)$$

$$\Rightarrow \quad |x|^2+|y|^2+\frac{|x\|y|}{2} \geq 0.$$

Since $|x|^2+|y|^2+\frac{|x\|y|}{2} \geq 0$, for every $x,y \in [-3,3]$, hence we conclude that T is (α,β)-nonexpansive mapping for $\alpha=0.5$, $\beta=0.5$.

5. Application in $L_1([0,1])$ Space

Consider the space $L_1([0,1])$ of real valued functions defined on $[0,1]$ with absolute value Lebesgue integrable, i.e., $\int_0^1 |f(x)|dx < \infty$. Also, $f=0$ if and only if the set $\{x \in [0,1] : f(x)=0\}$ has Lebesgue measure 0. For any $f \in L_1([0,1])$,

$$\|f\| = \int_0^1 |f(x)|dx.$$

The ordered intervals having the form

$$[f,\to) = \{g \in L_1([0,1]) : f \preceq g\} \text{ or } (\leftarrow,f] = \{g \in L_1([0,1]) : g \preceq f\}.$$

Moreover

$$[f,g] = \{h \in L_1([0,1]) : f \preceq h \preceq g\} = [f,\to) \cap (\leftarrow,g]$$

is closed and convex.

Lemma 5.1. *Let K be non-empty convex subset of $L_1([0,1])$ space. Let $T : K \to K$ be monotone nonexpansive mapping. Let $\{f_k\}$ be a sequence in K defined by (1.2) with $f_1 \in K$ such that $f_1 \preceq Tf_1$. Then*

(i) *$f_k \preceq g_k \preceq Tf_k$, for any $k \geq 2$,*

(ii) *$f_k \preceq f_{k+1}$, for any $k \geq 1$.*

Proof. (i): Start with induction on K. Since $f_1 \preceq Tf_1$, hence induction is true for $k=1$. Now suppose that $f_k \preceq Tf_k$, for any $k \geq 2$. Since the order intervals are convex, we have

$$f_k \preceq (1-a_k)f_k + a_k Tf_k \Rightarrow f_k \preceq g_k \preceq Tf_k.$$

Since T is monotone, $Tf_k \preceq Tg_k$, i.e., $Tf_k \preceq f_{k+1}$. Since $f_k \preceq Tf_k \preceq f_{k+1} \Rightarrow g_k \preceq f_{k+1}$. By monotonicity of T, $Tg_k \preceq Tf_{k+1} \Rightarrow f_{k+1} \preceq Tf_{k+1}$.

(ii): By part (i), $f_k \preceq g_k \Rightarrow Tf_k \preceq Tg_k$. By transitivity of order, $f_k \preceq f_{k+1}$. □

Lemma 5.2. *Let K be non-empty convex and compact subset of $L_1([0,1])$ space. Let $T: K \to K$ be monotone (α,β)-nonexpansive mapping. Let $\{f_k\}$ be a sequence in K defined by (1.2) with $f_1 \in K$ such that $f_1 \preceq Tf_1$. Then $\{f_k\}$ converges almost everywhere to some $f \in K$ and $\lim_{k\to\infty} \|f_k - Tf_k\| = 0$.*

Proof. Let $\{f_k\}$ be a sequence in K. Since K is compact, there exists a subsequence $\{f_{k_n}\}$ of $\{f_k\}$ converges almost everywhere to some $f \in K$. Since the order intervals are closed, $f \in [f_k, \to)$, i.e. $f_k \preceq f$ for all $k \geq 1$, i.e., $\{f_k\}$ converges almost everywhere to some $f \in K$. Now suppose that $\lim_{k\to\infty} \|f_k - Tf_k\| = s$. Then obviously $s = 0$. If $s \neq 0$, then, since $f_1 \preceq f_k \preceq f$, for any $k \geq 1$, hence f_k and f are comparable. Moreover, $0 \leq f_k - f_1 \leq f - f_1 \Rightarrow \|f_k - f_1\| \leq \|f - f_1\|$. Now

$$\begin{aligned}\|Tf_{k+n} - f_k\| &\leq \|Tf_{k+n} - Tf_1\| + \|Tf_1 - Tf_k\| + \|Tf_k - f_k\| \\ &\leq \|f_{k+n} - f_1\| + \|f_1 - f_k\| + \|Tf_k - f_k\| \\ &\leq 2\|f - f_1\| + |Tf_k - f_k\|\end{aligned}$$

Taking $k \to \infty$, we have $\|f_k - f_1\| \leq 0$, which gives a contradiction. Hence $s = 0$ and thus we have result. □

Theorem 5.3. *Let K be non-empty convex and compact subset of $L_1([0,1])$ space. Let $T: K \to K$ be monotone (α,β)-nonexpansive mapping. Let $\{f_k\}$ be a sequence in K defined by (1.2) with $f_1 \in K$ such that $f_1 \preceq Tf_1$. Suppose that all conditions of Lemma 5.2 satisfies. Then $T(f) = f$.*

Proof. Let $f_1 \preceq Tf_1$. From Lemma 5.2, $\{f_k\}$ converges almost everywhere to $f \in K$, where f_k and f are comparable for any $k \geq 1$. Using Lemma 3.1, we have

$$\begin{aligned}\|Tf - Tf_k\|^2 &\leq \|f - f_k\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tf_k - f_k\|^2 + \frac{2}{1-\beta}\|Tf_k - f_k\|[|\alpha|\|f - fk_k\| + |\beta|\|Tf - Tf_k\|] \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Now

$$\begin{aligned}\|Tf - f\| &\leq \|Tf - Tf_k\| + \|Tf_k - f_k\| + \|f_k - f\| \\ &\to 0 \text{ as } k \to \infty,\end{aligned}$$

which implies that $\|f - Tf\| = 0$. □

6. Conclusion

Some convergence results are obtained for a sequence generated by Picard-Mann hybrid iteration scheme for monotone (α,β)-nonexpansive mappings in the setting of ordered hyperbolic metric spaces. The main results are justified by numerical examples. Also, an application in $L_1([0,1])$ space is also discussed.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] K. Aoyama and F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, *Nonlinear Analysis: Theory, Methods & Applications* **74**(13) (2011), 4387 – 4391, DOI: 10.1016/j.na.2011.03.057.

[2] M. Bachar and M. A. Khamsi, On common approximate fixed points of monotone nonexpansive semigroups in Banach spaces, *Fixed Point Theory and Applications* **2015** (2015), Article number: 160, DOI: 10.1186/s13663-015-0405-3.

[3] S.-S. Chang, G. Wang, L. Wang, Y. K. Tang and Z. L. Ma, Δ-convergence theorems for multi-valued nonexpansive mappings in hyperbolic spaces, *Applied Mathematics and Computation* **249** (2014), 535 – 540, DOI: 10.1016/j.amc.2014.10.076.

[4] B. A. B. Dehaish and M. A. Khamsi, Mann iteration process for monotone nonexpansive mappings, *Fixed Point Theory and Applications* **2015** (2015), Article number: 177, DOI: 10.1186/s13663-015-0416-0.

[5] B. A. B. Dehaish and M. A. Khamsi, Browder and Göhde fixed point theorem for monotone nonexpansive mappings, *Fixed Point Theory and Applications* **2016** (2016), Article number: 20, DOI: 10.1186/s13663-016-0505-8.

[6] M. A. Khamsi and A. R. Khan, On monotone nonexpansive mappings in $L_1([0,1])$, *Fixed Point Theory and Applications* **2015** (2015), Article number: 94, DOI: 10.1186/s13663-015-0346-x.

[7] S. H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory and Applications* **2013** (2013), Article number: 69, DOI: 10.1186/1687-1812-2013-69.

[8] A. Kalsoom, N. Saleem, H. Işık, T. M. Al-Shami, A. Bibi and H. Khan, Fixed point approximation of monotone nonexpansive mappings in hyperbolic spaces, *Journal of Function Spaces* **2021** (2021), 3243020, 14 pages, DOI: 10.1155/2021/3243020.

[9] A. Kalsoom, M. Rashid, T.-C. Sun, A. Bibi, A. Ghaffar, M. Inc and A. A. Aly, Fixed points of monotone total asymptotically nonexpansive mapping in hyperbolic space via new algorithm, *Journal of Function Spaces* **2021** (2021), 8482676, 10 pages, DOI: 10.1155/2021/8482676.

[10] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *Transactions of the American Mathematical Society* **357**(1) (2005), 89 – 128, DOI: 10.1090/S0002-9947-04-03515-9.

[11] L. Leuştean, Nonexpansive iterations in uniformly convex W-hyperboloc spaces, *Nonlinear Analysis and Optimization I: Nonlinear Analysis*, A. Leizarowitz, B. S. Mordukhovich, I. Shafrir and A. J. Zaslavski (editors), Contemporary Mathematics series, Vol. 513, pp. 193 – 210, (2010), DOI: 10.1090/conm/513/10084.

[12] W. R. Mann, Mean value methods in iteration, *Proceedings of the American Mathematical Society* **4** (1953), 506 – 510, DOI: 10.1090/S0002-9939-1953-0054846-3.

[13] K. Muangchoo-in, D. Thongtha, P. Kumam and Y. J. Cho, Fixed point theorems and convergence theorems for monotone (α,β)-nonexpansive mappings in ordered Banach spaces, *Creative Mathematics and Informatics* **26**(2) (2017), 163 – 180, DOI: 10.37193/CMI.2017.02.06.

[14] Y. Song, P. Kumam and Y. J. Cho, Fixed point theorems and iterative approximations for monotone nonexpansive mappings in ordered Banach spaces, *Fixed Point Theory and Applications* **2016** (2016), Article number: 73, DOI: 10.1186/s13663-016-0563-y.

[15] C. Suanoom and C. Klin-eam, Remark on fundamentally non-expansive mappings in hyperbolic spaces, *Journal of Nonlinear Sciences and Applications* **9**(5) (2016), 1952 – 1956, DOI: 10.22436/jnsa.009.05.01.

[16] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *Journal of Mathematical Analysis and Applications* **340**(2) (2008), 1088 – 1095, DOI: 10.1016/j.jmaa.2007.09.023.

[17] I. Uddin, C. Garodia and J. J. Nietro, Mann iteration for monotone nonexpansive mappings in ordered CAT(0) space with an application to integral equations, *Journal of Inequalities and Applications* **2018** (2018), Article number: 339, DOI: 10.1186/s13660-018-1925-2.