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Research Article

On the Hyper Zagreb Indices of the Subdivision Related Composite Graphs

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Abstract. Complex network structures are developed from basic graphs by using the graphical operations. The topological indices serves a crucial part for the prediction of any molecular compound in terms of toxicity of any chemical and the utility of the same for the pharmaceutical and the therapeutic industry. To understand the topology of a molecule, one needs to convert the information contained in a molecule to a numerical value which is when the topological indices comes into picture. In this article, we determine the explicit expressions of the first and second hyper Zagreb indices by the method of combinatorial inequalities for the corona products of the subdivision related graphs notably the subdivision vertex, subdivision edge, subdivision neighborbood vertex, subdivision neighborhood edge and subdivision double corona product of graphs. These graphical operations will facilitate in understanding the underlying topologies of certain complex network graphs.

Keywords. Hyper Zagreb indices, Corona, Subdivision graphs, Graph operations

Mathematics Subject Classification (2020). 05C09, 05C76, 05C92

1. Introduction

Molecular descriptors are significant units linked with some network graph for theoretical chemistry. Numerous properties of networks are characterized by using the invariants in graphs. Applications of descriptors in the quantitative structure property and activity interactions are particularly intriguing.

All through the paper, we are working with the simple, connected and undirected molecular graphs. A molecular graph represents a chemical compound's compositional formula. The atoms and molecular bonds are represented by the vertices and edges of the molecular networks respectively. The vertex and edge set of a molecular graph, G are represented by $V(G)$ and $E(G)$ respectively. For the graph G, $d_G(w)$ denotes the degree of vertex w and $e = vw$ is the edge joining v with w.

While describing the π-electron energy for the conjugated molecules, Gutman [13] initiated the study on the first and second Zagreb indices which turned out to be one among the extensively explored molecular descriptors (Gutman and Trinajstić [13], and Gutman *et al*. [14]). Further, many characteristics related to the descriptors have been studied by Zhou [23], and Zhou and Gutman [24]. The first hyper Zagreb index was defined by Shirdel *et al*. [21] and exact expressions of the invariant with respect to various graph operations were computed (Basavanagoud and Patil [4], and Gao *et al*. [10]). Bounds related to the first hyper Zagreb index have also been determined by Falahati-Nezhad and Azari [8],

$$HM_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^2.$$

Extensive research in this domain led to the determination of another descriptor called the second hyper Zagreb index (Gao *et al*. [11]) and the formulations of the indices with respect to the graph products and operational series have also been obtained by Alameri [2], and Modabish *et al*. [19]. Certain bounds related to the hyper Zagreb indices have been identified and generated by Wang *et al*. [22],

$$HM_2(G) = \sum_{uv\in E(G)} [d_G(u)\cdot d_G(v)]^2.$$

Analysis of various graph operations has always given a wide range of research opportunities in relevant areas. $S(G)$ or subdivision graph is obtained by substituting every edge of the graph with a degree two vertex, keeping the original vertices unaltered. The corona product of graphs $G_1(p_{G_1}, q_{G_1})$ and $G_2(p_{G_2}, q_{G_2})$ is obtained while taking a copy of G_1 with p_{G_1} replicas of G_2 along with linking every node of the pth duplicate of G_2 to the pth node of G_1; $1 \le p \le p_{G_1}$.

Combinatorial bounds have been obtained for the general sum-connectivity index and the first entire Zagreb index with respect to the corona products (Akhter *et al*. [1], and Maji and Ghorai [18]). Also, inequalities related to the subdivision graph and the corona product variants have been determined related to SK index by Sheeja *et al*. [20]. Discrete inequalities related to some graph operation formulations have been analysed for certain other molecular descriptors (Aruvi *et al*. [3], Bharali *et al*. [5], Das *et al*. [6], Fath-Tabar and Vaez-Zadeh [9], and Gao *et al*. [12]).

2. Methodology

The exact expressions of the second hyper Zagreb indices of the subdivision related composite graphs such as the subdivision vertex, subdivision edge, subdivision vertex neighborhood, subdivision edge neighborhood corona products is obtained. The explicit expressions of the first hyper Zagreb index for the subdivision vertex corona and subdivision edge corona of graphs have been determined by Devi and Kaladevi [7], and Luo and Juo [17]. Further, two new complex graph operations have also been identified by Hameed *et al.* [15], i.e., the subdivision double and subdivision double neighborhood corona products and here we are obtaining the precise formulations of the first and second hyper Zagreb indices with respect to the above graph variants.

2.1 Subdivision-Vertex Corona Product

Introduced by Lu and Miao [1], for $G(n_G,m_G)$ and $H(n_H,m_H)$, the *subdivision-vertex corona product* indicated as $G\odot H$ is acquired by taking one replica of distinct vertex graph $S(G)$ and n_G replicas of H and linking a node of $V(G)$ on the ith location in $S(G)$ to each node in the ith replica of H. $|V(G\odot H)| = m_G+n_G+n_Gn_H$ and $|E(G\odot H)| = 2m_G+n_Gn_H+n_Gm_H$.

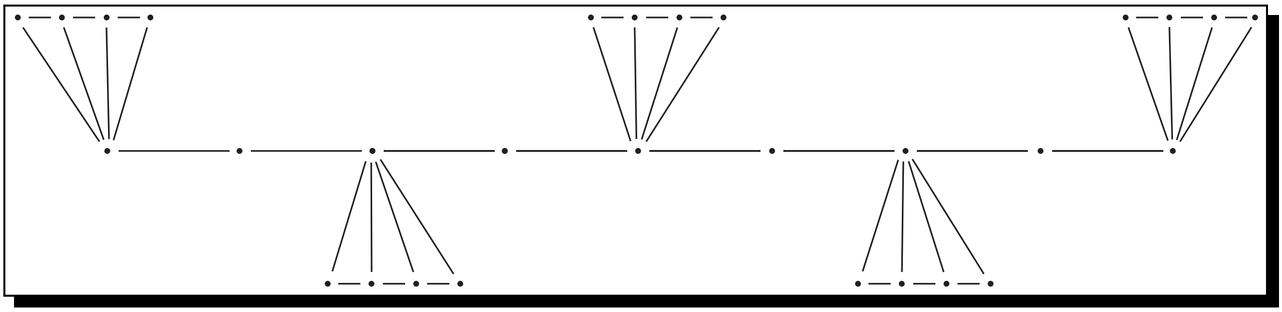

Figure 1. Subdivision vertex corona product $P_5\odot P_4$

Lemma 2.1. *The degree behaviour of the vertices is:*

$$d_{G\odot H}(v)=\begin{cases} d_G(v)+n_H, & \text{if } v\in V(G),\\ 2, & \text{if } v\in I(G),\\ d_H(v)+1, & \text{if } v\in V(H).\end{cases}$$

Theorem 2.1. *Assume that G and H are arbitrary graphs. Then*

$$\begin{aligned} HM_2(G\odot H) &= M_1(G)[M_1(H)+4m_H+9n_H]+M_1(H)[n_H(n_Gn_H+4m_G)+2n_G]+4F(G)\\ &\quad +n_G[F(H)+2(2M_2(H)+ReZG_3(H))+HM_2(H)+m_H]\\ &\quad +n_H[n_Hn_G(4m_H+n_H)+4m_G(4m_H+3n_H)]. \end{aligned}$$

Proof.

$$\begin{aligned} HM_2(G\odot H) &= \sum_{uv\in E(G\odot H)} (d_{G\odot H}(u).\, d_{G\odot H}(v))^2\\ &= \sum_{\substack{uv\in E(G\odot H)\\ u\in V(G)\\ v\in V(H)}} (d(u).\, d(v))^2+\sum_{\substack{uv\in E(G\odot H)\\ u\in V(G)\\ v\in I(G)}} (d(u)\cdot d(v))^2+\sum_{\substack{uv\in E(G\odot H)\\ u\in V(H)\\ v\in I(H)}} (d(u)\cdot d(v))^2\\ &= \sum f+\sum g+\sum h. \end{aligned}$$

[1]P. Lu and Y. Miao, Spectra of the subdivision-vertex and subdivision-edge coronae, *arXiv:1302.0457v2*, URL: https://arxiv.org/pdf/1302.0457.pdf.

For the computation of $\sum f$,

$$\sum f = \sum_{u\in V(G)}\sum_{v\in V(H)} ((d_G(u)+n_2)\cdot(d_H(v)+1))^2$$

$$= \sum_{u\in V(G)}\sum_{v\in V(H)} (d_G(u)^2+2n_H d_G(u)+n_H^2)(d_H(v)^2+2d_H(v)+1)$$

$$\Rightarrow \quad \sum f = M_1(G)[M_1(H)+4m_H+n_H]+n_H M_1(H)[n_G n_H+4m_G] + n_H[n_G n_H(4m_H+n_H)+4m_G(4m_H+n_H)].$$

To determine $\sum g$,

$$\sum g = \sum_{\substack{uv\in E(G\odot H)\\ u\in V(G)\\ v\in I(G)}} ((d_G(u)+n_H)\cdot 2)^2$$

$$= \sum_{u\in V(G)} 4[d_G(u)^3+2n_H d_G(u)^2+n_H^2 d_G(u)]$$

$$\Rightarrow \quad \sum g = 4[F(G)+2n_H M_1(G)+2m_G n_H^2].$$

Similarly for $\sum h$,

$$\sum h = \sum_{\substack{uv\in E(G\odot H)\\ u\in V(H)\\ v\in I(H)}} [(d_H(u)+1)\cdot(d_H(v)+1)]^2$$

$$= n_G \sum_{uv\in E(H)} (d_H(u)^2+2d_H(u)+1)\cdot(d_H(v)^2+2d_H(v)+1)$$

$$\Rightarrow \quad \sum h = n_G[HM_2(H)+2ReZG_3(H)+F(H)+4M_2(H)+2M_1(H)+m_H].$$

From all the computations,

$$HM_2(G\odot H) = M_1(G)[M_1(H)+4m_H+9n_H]+M_1(H)[n_H(n_G n_H+4m_G)+2n_G]+4F(G) + n_G[F(H)+2(2M_2(H)+ReZG_3(H))+HM_2(H)+m_H] + n_H[n_H n_G(4m_H+n_H)+4m_G(4m_H+3n_H)].$$

This concludes the proof. □

Corollary 2.1. *The Second Hyper Zagreb Index of Subdivision Vertex Corona Product of two path graphs P_n, P_m and cycle graphs C_n, C_m:*

$$HM_2(P_n\odot P_m) = 9m^3n+34m^2n-44m^2+109mn-62m-179n+4,$$

$$HM_2(C_n\odot C_m) = 9m^3n+44m^2n+149mn+32n,$$

$$HM_2(P_n\odot C_m) = 9m^3n+44m^2n-44m^2+149mn-102m+32n-56.$$

2.2 Subdivision Edge Corona Product

Introduced by Lu and Miao[1], for $G(n_G,m_G)$ and $H(n_H,m_H)$, the *subdivision-edge corona product* indicated as $G\ominus H$ is acquired by taking a replica of distinct vertex graph $S(G)$ and m_G replicas of H and linking a node of $I(G)$ on ith location in $R(G)$ to each node in the ith replica of H. $|V(R(G)\ominus H)| = m_G(1+n_H)+n_G$ and $|E(R(G)\ominus H)| = m_G(n_H+m_H+2)$.

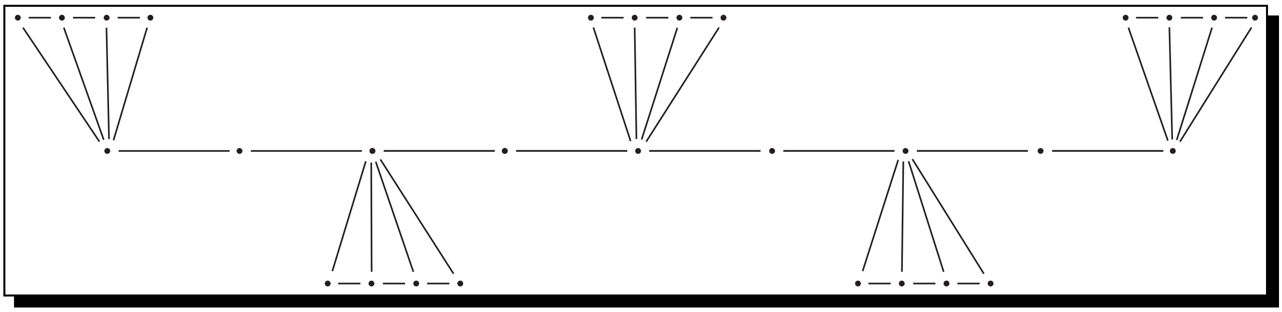

Figure 2. Subdivision edge corona product $P_5 \ominus P_4$

Lemma 2.2. *The degree behaviour of the vertices is:*

$$d_{G\ominus H}(v) = \begin{cases} d_G(v), & \text{if } v \in V(G), \\ n_H + 2, & \text{if } v \in I(G), \\ d_H(v)+1, & \text{if } v \in V(H). \end{cases}$$

Theorem 2.2. *Assume that G and H are arbitrary graphs. Then*

$$HM_2(G\ominus H) = (n_H+2)^2 F(G) + m_G[(n_H+2)^2+2]M_1(H) + m_G[2(2M_2(H)+ReZG_3(H)) + F(H) + HM_2(H) + m_H + (n_H+2)^2(4m_H+n_H)].$$

Proof.

$$\begin{aligned} HM_2(G\ominus H) &= \sum_{uv\in E(G\ominus H)} (d_{G\ominus H}(u).\, d_{G\ominus H}(v))^2 \\ &= \sum_{\substack{uv\in E(G\ominus H)\\ u\in I(G)\\ v\in V(H)}} (d(u)\cdot d(v))^2 + \sum_{\substack{uv\in E(G\ominus H)\\ u\in V(H)\\ v\in V(H)}} (d(u)\cdot d(v))^2 + \sum_{\substack{uv\in E(G\ominus H)\\ u\in V(G)\\ v\in I(G)}} (d(u)\cdot d(v))^2 \\ &= \sum f + \sum g + \sum h. \end{aligned}$$

For the computation of $\sum f$,

$$\begin{aligned} \sum f &= m_G \sum_{v\in V(H)} (n_H+2)^2 (d_H(v)+1)^2 \\ &= m_G \sum_{v\in V(H)} (d_H(v)^2 + 2d_H(v)+1) \end{aligned}$$

$$\Rightarrow \quad \sum f = m_G (n_H+2)^2 [M_1(H) + 4m_H + n_H]$$

To determine $\sum g$,

$$\begin{aligned} \sum g &= m_G \sum_{\substack{uv\in E(H)\\ u,v\in V(H)}} ((d_H(u)+1)\cdot(d_H(v)+1))^2 \\ &= m_G \sum_{u,v\in V(H)} ((d_H(u)^2 + 2d_H(u)+1)\cdot(d_H(v)^2+2d_H(v)+1)) \end{aligned}$$

$$\Rightarrow \quad \sum g = m_G[HM_2(H) + F(H) + 2(M_1(H) + 2M_2(H) + ReZG_3(H)) + m_H].$$

Similarly for $\sum h$,

$$\sum h = \sum_{\substack{uv\in E(G\ominus H)\\ u\in V(G)\\ v\in I(G)}} [(d_G(u))\cdot(n_H+2)]^2$$

$\Rightarrow \qquad \sum h = (n_H+2)^2 F(G).$

From all the computations,

$$HM_2(G \ominus H) = (n_H+2)^2 F(G) + m_G[(n_H+2)^2+2]M_1(H) + m_G[2(2M_2(H)+ReZG_3(H))+F(H)+HM_2(H)+m_H+(n_H+2)^2(4m_H+n_H)].$$

This concludes the proof. □

Corollary 2.2. *The second hyper Zagreb index of subdivision edge corona product of two path graphs P_n, P_m and cycle graphs C_n, C_m:*

$$HM_2(P_n \ominus P_m) = 9m^3n - 9m^3 + 34m^2n - 40m^2 + 109mn - 133m - 179n + 155,$$
$$HM_2(C_n \ominus C_m) = 9m^3n + 44m^2n + 149mn + 32n,$$
$$HM_2(P_n \ominus C_m) = 9m^3n - 9m^3 + 44m^2n - 50m^2 + 133mn - 157m + 32n - 56.$$

2.3 Subdivision Vertex Neighborhood Corona Product

For $G(n_G, m_G)$ and $H(n_H, m_H)$, the *subdivision vertex neighborhood corona product* indicated as $G \boxdot H$ is acquired by taking one replica of distinct vertex graph $S(G)$ and n_G replicas of H and linking neighboring vertices of G on the ith position in $S(G)$ to each node in the ith replica of H. $|V(G \boxdot H)| = n_G + m_G + n_G n_H$ and $|E(G \boxdot H)| = 2m_G + n_G m_H + 2m_G n_H$ (Liu and Lu [16]).

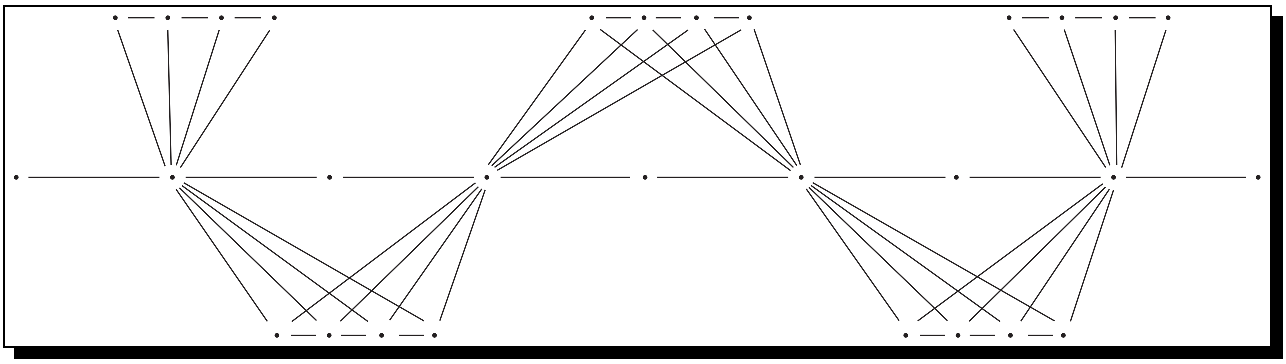

Figure 3. Subdivision vertex neighborhood corona product $P_5 \boxdot P_4$

Lemma 2.3. *The degree behaviour of the vertices is:*

$$d_{G\boxdot H}(v) = \begin{cases} d_G(v), & \text{if } v \in V(G), \\ 2(n_H+1), & \text{if } v \in I(G), \\ d_H(v) + d_G(w), & \text{if } v \in V(H),\ w \in V(G). \end{cases}$$

Theorem 2.3. *Assume that G and H are arbitrary graphs. Then*

$$HM_2(G \boxdot H) = n_G HM_2(H) + M_1(G)[4HM_2(H) + F(H)] + 2[F(G)M_1(H) + 2m_G ReZG_3(H)] + m_H Y(G) + 4(n_H+1)^2[n_G M_1(H) + n_H M_1(G) + F(G) + 8m_G m_H].$$

Proof.

$$HM_2(G \boxdot H) = \sum_{uv \in E(G\boxdot H)} (d_{G\boxdot H}(u) \cdot d_{G\boxdot H}(v))^2$$

$$= \sum_{\substack{uv\in E(G\boxdot H)\\ u,v\in V(H)}} (d(u)\cdot d(v))^2 + \sum_{\substack{uv\in E(G\boxdot H)\\ u\in V(G)\\ v\in I(G)}} (d(u)\cdot d(v))^2 + \sum_{\substack{uv\in E(G\boxdot H)\\ u\in I(G)\\ v\in V(H)}} (d(u)\cdot d(v))^2$$

$$= \sum f + \sum g + \sum h.$$

For the computation of $\sum f$,

$$\sum f = \sum_{w\in V(G)} \sum_{\substack{uv\in E(H)\\ u,v\in V(H)}} ((d_H(u)+d_G(w))\cdot(d_H(v)+d_G(w)))^2$$

$$= \sum_{w\in V(G)} \sum_{\substack{uv\in E(H)\\ u,v\in V(H)}} (d_H(u)^2+2d_H(u)d_G(w)+d_G(w)^2)(d_H(v)^2+2d_H(v)d_G(w)+d_G(w)^2)$$

$$\Rightarrow \quad \sum f = n_G HM_2(H)+M_1(G)F(H)+4m_G ReZG_3(H)+m_H Y(G)+2F(G)M_1(H)+4M_1(G)M_2(H).$$

To determine $\sum g$,

$$\sum g = \sum_{\substack{uv\in E(H)\\ u\in V(G)\\ v\in I(G)}} ((d_G(u))\cdot(2n_H+2))^2$$

$$\Rightarrow \qquad \sum g = 4(n_H+1)^2 F(G)$$

Similarly for $\sum h$,

$$\sum h = \sum_{\substack{uv\in E(G\boxdot H)\\ u\in I(G)\\ v\in V(H)}} ((2n_H+2)\cdot(d_H(v)+d_G(w)))^2$$

$$= 4(n_H+1)^2 \sum_{w\in V(G)} \sum_{v\in V(H)} d_H(v)^2+2d_H(v)d_G(w)+d_G(w)^2$$

$$\Rightarrow \qquad \sum h = 4(n_H+1)^2[n_G M_1(H)+n_H M_1(G)+8m_G m_H]$$

From all the computations,

$$HM_2(G\boxdot H) = n_G HM_2(H)+M_1(G)[4HM_2(H)+F(H)]+2[F(G)M_1(H)+2m_G ReZG_3(H)]$$
$$+m_H Y(G)+4(n_H+1)^2[n_G M_1(H)+n_H M_1(G)+F(G)+8m_G m_H].$$

This concludes the proof. □

Corollary 2.3. *The second hyper Zagreb index of subdivision vertex neighborhood corona product of two path graphs P_n,P_m and cycle graphs C_n, C_m:*

$$HM_2(P_n \boxdot P_m) = 64m^3n - 56m^3 + 104m^2n - 136m^2 + 272mn - 454m - 504n + 594,$$
$$HM_2(C_n \boxdot C_m) = 64m^3n + 160m^2n + 384mn + 32n,$$
$$HM_2(P_n \boxdot C_m) = 64m^3n - 56m^3 + 160m^2n - 168m^2 + 384mn - 518m + 32n - 56.$$

2.4 Subdivision Edge Neighborhood Corona Product

For $G(n_G,m_G)$ and $H(n_H,m_H)$, the *subdivision edge neighborhood corona product* indicated as $G\boxminus H$ is acquired by taking the distinct vertex graph $S(G)$ and m_G replicas of H and linking neighboring nodes of $I(G)$ on the ith location in $S(G)$ to each node in ith replica of H. $|V(G\boxminus H)| = m_G(1+n_H)+n_G$ and $|E(R(G)\boxminus H)| = 2m_G + m_G m_H + 2m_G n_H$ (Liu and Lu [16]).

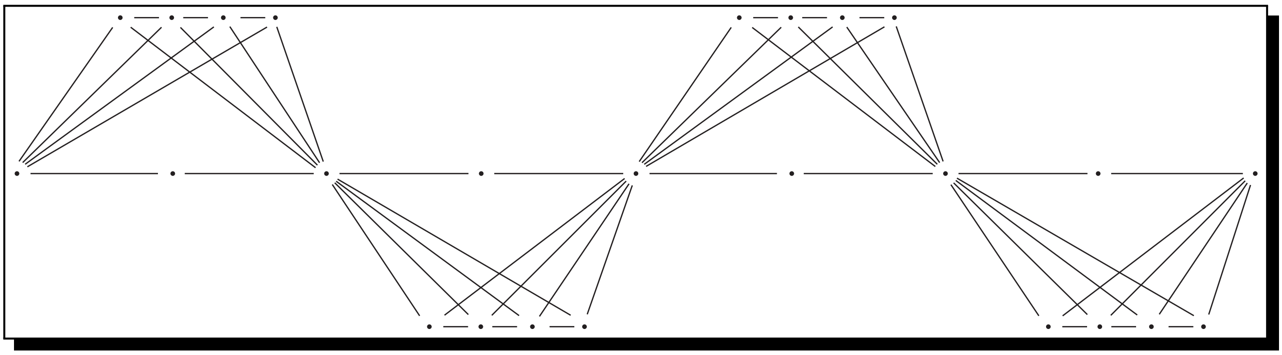

Figure 4. Subdivision edge neighborhood corona product $P_5 \boxminus P_4$

Lemma 2.4. *The degree behaviour of the vertices is:*

$$d_{G\boxminus H}(v) = \begin{cases} (n_H+1)d_G(v), & \text{if } v \in V(G), \\ 2, & \text{if } v \in I(G), \\ d_H(v)+2, & \text{if } v \in V(H). \end{cases}$$

Theorem 2.4. *Assume that G and H are arbitrary graphs. Then*

$$HM_2(G\boxminus H) = m_G[HM_2(H)+4(ReZG_3(H)+F(H))+16(M_1(H)+M_2(H)+m_H)] \\ +(n_H+1)^2[M_1(H)+4(n_H+2m_H+1)]F(G).$$

Proof.

$$HM_2(G\boxminus H) = \sum_{uv\in E(G\boxminus H)} (d_{G\boxminus H}(u)\cdot d_{G\boxminus H}(v))^2$$

$$= \sum_{\substack{uv\in E(G\boxminus H)\\ u,v\in V(H)}} (d(u)\cdot d(v))^2 + \sum_{\substack{uv\in E(G\boxminus H)\\ u\in V(G)\\ v\in I(G)}} (d(u)\cdot d(v))^2 + \sum_{\substack{uv^i\in E(G\boxminus H)\\ u\in V(G)\\ v^i\in V(H^i)}} (d(u)\cdot d(v))^2$$

$$= \sum f + \sum g + \sum h.$$

For the computation of $\sum f$,

$$\sum f = m_G \sum_{\substack{uv\in E(H)\\ u,v\in V(H)}} ((d_H(u)+2)\cdot(d_H(v)+2))^2$$

$$= m_G \sum_{\substack{uv\in E(H)\\ u,v\in V(H)}} (d_H(u)^2+4d_H(u)+4)(d_H(v)^2+4d_H(v)+4)$$

$$\Rightarrow \quad \sum f = m_G[HM_2(H)+4ReZG_3(H)+4F(H)+16M_2(H)+16M_1(H)+16m_H]$$

To determine $\sum g$,

$$\sum g = \sum_{\substack{uv\in E(S(G))\\ u\in V(G)\\ v\in I(G)}} (((n_H+1)d_G(u))\cdot 2)^2$$

$$\Rightarrow \quad \sum g = 4(n_H+1)^2F(G).$$

Similarly for $\sum h$,

$$\sum h = \sum_{\substack{uv^i \in E(G \boxminus H) \\ u \in V(G) \\ v^i \in V(H^i)}} ((d_G(u)(n_H+1)) \cdot (d_H(v)+2))^2$$

$$= \sum_{u \in V(G)} \sum_{v \in V(H)} (n_H+1)^2 d_G(u)^3 (d_H(v)^2 + 4d_H(v) + 4)$$

$$\Rightarrow \quad \sum h = (n_H+1)^2 [M_1(H) + 8m_H + 4n_H] F(G).$$

From all the computations,

$$HM_2(G \boxminus H) = m_G[HM_2(H) + 4(ReZG_3(H) + F(H)) + 16(M_1(H) + M_2(H) + m_H)] \\ + (n_H+1)^2[M_1(H) + 4(n_H + 2m_H + 1)]F(G).$$

This concludes the proof. □

Corollary 2.4. *The second hyper Zagreb index of subdivision edge neighborhood corona product of two path graphs P_n, P_m and cycle graphs C_n, C_m:*

$$HM_2(P_n \boxminus P_m) = 128m^3n - 224m^3 + 176m^2n - 308m^2 + 224mn - 200m - 560n + 620,$$

$$HM_2(C_n \boxminus C_m) = 128m^3n + 288m^2n + 448mn + 32n,$$

$$HM_2(P_n \boxminus C_m) = 128m^3n - 224m^3 + 288m^2n - 504m^2 + 448mn - 592m + 32n - 56.$$

2.5 Subdivision Double Corona Product

The *subdivision double corona product* for the graphs $\Upsilon(n,m)$, $\Upsilon_1(n_1,m_1)$ and $\Upsilon_2(n_2,m_2)$ indicated as $\Upsilon^s \circ (\Upsilon_1, \Upsilon_2)$ is determined by taking a replica of Υ^s, n replicas of Υ_1 and m replicas of Υ_2 and linking a node of Υ in the ith location in Υ^s to each node in the ith replica of Υ_1 and also linking the new inserted nodes of Υ^s in the jth location to each node in the jth replica of Υ_2 (Hameed *et al.* [15]).

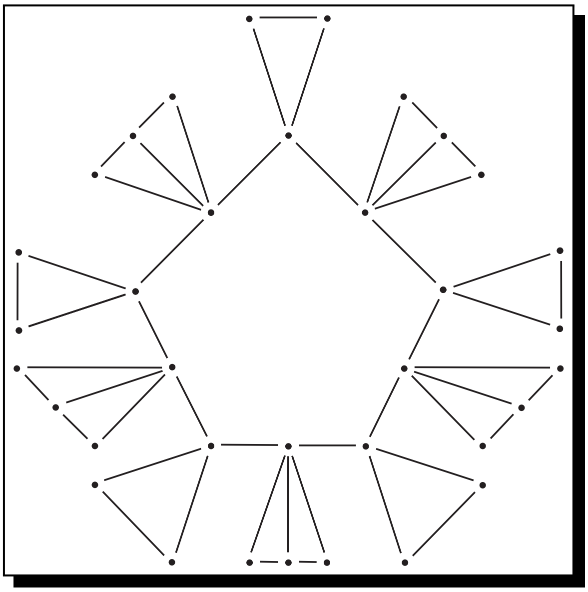

Figure 5. Subdivision double corona product $C_5^s \circ (P_2, P_3)$

Lemma 2.5. *The degree behaviour of the vertices is:*

$$d_{\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)}(v)=\begin{cases} d_{\Upsilon}(v)+n_1, & \text{if } v\in V(\Upsilon),\\ d_{\Upsilon_1}(v)+1, & \text{if } v\in V(\Upsilon_1),\\ d_{\Upsilon_2}(v)+1, & \text{if } v\in V(\Upsilon_2),\\ n_2+2, & \text{if } v\in V(\Upsilon^s).\end{cases}$$

Theorem 2.5. *Assume that $\Upsilon(n,m)$, $\Upsilon_1(n_1,m_1)$ and $\Upsilon_2(n_2,m_2)$ are arbitrary graphs. Then*

$$\begin{aligned} HM_1(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)) &= n[HM_1(\Upsilon_1)+5M_1(\Upsilon_1)]+m[HM_2(\Upsilon_2)+5M_1(\Upsilon_2)]\\ &\quad +n_1M_1(\Upsilon)+2m[(n_1+n_2+2)^2+4m_1+2n_1(n_1+1)]\\ &\quad +4(nm_1+mm_2)+n(n_1+1)[n_1(n_1+1)+4m_1]\\ &\quad +m(n_2+3)[n_2(n_2+3)+4m_2]\\ &\quad +\sum_{uv\in E(\Upsilon^s)}[d_{\Upsilon}^2+2(n_1+n_2+2)d_{\Upsilon}]. \end{aligned}$$

Proof.

$$\begin{aligned} HM_1(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)) &= \sum_{uv\in E(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2))}(d_{\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)}(u)+d_{\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)}(v))^2\\ &= \sum_{uv\in E(\Upsilon^s)}((d_{\Upsilon}(u)+n_1)+(n_2+2))^2\\ &\quad +n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+1)+(d_{\Upsilon_1}(v)+1))^2\\ &\quad +m\sum_{uv\in E(\Upsilon_2)}((d_{\Upsilon_2}(u)+1)+(d_{\Upsilon_2}(v)+1))^2\\ &\quad +\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}((d_{\Upsilon}(u)+n_1)+(d_{\Upsilon_1}(v)+1))^2\\ &\quad +\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}((n_2+2)+(d_{\Upsilon_2}(v)+1))^2\\ &= \sum f+\sum g+\sum h+\sum i+\sum j. \end{aligned}$$

For the computation of $\sum f$,

$$\begin{aligned} \sum f &= \sum_{uv\in E(\Upsilon^s)}((d_{\Upsilon}(u)+n_1)+(n_2+2))^2\\ &= \sum_{uv\in E(\Upsilon^s)}(d_{\Upsilon}(u)^2+2(n_1+n_2+2)d_{\Upsilon}(u)+(n_1+n_2+2)^2)\\ \Rightarrow\quad \sum f &= \sum_{uv\in E(\Upsilon^s)}(d_{\Upsilon}(u)^2+2(n_1+n_2+2)d_{\Upsilon}(u))+2m(n_1+n_2+2)^2. \end{aligned}$$

Also for the computation of $\sum g$,

$$\begin{aligned} \sum g &= n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+1)+(d_{\Upsilon_1}(v)+1))^2\\ &= n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v))^2+4(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v))+4)\\ \Rightarrow\quad \sum g &= n[HM_1(\Upsilon_1)+4M_1(\Upsilon_1)+4m_1]. \end{aligned}$$

Similarly, for $\sum h$,

$$\sum h = m \sum_{uv\in E(\Upsilon_2)} ((d_{\Upsilon_2}(u)+1)+(d_{\Upsilon_2}(v)+1))^2$$

$$= m \sum_{uv\in E(\Upsilon_2)} ((d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v))^2+4(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v))+4)$$

$$\Rightarrow \quad \sum h = n[HM_1(\Upsilon_2)+4M_1(\Upsilon_2)+4m_2].$$

In order to determine $\sum i$,

$$\sum i = \sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}} ((d_{\Upsilon}(u)+n_1)+(d_{\Upsilon_1}(v)+1))^2$$

$$= \sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}} d_{\Upsilon}(u)^2+d_{\Upsilon_1}(v)^2+2d_{\Upsilon}(u)d_{\Upsilon_1}(v)+2(n_1+1)d_{\Upsilon}(u)+2(n_1+1)d_{\Upsilon_1}(v)+(n_1+1)^2$$

$$\Rightarrow \quad \sum i = n_1M_1(\Upsilon)+nM_1(\Upsilon_1)+8mm_1+4mn_1(n_1+1)+4nm_1(n_1+1)+nn_1(n_1+1)^2$$

Similarly for $\sum j$,

$$\sum j = \sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}} ((n_2+2)+(d_{\Upsilon_2}(v)+1))^2$$

$$= \sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}} d_{\Upsilon_2}(v)^2+2(n_2+3)d_{\Upsilon_2}(v)+(n_2+3)^2$$

$$\Rightarrow \quad \sum j = mM_1(\Upsilon_2)+4mm_2(n_2+3)+mn_2(n_2+3)^2$$

From all the computations,

$$\begin{aligned}HM_1(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)) = {} & n[HM_1(\Upsilon_1)+5M_1(\Upsilon_1)]+m[HM_2(\Upsilon_2)+5M_1(\Upsilon_2)]+n_1M_1(\Upsilon)\\ & +2m[(n_1+n_2+2)^2+4m_1+2n_1(n_1+1)]+4(nm_1+mm_2)\\ & +n(n_1+1)[n_1(n_1+1)+4m_1]+m(n_2+3)[n_2(n_2+3)+4m_2]\\ & +\sum_{uv\in E(\Upsilon^s)} d_{\Upsilon}^2+2(n_1+n_2+2)d_{\Upsilon}.\end{aligned}$$

This concludes our result. □

Theorem 2.6. *Assume that $\Upsilon(n,m)$, $\Upsilon_1(n_1,m_1)$ and $\Upsilon_2(n_2,m_2)$ are arbitrary graphs. Then*

$$\begin{aligned}HM_2(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)) = {} & n[HM_2(\Upsilon_1)+HM_1(\Upsilon_1)+2ReZG_3(\Upsilon_1)+2(M_1(\Upsilon_1)+M_2(\Upsilon_1))+m_1]\\ & +m[HM_2(\Upsilon_2)+HM_1(\Upsilon_2)+2ReZG_3(\Upsilon_2)+2(M_1(\Upsilon_2)+M_2(\Upsilon_2))+m_2]\\ & +M_1(\Upsilon)[M_1(\Upsilon_1)+4m_1+n_1]+n_1M_1(\Upsilon_1)[n_1^2+4m]\\ & +m(n_2+2)^2[M_1(\Upsilon_2)+4m_2+n_2+2n_1^2]\\ & +n_1[n_1(nn_1+4(m+nm_1))+16mm_1].\end{aligned}$$

Proof.

$$HM_2(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)) = \sum_{uv\in E(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2))} (d_{\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)}(u)\cdot d_{\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)}(v))^2$$

$$= \sum_{uv\in E(\Upsilon^s)} ((d_{\Upsilon}(u)+n_1)\cdot(n_2+2))^2+n\sum_{uv\in E(\Upsilon_1)} ((d_{\Upsilon_1}(u)+1)\cdot(d_{\Upsilon_1}(v)+1))^2$$

$$+m\sum_{uv\in E(\Upsilon_2)}((d_{\Upsilon_2}(u)+1)\cdot(d_{\Upsilon_2}(v)+1))^2+\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}((d_{\Upsilon}(u)+n_1)\cdot(d_{\Upsilon_1}(v)+1))^2$$

$$+\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}((n_2+2)\cdot(d_{\Upsilon_2}(v)+1))^2$$

$$=\sum f+\sum g+\sum h+\sum i+\sum j.$$

For the computation of $\sum f$,

$$\sum f=\sum_{uv\in E(\Upsilon^s)}((d_{\Upsilon}(u)+n_1)\cdot(n_2+2))^2$$

$$=\sum_{uv\in E(\Upsilon^s)}(n_2+2)^2(d_{\Upsilon}(u)^2+2n_1d_{\Upsilon}(u)+n_1^2)$$

$$\Rightarrow\quad \sum f=\sum_{uv\in E(\Upsilon^s)}(n_2+2)^2(d_{\Upsilon}(u)^2+2n_1d_{\Upsilon}(u))+2mn_1^2(n_2+2)^2$$

Also for the computation of $\sum g$,

$$\sum g=n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+1)\cdot(d_{\Upsilon_1}(v)+1))^2$$

$$=n\sum_{uv\in E(\Upsilon_1)}(d_{\Upsilon_1}(u)\cdot d_{\Upsilon_1}(v)+(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v))+1)^2$$

$$\Rightarrow\quad \sum g=n[HM_2(\Upsilon_1)+HM_1(\Upsilon_1)+2ReZG_3(\Upsilon_1)+2(M_1(\Upsilon_1)+M_2(\Upsilon_1))+m_1]$$

Similarly, for $\sum h$,

$$\sum h=m\sum_{uv\in E(\Upsilon_2)}((d_{\Upsilon_2}(u)+1)\cdot(d_{\Upsilon_2}(v)+1))^2$$

$$=m\sum_{uv\in E(\Upsilon_2)}(d_{\Upsilon_2}(u)\cdot d_{\Upsilon_2}(v)+(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v))+1)^2$$

$$\Rightarrow\quad \sum h=m[HM_2(\Upsilon_2)+HM_1(\Upsilon_2)+2ReZG_3(\Upsilon_2)+2(M_1(\Upsilon_2)+M_2(\Upsilon_2))+m_2]$$

In order to determine $\sum i$,

$$\sum i=\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}((d_{\Upsilon}(u)+n_1)\cdot(d_{\Upsilon_1}(v)+1))^2$$

$$=\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}(d_{\Upsilon}(u)\cdot d_{\Upsilon_1}(v)+d_{\Upsilon}(u)+n_1d_{\Upsilon_1}(v)+n_1)^2$$

$$\Rightarrow\quad \sum i=M_1(\Upsilon)[M_1(\Upsilon_1)+4m_1+n_1]+n_1M_1(\Upsilon_1)[n_1^2+4m]$$

$$+n_1[n_1(nn_1+4(m+nm_1))+16mm_1]$$

Similarly for $\sum j$,

$$\sum j=\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}((n_2+2)\cdot(d_{\Upsilon_2}(v)+1))^2$$

$$=\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}((n_2+2)^2)(d_{\Upsilon_2}(v)^2+2d_{\Upsilon_2}(v)+1)$$

$$\Rightarrow\quad \sum j=m(n_2+2)^2[M_1(\Upsilon_2)+4m_2+n_2]$$

From all the computations,

$$\begin{aligned}HM_2(\Upsilon^s\circ(\Upsilon_1,\Upsilon_2)) &= n[HM_2(\Upsilon_1)+HM_1(\Upsilon_1)+2ReZG_3(\Upsilon_1)+2(M_1(\Upsilon_1)+M_2(\Upsilon_1))+m_1]\\ &\quad +m[HM_2(\Upsilon_2)+HM_1(\Upsilon_2)+2ReZG_3(\Upsilon_2)+2(M_1(\Upsilon_2)+M_2(\Upsilon_2))+m_2]\\ &\quad +M_1(\Upsilon)[M_1(\Upsilon_1)+4m_1+n_1]+n_1M_1(\Upsilon_1)[n_1^2+4m]\\ &\quad +m(n_2+2)^2[M_1(\Upsilon_2)+4m_2+n_2+2n_1^2]\\ &\quad +n_1[n_1(nn_1+4(m+nm_1))+16mm_1].\end{aligned}$$

This concludes our result. □

2.6 Subdivision Double Neighborhood Corona Product

The *subdivision double neighborhood corona product* for the graphs $\Upsilon(n,m)$, $\Upsilon_1(n_1,m_1)$ and $\Upsilon_2(n_2,m_2)$ indicated as $\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)$ is determined by taking a replica of Υ^s, n replicas of Υ_1 and m replicas of Υ_2 and connecting neighboring vertices of Υ in the ith location in Υ^s to every node in the ith replica of Υ_1 and also connecting the neighboring nodes of the jth new inserted node of Υ^s to every node in the jth replica of Υ_2 (Hameed *et al*. [15]).

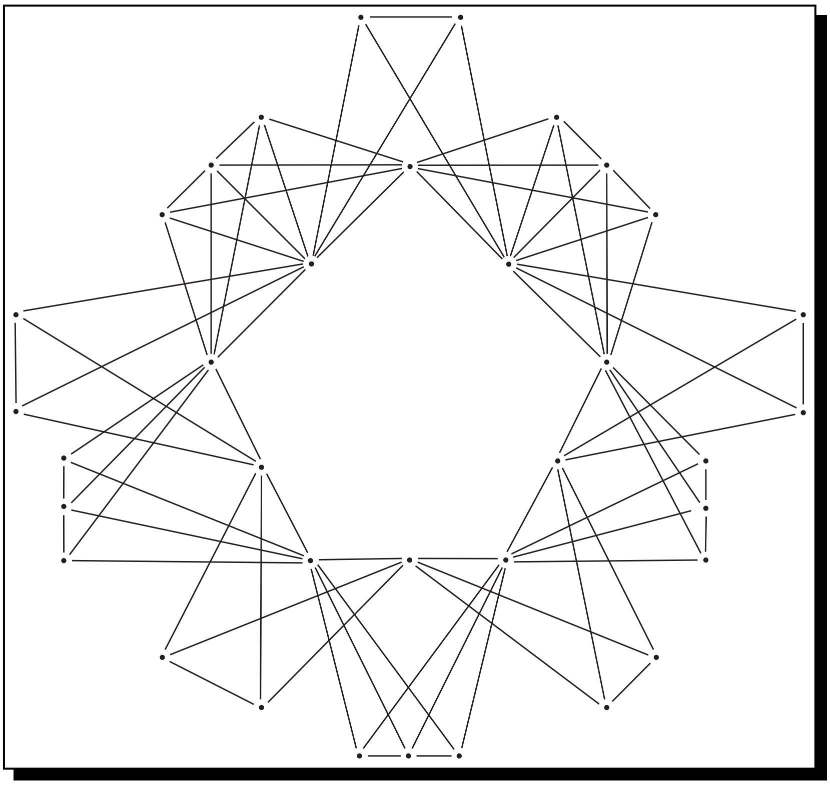

Figure 6. Subdivision double neighborhood corona product $C_5^s\bullet(P_2,P_3)$

Lemma 2.6. *The degree behaviour of the vertices is:*

$$d_{\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)}(v)=\begin{cases} d_\Upsilon(v)+2n_2, & \text{if } v\in V(\Upsilon),\\ d_{\Upsilon_1}(v)+2, & \text{if } v\in V(\Upsilon_1),\\ d_{\Upsilon_2}(v)+2, & \text{if } v\in V(\Upsilon_2),\\ 2(n_1+1), & \text{if } v\in V(\Upsilon^s).\end{cases}$$

Theorem 2.7. *Assume that $\Upsilon(n,m)$, $\Upsilon_1(n_1,m_1)$ and $\Upsilon_2(n_2,m_2)$ are arbitrary graphs. Then*

$$\begin{aligned}HM_1(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)) &= n[HM_1(\Upsilon_1)+8(M_1(\Upsilon_1]+2m_1)]+m[HM_1(\Upsilon_2)+8(M_1(\Upsilon_2]+2m_2)]\\ &\quad+2[n_2M_1(\Upsilon)+nM_1(\Upsilon_2)+mM_1(\Upsilon_1)]+8(n_2+1)[nn_2(n_2+1)+2m_2+2mn_2]\\ &\quad+8m[(n_1+n_2+1)^2+2m_2]+8m(n_1+2)[n_1(n_1+2)+2m_1]\\ &\quad+\sum_{uv\in E(\Upsilon^s)} d_\Upsilon(u)^2+4(n_1+n_2+1)d_\Upsilon(u).\end{aligned}$$

Proof.

$$\begin{aligned}HM_1(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)) &= \sum_{uv\in E(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2))}(d_{\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)}(u)+d_{\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)}(v))^2\\ &= \sum_{uv\in E(\Upsilon^s)}((d_\Upsilon(u)+2n_2)+(2n_1+2))^2+n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+2)+(d_{\Upsilon_1}(v)+2))^2\\ &\quad+m\sum_{uv\in E(\Upsilon_2)}((d_{\Upsilon_2}(u)+2)+(d_{\Upsilon_2}(v)+2))^2\\ &\quad+2\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}((d_\Upsilon(u)+2n_2)+(d_{\Upsilon_2}(v)+2))^2\\ &\quad+2\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}((2n_1+2)+(d_{\Upsilon_1}(v)+2))^2\\ &= \sum f+\sum g+\sum h+\sum i+\sum j.\end{aligned}$$

For the computation of $\sum f$,

$$\begin{aligned}\sum f &= \sum_{uv\in E(\Upsilon^s)}((d_\Upsilon(u)+2n_2)+(2n_1+2))^2\\ &= \sum_{uv\in E(\Upsilon^s)}(d_\Upsilon(u)^2+4(n_1+n_2+1)^2+4(n_1+n_2+1)d_\Upsilon(u))\end{aligned}$$

$$\Rightarrow \quad \sum f = \sum_{uv\in E(\Upsilon^s)}(d_\Upsilon(u)^2+4(n_1+n_2+1)d_\Upsilon(u))++8m(n_1+n_2+1)^2$$

Also for the computation of $\sum g$,

$$\begin{aligned}\sum g &= n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+2)+(d_{\Upsilon_1}(v)+2))^2\\ &= n\sum_{uv\in E(\Upsilon_1)}((d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v)+4))^2\end{aligned}$$

$$\Rightarrow \quad \sum g = n[HM_1(\Upsilon_1)+8(M_1(\Upsilon_1]+2m_1)].$$

Similarly, for $\sum h$,

$$\begin{aligned}\sum h &= m\sum_{uv\in E(\Upsilon_2)}((d_{\Upsilon_2}(u)+2)+(d_{\Upsilon_2}(v)+2))^2\\ &= m\sum_{uv\in E(\Upsilon_2)}(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)+4)^2\end{aligned}$$

$$\Rightarrow \quad \sum h = m[HM_1(\Upsilon_2)+8(M_1(\Upsilon_2]+16m_2)]$$

In order to determine $\sum i$,

$$\sum i = 2\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}} ((d_\Upsilon(u)+2n_2)+(d_{\Upsilon_2}(v)+2))^2$$

$$= 2\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}} d_\Upsilon(u)^2 + d_{\Upsilon_2}(v)^2 + 4(n_2+1)^2 + 2d_\Upsilon(u)d_{\Upsilon_2}(v) + 4(n_2+1)d_{\Upsilon_2}(v) + 4(n_2+1)d_\Upsilon(u)$$

$$\Rightarrow \quad \sum i = 2[n_2M_1(\Upsilon)+nM_1(\Upsilon_2)+4(n_2+1)(nn_2(n_2+1)+2nm_2+2mn_2)+8mm_2]$$

Similarly for $\sum j$,

$$\sum j = 2\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}} ((2n_1+2)+(d_{\Upsilon_1}(v)+2))^2$$

$$= 2\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}} (d_{\Upsilon_1}(v)^2 + 2(4+2n_1)d_{\Upsilon_1}(v) + (4+2n_1)^2)$$

$$\Rightarrow \quad \sum j = 2[mM_1(\Upsilon_1)+m(4+2n_1)(n_1(4+2n_1)+4m_1)]$$

From all the computations,

$$\begin{aligned} HM_1(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)) &= n[HM_1(\Upsilon_1)+8(M_1(\Upsilon_1]+2m_1)]+m[HM_1(\Upsilon_2)+8(M_1(\Upsilon_2]+2m_2)] \\ &\quad + 2[n_2M_1(\Upsilon)+nM_1(\Upsilon_2)+mM_1(\Upsilon_1)] \\ &\quad + 8(n_2+1)[nn_2(n_2+1)+2m_2+2mn_2] \\ &\quad + 8m[(n_1+n_2+1)^2+2m_2]+8m(n_1+2)[n_1(n_1+2)+2m_1] \\ &\quad + \sum_{uv\in E(\Upsilon^s)} d_\Upsilon(u)^2+4(n_1+n_2+1)d_\Upsilon(u). \end{aligned}$$

This concludes our result. □

Theorem 2.8. *Assume that $\Upsilon(n,m)$, $\Upsilon_1(n_1,m_1)$ and $\Upsilon_2(n_2,m_2)$ are arbitrary graphs. Then*

$$\begin{aligned} HM_2(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)) &= 4(n_1+1)^2\sum_{uv\in E(\Upsilon^s)} d_\Upsilon(u)^2+4n_2d_\Upsilon(u) \\ &\quad + n[HM_2(\Upsilon_1)+4HM_1(\Upsilon_1)+8M_2(\Upsilon_1)+16M_1(\Upsilon_1)+4ReZG_3(\Upsilon_1)+16m_1] \\ &\quad + m[HM_2(\Upsilon_2)+4HM_1(\Upsilon_2)+8M_2(\Upsilon_2)+16M_1(\Upsilon_2)+4ReZG_3(\Upsilon_2)+16m_2] \\ &\quad + 2[M_1(\Upsilon)M_1(\Upsilon_2)+4n_2(M_1(\Upsilon)+nn_2M_1(\Upsilon_2))+8(m_2M_1(\Upsilon)+mn_2M_1(\Upsilon_2))] \\ &\quad + 32n_2(n_2+2m_2)(nn_2+2m)+8(n_1+1)^2[mM_1(\Upsilon_1)+2m(m_1+2n_1)+4mn_2^2]. \end{aligned}$$

Proof.

$$\begin{aligned} HM_2(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)) &= \sum_{uv\in E(\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2))} (d_{\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)}(u)\cdot d_{\Upsilon^s\bullet(\Upsilon_1,\Upsilon_2)}(v))^2 \\ &= \sum_{uv\in E(\Upsilon^s)} ((d_\Upsilon(u)+2n_2)\cdot(2n_1+2))^2 + n\sum_{uv\in E(\Upsilon_1)} ((d_{\Upsilon_1}(u)+2)\cdot(d_{\Upsilon_1}(v)+2))^2 \\ &\quad + m\sum_{uv\in E(\Upsilon_2)} ((d_{\Upsilon_2}(u)+2)\cdot(d_{\Upsilon_2}(v)+2))^2 + 2\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}} ((d_\Upsilon(u)+2n_2)\cdot(d_{\Upsilon_2}(v)+2))^2 \end{aligned}$$

$$+2\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}((2n_1+2)\cdot(d_{\Upsilon_1}(v)+2))^2$$

$$=\sum f+\sum g+\sum h+\sum i+\sum j.$$

For the computation of $\sum f$,

$$\sum f=\sum_{uv\in E(\Upsilon^s)}((d_\Upsilon(u)+2n_2)\cdot(2n_1+2))^2$$

$$=4(n_1+1)^2\sum_{uv\in E(\Upsilon^s)}(d_\Upsilon(u)^2+4n_2d_\Upsilon(u)+4n_2^2)$$

$$\Rightarrow \quad \sum f=4(n_1+1)^2\sum_{uv\in E(\Upsilon^s)}(d_\Upsilon(u)^2+4n_2d_\Upsilon(u))+8mn_2^2$$

Also for the computation of $\sum g$,

$$\sum g=n\sum_{uv\in E(\Upsilon_1)}[(d_{\Upsilon_1}(u)+2)\cdot(d_{\Upsilon_1}(v)+2)]^2$$

$$=n\sum_{uv\in E(\Upsilon_1)}[(d_{\Upsilon_1}(u)\cdot d_{\Upsilon_1}(v))+2(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v))+4)]^2$$

$$\Rightarrow \quad \sum g=n[HM_2(\Upsilon_1)+4HM_1(\Upsilon_1)+8M_2(\Upsilon_1)+16M_1(\Upsilon_1)+4ReZG_3(\Upsilon_1)+16m_1]$$

Similarly, for $\sum h$,

$$\sum h=m\sum_{uv\in E(\Upsilon_2)}((d_{\Upsilon_2}(u)+2)\cdot(d_{\Upsilon_2}(v)+2))^2$$

$$=m\sum_{uv\in E(\Upsilon_2)}[(d_{\Upsilon_2}(u)\cdot d_{\Upsilon_2}(v))+2(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)+4)]^2$$

$$\Rightarrow \quad \sum h=m[HM_2(\Upsilon_2)+4HM_1(\Upsilon_2)+8M_2(\Upsilon_2)+16M_1(\Upsilon_2)+4ReZG_3(\Upsilon_2)+16m_2]$$

In order to determine $\sum i$,

$$\sum i=2\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}((d_\Upsilon(u)+2n_2)\cdot(d_{\Upsilon_2}(v)+2))^2$$

$$=2\sum_{\substack{u(o)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}[(d_\Upsilon(u)\cdot d_{\Upsilon_2}(v))^2+4d_\Upsilon(u)^2+4n_2^2d_{\Upsilon_2}(v)^2+16n_2^2+4d_\Upsilon(u)^2d_{\Upsilon_2}(v)$$

$$+4n_2d_\Upsilon(u)d_{\Upsilon_2}(v)^2+16n_2d_\Upsilon(u)d_{\Upsilon_2}(v)+16n_2d_\Upsilon(u)+16n_2^2d_{\Upsilon_2}(v)]$$

$$\Rightarrow \quad \sum i=2[M_1(\Upsilon)M_1(\Upsilon_2)+4n_2(M_1(\Upsilon)+nn_2M_1(\Upsilon_2)\big)+8m_2M_1(\Upsilon)+8mn_2M_1(\Upsilon_2)$$

$$+64mm_2n_2+32mn_2^2+32nm_2n_2^2+16nn_2^3]$$

Similarly for $\sum j$,

$$\sum j=2\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_1)}}((2n_1+2)\cdot(d_{\Upsilon_1}(v)+2))^2$$

$$=8(n_1+1)^2\sum_{\substack{u(n)\in V(\Upsilon^s)\\ v\in V(\Upsilon_2)}}(d_{\Upsilon_1}(v)^2+4d_{\Upsilon_1}(v)+4)$$

$$\Rightarrow \quad \sum j=8(n_1+1)^2[mM_1(\Upsilon_1)+2m(m_1+2n_1)]$$

From all the computations,

$$\begin{aligned}HM_2(\Upsilon^s \bullet (\Upsilon_1,\Upsilon_2)) = 4(n_1+1)^2 \sum_{uv\in E(\Upsilon^s)} d_\Upsilon(u)^2 + 4n_2 d_\Upsilon(u) \\ + n[HM_2(\Upsilon_1) + 4HM_1(\Upsilon_1) + 8M_2(\Upsilon_1) + 16M_1(\Upsilon_1) + 4ReZG_3(\Upsilon_1) + 16m_1] \\ + m[HM_2(\Upsilon_2) + 4HM_1(\Upsilon_2) + 8M_2(\Upsilon_2) + 16M_1(\Upsilon_2) + 4ReZG_3(\Upsilon_2) + 16m_2] \\ + 2[M_1(\Upsilon)M_1(\Upsilon_2) + 4n_2(M_1(\Upsilon) + nn_2M_1(\Upsilon_2)) + 8(m_2M_1(\Upsilon) + mn_2M_1(\Upsilon_2))] \\ + 32n_2(n_2+2m_2)(nn_2+2m) + 8(n_1+1)^2[mM_1(\Upsilon_1) + 2m(m_1+2n_1) + 4mn_2^2].\end{aligned}$$

This concludes our result. □

3. Conclusion

For the advancements and progressions in chemical graph theory, exploration of every molecular attribute is essential and the process can be facilitated by the research results in the domain of topological indices.

Accordingly, this paper monitors and analyses the proposed graph operational products notably the subdivision vertex, subdivision edge, subdivision vertex neighborhood and subdivision edge neighborhood corona product of two graphs by exploring the exact expressions of the second hyper Zagreb index involving the subdivision graph. Further, the formulations of the first and second hyper Zagreb indices with respect to the subdivision double and subdivision double neighborhood corona products of the graphs. The procured outcomes contributes towards further research for the exploration in the realm of degree, distance dependent descriptors and series of graph operational variants.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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