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Research Article

Some Aspects of Theory of Schrödinger Operators on Riemannian Manifold

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Abstract. This paper deals with a given Riemannian manifold $\mathcal{M}$. One of the main tasks is description of spectrum of several classes of Schrödinger operator $P = \frac{-h^2}{2}\Delta_g + V$ where Δ_g is Laplace Beltrami operator and V is potential on manifold. We illustrate the inverse and direct problems of Δ_g and the way to discover the geometry of Riemannian manifold from spectral data.

Keywords. Manifolds, Spectral theory, Laplacian, Spectral geometry

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1. Introduction

Let $\mathcal{M}$ be a connected manifold of n-dimension. We endow $\mathcal{M}$ with a Riemannian metric $g = (g_{jk})$, the pair $(\mathcal{M}, g)$ is called *Riemannian manifold*. In this paper, we present some important theorems in Spectral theory of Schrödinger operator $P = \frac{-h^2}{2}\Delta_g + V$. P is a linear unbounded operator on the set of smooth compact supports real valued functions $C_c^\infty(\mathcal{M}) \subset L^2(\mathcal{M})$, where h denotes the Planck constant, let $h = \sqrt{2}$ and V is a potential function on $\mathcal{M}$. In Riemannian geometry, we use Laplace Beltrami operator Δ_g which is the generalization of

Laplace operator $\Delta_{\mathbb{R}^n}$:

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(\sqrt{g} g^{jk} \frac{\partial f}{\partial x^k}\right), \tag{1.1}$$

where $(x^1, x^2, \dots, x^n)$ are local coordinates, $g = \det(g_{jk})$, g^{jk} is the inverse matrix to (g_{jk}) and $f : \mathcal{M} \to \mathbb{R}$ a smooth function.

The main object of our paper is the Schrödinger operator

$$P = -\Delta_g + V. \tag{1.2}$$

We describe the analytic and geometric aspects of spectrum of several classes of Schrödinger operator. Let us give here some classes of (1.2) by giving different potentials:

Free motion potential $V = 0$ the operator is just the Laplace-Beltram operator which is used in Riemannian geometry.

The hydrogen atom potential $V = \frac{-k}{\|x\|}$ $(k > 0)$.

The harmonic oscillator $V = \frac{\|x\|^2}{2}$.

We define the discrete spectrum of self adjoint operator A as $\sigma_{\text{disc}}(A)$ the set of eigenvalues (λ_k) of A having finite multiplicity and being isolated points of the spectrum. The essential spectrum of A is the set $\sigma_{\text{ess}}(A) = \sigma(A) \setminus \sigma_{\text{disc}}(A)$. The spectrum behaviour on Riemannian manifold $(\mathcal{M}, g)$ under topological perturbation has been the subject of a vast literature over the last two decades. Kac [16], Rauch and Taylor [22] were the first who studied the spectrum of $\Delta_{\mathbb{R}^n}$; they showed that the spectrum of $\Delta_{\mathbb{R}^n}$ in a compact set $\mathcal{M}$ of $\mathbb{R}^n$ is invariant under topological excision of a compact subset with a Newtonian capacity zero. Chavel and Feldman [5,6] dealt with Riemannian manifold case. In [13], Gesztesy and Zhao investigated the case of a Schrödinger operator in $\mathbb{R}^n$ with Dirichlet boundary condition. Lablée ([1]) focused on Eigenvalues for a Schrödinger operator on a closed Riemannian manifold with holes.

This paper is organized as following: In Section 2 we start the paper from the spectral theory background for Multidimensional Schrödinger operator $P = -\Delta_{\mathbb{R}^n} + V$. In Section 3 we give the concept of Sobolev space on Riemannian manifold and describe in details the spectrum properties of Schrödinger operator on Riemannian manifold. The way to discover the geometry of Riemannian manifold from spectral data is detailed in Section 4. We refer to [4], [18] for more details of self-adjointness of Schrödinger operator. The references [5, 6, 10, 13, 16, 22] covers many results of spectral theory on manifolds. In this paper, we will assume that $(\mathcal{M}, g)$ is a manifold of bounded geometry. We mention that a closed manifold means compact manifold without boundary.

2. Spectral Problem Background for Schrödinger Operator

Definition 2.1. In order to state the results we have to define some spaces:

$$C(U) = \{u : U \to \mathbb{R} \mid u \text{ is continuous}\},$$

[1] O. Lablée, Eigenvalues for a Schrödinger operator on a closed Riemannian manifold with holes, *arXiv preprint*, arXiv:1301.6909 (2013), URL: https://arxiv.org/pdf/1301.6909.pdf.

$C^k(U) = \{u : U \to \mathbb{R} \mid u \text{ is } k\text{-times continuously differentiable}\}$,

$C^\infty(U) = \{u : U \to \mathbb{R} \mid u \text{ is indefinitely differentiable}\}$,

$C_c^k(U) = \{f \in C^k(U) \text{ with compact support}\}$.

We said that V is compactly contained in U if there is a compact set K such that $V \subset K \subset U$, we denoted it by $V \Subset U$.

$L^p_{\text{loc}}(U) = \{u : U \to \mathbb{R} \mid u \in L^p(V) \text{ for } V \Subset U\}$.

We define $-\Delta_{\mathbb{R}^n}$ as a positive self adjoint operator on the Sobolev space

$H^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \le 2\}$.

For multidimensional Schrödinger operator, consider $P = -\Delta_{\mathbb{R}^n} + V$ as a perturbation of $-\Delta_{\mathbb{R}^n}$ by the potential V with domain $D(P) = H^2(\mathbb{R}^n) \cap \{f \in L^2(\mathbb{R}^n) : fV \in L^2(\mathbb{R}^n)\}$ such that V is locally bounded on $\mathbb{R}^n$, real valued function. We will discuss two classes:

(i) $V(x) \to \infty$ as $x \to \infty$,

(ii) $V(x) \to 0$ as $x \to \infty$.

For the case $\lim\limits_{|x|\to\infty} V(x) = \infty$ the most important example is the harmonic oscillator for which $V(x) = |x|^{2k}$, $k \in \mathbb{N}^*$. We also give some important spectral results for Schrödinger operator in the case of $\lim\limits_{|x|\to 0} V(x) = \infty$, we focus in particular on the Coulomb potential.

Lemma 2.1. *If $V \ge 0$ is a non-negative potential and W is multiplication by a bounded function of compact support, then W is P-compact, i.e., $W(I+P)^{-1}$ is compact operator on $L^2(\mathbb{R}^n)$.*

Proof. We have $-\Delta_{\mathbb{R}^n}$ is positive self adjoint operator on $H^2(\mathbb{R}^n)$ thus $I+P$ is positive operator, then $(I+P)^{-1}$ is bounded on $L^2(\mathbb{R}^n)$.

Furthermore, $W(I+P)^{\frac{-1}{2}} = W(I+P_0^{\frac{1}{2}})^{-1}(I+P_0^{\frac{1}{2}})(I+P)^{\frac{-1}{2}}$, $P_0 = -\Delta_{\mathbb{R}^n}$. We know that the product of two factors one is compact and second is bounded. So, $W(I+P)^{\frac{-1}{2}}$ is compact. Now, if we multiply by the bounded operator $(I+P)^{\frac{-1}{2}}$ on the right, we conclude that: $W(I+P)^{\frac{-1}{2}}$ is compact. □

Theorem 2.1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) \to \infty$ as $|x| \to \infty$, then $\sigma_{\text{ess}}(-\Delta_{\mathbb{R}^n} + V)$ is empty.*

Proof. For any $E \in \mathbb{R}$, write $V - E = f - g$ where $f \ge 0$ and g has compact support. By Lemma 2.1, g is $(-\Delta_{\mathbb{R}^n} + f)$-compact, so by virtue of perturbation theorem of Weyl, we have $\sigma_{\text{ess}}(-\Delta_{\mathbb{R}^n} + f) = \sigma_{\text{ess}}(P - E)$. Since $f \ge 0$, we know that $\sigma_{\text{ess}}(P - E) \subset [0,\infty[$ and then $\sigma_{\text{ess}}(P) \subset [E,\infty[$. Since this is true for all E, we get that $\sigma_{\text{ess}}(P)$ is empty. □

Theorem 2.2. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be continuous with $V(x) \to \infty$ as $|x| \to \infty$ then the spectrum $\sigma(P)$ of P is an increasing sequence $(\lambda_k) \subset \mathbb{R}$ of eigenvalues of finite multiplicity and $\lambda_k \to \infty$ for $k \to \infty$, i.e. $\sigma(P) = \sigma_{\text{disc}}(P)$ and $\sigma_{\text{ess}}(P) = \phi$. The associated eigenfunctions form an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^n)$.*

Proof. $(P+I):D(P)\to L^2(\mathbb{R}^n)$ is bijective with compact inverse $(P+I)^{-1}$. We conclude that $\sigma(P)$ is either empty or $\sigma_{\text{disc}}(P)$ contains countably many eigenvalues λ_k. The results in this theorem is due to [12]. If $\sigma(P)$ is infinite, then λ_k is an increasing sequence of eigenvalues of finite multiplicity and $|\lambda_k|\to\infty$ as $k\to\infty$. For all $\lambda_k\in\sigma(P)$ the range $R(\lambda_k-P)$ of λ_k-P is closed, $\dim N(\lambda_k-P)=\operatorname{codim} R(\lambda_k-P)$, the eigenfunctions of $(P+I)^{-1}$ form an orthonormal basis of $L^2(\mathbb{R}^n)$, where $N(\lambda_k-P)$ denotes the kernel of (λ_k-P). Clearly, $\sigma_{\text{ess}}(P)=\phi$. □

For the case $V(x)\to 0$ as $|x|\to\infty$, we show that under certain conditions, the essential spectrum of $P=-\Delta_{\mathbb{R}^n}+V:D(P)\to L^2(\mathbb{R}^n)$ is in fact exactly the set of non-negative real numbers. For example Schrödinger operators with Coulomb potentials $V(x)=\frac{\gamma}{|x|}$, $\gamma>0$.

Theorem 2.3. *Let $V:\mathbb{R}^n\to\mathbb{R}$ is locally bounded and $V(x)\to 0$ as $|x|\to\infty$. Then $\sigma_{\text{ess}}(P)=[0,\infty[$.*

Proof. Theorem 2.2 allows to show that P can have only isolated eigenvalues of finite multiplicity on $]-\infty,0[$. It remains for us to show that $[0,\infty[\,\subset\sigma(P)$. Let $\lambda\geq 0$ be fixed. We shall construct a Weyl sequence $(\varphi_k)\subset H^2(\mathbb{R}^n)$ of P and λ. We have $-\Delta_{\mathbb{R}^n}e^{i\zeta x}=\lambda e^{i\zeta x}$, where $\zeta\in\mathbb{R}^n$, $|\zeta|=\sqrt{\lambda}$. Moreover, let$\psi\in C_c^\infty(\mathbb{R}^n)$ such that $\psi\geq 0$, $\psi(x)=1$ for $|x|\leq\frac{1}{2}$ and $\psi(x)=0$ for $|x|\geq 2$.

Let $\psi_k(x)=\psi\left(\frac{x-k}{\sqrt{k}}\right)$, $k\in\mathbb{N}^*$, then:

$$\operatorname{supp}\psi_k\subset\{x\in\mathbb{N},|x-k|\leq\sqrt{k}\}\text{ and }\lim_{k\to\infty}\sup_{x\in\operatorname{supp}\psi_k}V(x)=0.$$

Suppose $\varphi_k=\psi_k e^{i\zeta x}$, $k\in\mathbb{N}^*$

$$\|\varphi_k\|^2=\int_{\mathbb{R}^n}|\psi_k(x)|^2dx=k^{\frac{n}{2}}\|\psi\|^2. \tag{2.1}$$

Then $P\varphi_k=(-\Delta_{\mathbb{R}^n}\psi_k)e^{i\zeta x}-(\nabla\psi_k)(-\nabla e^{i\zeta x})+|\zeta|^2\psi_k(x)e^{i\zeta x}+V(x)\psi_k(x)e^{i\zeta x}$, and

$$(P-\lambda)\varphi_k=e^{i\zeta x}(P\psi_k-i\zeta\nabla\psi_k). \tag{2.2}$$

We have $|\nabla\psi_k|\leq\frac{\|\psi\|^2}{\sqrt{k}}$, $|\Delta_{\mathbb{R}^n}\psi_k|\leq\frac{\|\psi\|^2}{k}$, $k\in\mathbb{N}^*$. We conclude from $\lim\limits_{k\to\infty}\sup\limits_{x\in\mathbb{R}^n}|(P-\lambda)\varphi_k|=0$, we get $\lim\limits_{k\to\infty}\frac{\|(P-\lambda)\varphi_k\|^2}{k^{\frac{n}{2}}}=0$ and then we deduce $\lim\limits_{k\to\infty}\frac{\|(P-\lambda)\varphi_k\|^2}{\|\varphi_k\|}=0$ implies that $\sigma_{\text{ess}}(P)=[0,\infty[$. □

Theorem 2.4. *Let $V:\mathbb{R}^n\to\mathbb{R}$ be piece-wise continuous with $V(x)\to 0$ as $x\to\infty$. Assume that the multiplication operator by V is relatively bounded with respect to $-\Delta_{\mathbb{R}^n}$ with relative bound <1. Then $P=-\Delta_{\mathbb{R}^n}+V:D(P)=H^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is self-adjoint and $\sigma_{\text{ess}}(-\Delta_{\mathbb{R}^n})=\sigma_{\text{ess}}(P)=[0,\infty[$.*

Example 2.1. We focus in particular on the Coulomb potential $V(x)=\frac{-1}{|x|}$, $x\in\mathbb{R}^3\backslash\{0\}$. $P=-\Delta_{\mathbb{R}^3}-\frac{1}{|x|}$ in $L^2(\mathbb{R}^3)$ is the Schrödinger operator of the hydrogen atom. Hardy's inequality implies that the Coulomb potential in $\mathbb{R}^3$ is relatively bounded with respect to $-\Delta_{\mathbb{R}^3}$ with relative bound 0 and the perturbation theorem of Kato and Rellich shows that P is self-adjoint on $H^2(\mathbb{R}^3)$, we have also $\sigma_{\text{ess}}(P)=\sigma_{\text{ess}}(-\Delta_{\mathbb{R}^3})=[0,\infty[$.

For all $\psi \in H^2(\mathbb{R}^3)$ we first prove that $V\psi \in L^2(\mathbb{R}^3)$. Let $\phi = F^{-1}\psi \in L^2(\mathbb{R}^3)$, so $\psi = F\ \phi$ where F and F^{-1} are the Fourier transform and inverse Fourier transform, respectively on $L^2(\mathbb{R}^3)$. As the functions of $H^2(\mathbb{R}^3)$ are continuous and tend to zero at infinity, we deduce that ψ is essentially-bounded on $\mathbb{R}^3$:

$$\begin{aligned}\|\psi\|_\infty &= \operatorname*{esssup}_{x\in\mathbb{R}^3} |\psi(x)| \\ &= \sup_{x\in\mathbb{R}^3} \left|(2\pi)^{\frac{-3}{2}} \int_{\mathbb{R}^3} e^{-ix\xi}\phi(\zeta)d\zeta\right| \\ &\le (2\pi)^{\frac{-3}{2}} \int_{|\xi|\le 1} |\phi(\zeta)|d\zeta + (2\pi)^{\frac{-3}{2}} \int_{|\xi|>1} |\xi|^{-2}(|\xi|^2|\phi(\zeta)|).\end{aligned}$$

Hence, by applying the Cauchy-Schwarz inequality we obtain:

$$\|\psi\|_{L^\infty(\mathbb{R}^3)} \le c_1\|\psi\|_{L^2(\mathbb{R}^3)} + c_2\||\xi|^2|\phi|\|_{L^2(\mathbb{R}^3)},$$

where $c_1 = (2\pi)^{\frac{-3}{2}}\left(\frac{4\pi}{3}\right)^{\frac{1}{2}}$, $c_2 = (2\pi)^{\frac{-3}{2}}\left(\int_{|\xi|>1}\frac{d\zeta}{|\xi|^4}\right)^{\frac{1}{2}}$, $\psi \in H^2(\mathbb{R}^3)$ then $\Delta_{\mathbb{R}^3}\psi \in L^2(\mathbb{R}^3)$, and $|\zeta|^2\phi = F^{-1}(-\Delta_{\mathbb{R}^3}\psi) \in L^2(\mathbb{R}^3)$.

We know that Fourier transform is unitary operator on $L^2(\mathbb{R}^3)$, we obtain

$$\|\psi\|_{L^\infty(\mathbb{R}^3)} \le c_1\|\psi\|_{L^2(\mathbb{R}^3)} + c_1\|\Delta_{\mathbb{R}^3}\psi\|_{L^2(\mathbb{R}^3)} \tag{2.3}$$

for all $\varepsilon > 0$

$$\|V\psi\|_{L^2(\mathbb{R}^3)} = \int_{r=|x|\le\varepsilon} r^{-2}|\psi(x)|^2dx + \int_{r>\varepsilon} r^{-2}|\psi(x)|^2dx$$

or

$$\|V\psi\|_{L^2(\mathbb{R}^3)} \le \|\psi\|_{L^\infty(\mathbb{R}^3)}\sqrt{\int_{r\le\varepsilon} r^{-2}dx} + \varepsilon^{-1}\|\psi\|_{L^2(\mathbb{R}^3)},$$

we have by using (2.3)

$$\|V\psi\|_{L^2(\mathbb{R}^3)} \le a\|\Delta_{\mathbb{R}^3}\psi\|_{L^2(\mathbb{R}^3)} + b\|\psi\|_{L^2(\mathbb{R}^3)}, \tag{2.4}$$

where $a = c_2\sqrt{\int_{r\le\varepsilon} r^{-2}dx}$ and $b = \varepsilon^{-1} + c_1\sqrt{\int_{r\le\varepsilon} r^{-2}dx}$. Thus, V is Δ-bounded with relative bound a, make a small enough by choosing $\varepsilon \to 0$. For the Coulomb potential in $\mathbb{R}^3$ one obtains an infinite sequence of negative eigenvalues. As $V(x) = -\frac{1}{x}$ is spherically symmetric, $V(x) = V(r)$, with $|x| = r$, the main idea is to separate $P = -\Delta_{\mathbb{R}^3} - \frac{1}{|x|}$ in spherical coordinates to obtain the negative eigenvalues and the associated eigenfunctions. For any $x \in \mathbb{R}^3\backslash\{0\}$, $x = r\omega$ where $\omega = \frac{x}{|x|} \in S^2$ the unit sphere of $\mathbb{R}^3$.

The operator $-\Delta_{\mathbb{R}^3}$ on S^2, has compact resolvent and purely discrete spectrum $0 = v_0 < v_1 < \ldots < v_k \to \infty$ as $k \to \infty$, the associated eigenspaces to v_k have a basis of C^∞-functions $\psi_{k,l} : S^2 \to \mathbb{R}$, $l = 1,\ldots,m_k$ called the spherical harmonics, where $m_k = k(k+1)$, $k \in \mathbb{N}^*$, is the dimension of the eigenspace.

Using separation of variables $u(x) = f(r)\psi_{k,l}$ for the eigenfunctions u and eigenvalues λ of $P = -\Delta_{\mathbb{R}^3} - \frac{1}{|x|}$ in the Hilbert space $L^2(\mathbb{R}^3) = L^2(]0,\infty[, r^2dr) \oplus L^2(S^2, d\sigma_2)$, leads to the Bessel differential equation for f:

$$-''f(r) - \frac{2}{r}\acute{f}(r) - \frac{1}{|x|}f(r) + \frac{v_k}{r^2}f(r) = \lambda f(r), \quad r \in \,]0,\infty[\,.$$

We remark that one can show by power-series methods that for any $k \in \mathbb{N}^*$, there exists a solution $f = f_k \in L^2(]0,\infty[, r^2 dr)$ and an infinite sequence λ_k of negative eigenvalues of P, $\lambda_k = \frac{-1}{4k^2}$, $k \in \mathbb{N}^*$ (see Landau and Lifshitz [19]). So, we see that the hydrogen atom has an infinite number of bound states below the essential spectrum $\sigma_{\text{ess}}(P) \subset [0,\infty[$, which accumulate at zero.

3. Spectrum of Schrödinger Operator on Riemannian Manifold

Let $(\mathcal{M}, g)$ be a smooth, connected compact Riemannian manifold with boundary $\partial\mathcal{M}$. For a function $f \in C^2(\mathcal{M})$, we define the Laplace Beltrami operator by $\Delta_g f : -\text{divgrad}\, f$.

In local coordinates $\{x^i\}$, the Laplace Beltrami reads

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jk}\frac{\partial f}{\partial x^k}\right).$$

We present a class of eigenvalue problems as follows:

Closed problem	$\Delta_g f = \lambda f$ in $\mathcal{M}$	$\partial\mathcal{M} = \phi$
Dirchlet problem	$\Delta_g f = \lambda f$ in $\mathcal{M}$	$f\|_{\partial\mathcal{M}} = 0$
Neumann problem	$\Delta_g f = \lambda f$ in $\mathcal{M}$	$\frac{\partial f}{\partial N}\|_{\partial\mathcal{M}} = 0$

where N is outward oriented unit vector field normal to boundary. Let us motivate the paper by introduce the definition of Sobolev space on Riemannian manifold $(\mathcal{M}, g)$.

Definition 3.1. The space of all smooth functions $u \in C^\infty(\mathcal{M})$ such that $|\nabla^k u| \in L^p(\mathcal{M})$ is denoted by $C_k^p(\mathcal{M})$

$$C_k^p(\mathcal{M}) = \left\{u \in C^\infty(\mathcal{M}) \;\middle|\; \int_{\mathcal{M}} |\nabla^k u|^p dV_g < \infty\right\}$$

$dV_g = \sqrt{\det(g_{ij})}dx$, dx is the Lebesgue's volume element of $\mathbb{R}^n$.

The completion of $C_k^p(\mathcal{M})$ with respect to the norm

$$\|u\| = \left(\sum_{i=0}^{k} \int_{\mathcal{M}} |\nabla^k u|^p dV_g\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

is called Sobolev space $H^{k,p}(\mathcal{M})$ on Riemannian manifold. Note that we use covariant derivative as the case when use weak derivative for Sobolev space over $\mathbb{R}^n$. In particular, in the case of $P = 2$ we denote to Sobolev space by $H^k(\mathcal{M})$.

Theorem 3.1. *$H^{k,p}(\mathcal{M})$ is reflexive Banach space* $(1 < p < \infty)$.

Proof. The closed subspace of a reflexive Banach space is also reflexive.$H^{k,p}(\mathcal{M})$ is closed subspace of $L^p(\mathcal{M}) \times L^p(\mathcal{M})$ and since $L^p(\mathcal{M})$ is a reflexive Banach space for $(1 < p < \infty)$, thus the finite Cartesian product space is reflexive space. Hence we get $H^{k,p}(\mathcal{M})$ is reflexive space. □

Corollary 3.1. *$H^k(\mathcal{M})$ is Hilbert space with the norm*

$$\|f\|_{H^k} = \sqrt{\sum_{i=0}^{k}\left(\int_{\mathcal{M}} |\nabla^k f|^2 dV_g\right)}.$$

Proof. We can easily check the following conditions:

(i) $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$,

(ii) $\langle \alpha u, w\rangle = \alpha\langle u,w\rangle$,

(iii) $\langle u,w\rangle = \langle w,u\rangle$,

(iv) $\langle u,u\rangle = 0$ if $u=0$,

for $u \in H^k$, if $\langle u,u\rangle = 0$ then,

$$\langle u,u\rangle = \sum_{m=0}^{k}\int_{\mathcal{M}} (g^{i_1 j}\dots g^{i_m j_m}(\nabla^m u)_{i_1\dots i_m}(\nabla^m u)_{j_1\dots j_m})dV_g = 0$$

for $m=0$, we get $\int_M (U)^2 dV_g = 0$. Hence, $u=0$ a.e. on $\mathcal{M}$. We get that $H^k(\mathcal{M})$ is inner product space.

In general, let us prove that $H^{k,p}(\mathcal{M})$ is complete space, the scalar product $\langle\cdot\rangle$ associated to $\|\cdot\|$ is defined by

$$\langle u,v\rangle = \sum_{m=0}^{k}\int_{\mathcal{M}} (g^{i_1 j}\dots g^{i_m j_m}(\nabla^m u)_{i_1\dots i_m}(\nabla^m u)_{j_1\dots j_m})dV_g.$$

Any Cauchy sequence in $(C_k^p(\mathcal{M}), \|\cdot\|_{H^{k,p}})$ is a Cauchy sequence in the Lebesgue space $(L^p(\mathcal{M}), \|\cdot\|_p)$. We can look at $H^{k,p}$ as a subspace of $L^p(\mathcal{M})$ made of functions $u \in L^p(\mathcal{M})$ which are limits in $(L^p(\mathcal{M}), \|\cdot\|_p)$ of a cauchy sequence (u_m) in $(C_k^p(\mathcal{M}), \|\cdot\|_{H^{k,p}})$ and define $\|u\|_{H^{k,p}}$ as before, where $(\nabla^j u)$, $0 \le j \le k$, is now the limit in $(L^p(\mathcal{M}), \|\cdot\|_p)$ of the Cauchy sequence $(\nabla^j u_m)$ so $H^{k,p}(\mathcal{M})$ is complete space. So, H^k is Hilbert space. □

The main question about spectrum is self-adjointness of the Schrödinger operator $P = -\Delta_g + V$, recall that an unbounded linear operator P essentially self-adjoint if its closure $\bar{P}$ is self-adjoint. In the case of $\mathcal{M} = \mathbb{R}^n$ with standard metric, Cartier [4] showed that if the function V is locally bounded and if there exists C such that $V \ge C$ on $\mathcal{M}$, then the Schrödinger operator H is essentially self-adjoint. Later, Kato [18] proved that it is possible to replace the hypothesis $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ by $V \in L^2_{\text{loc}}(\mathbb{R}^n)$. Next, in the works of Oleinik [21] we can find a general theorem with complex hypotheses on V as following:

Theorem 3.2. *Let $(\mathcal{M}, g)$ be a complete connected manifold of dimension $n \ge 1$ and $V \in L^\infty_{\text{loc}}(\mathcal{M})$ be a potential such that for all $x \in \mathcal{M}$, $V(x) \ge C$, where C is a real constant. Then the operator $P = -\Delta_g + V(x)$ on $(\mathcal{M}, g)$ is essentially self-adjoint.*

Determination of spectrum under topological perturbation has been studied by Kac [16] later, for the case of $\Delta_{\mathbb{R}^n}$, Rauch and Taylor [22] proved that spectrum is invariant in a compact subset of $\mathbb{R}^n$. Then, Chavel and Feldman [5,6] used the technique of complex probabilistic for Riemannian manifold. Courtois [10] dealt with Δ_g on $(\mathcal{M}, g)$.

The following theorem deals with discrete spectrum in case of compact setting for $(\mathcal{M},g)$ be a compact connected manifold of dimension $n \geq 1$.

Theorem 3.3. *Let* $(\mathcal{M},g)$ *be a compact connected manifold of dimension* $n \geq 1$, $V \in L^\infty_{\text{loc}}(\mathcal{M})$ *the spectrum of* $P = -\Delta_g + V(x)$ *is discrete. Not only that it consists of an infinite increasing sequence of eigenvalues with finite multiplicity,*

$$\inf_{x\in M} V(x) \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots + \infty$$

associated with eigen functions (e_k) *which forms a Hilbert basis of the space* $L^2(\mathcal{M})$.

For non-compact case we have the theorem:

Theorem 3.4. *Let* $(\mathcal{M},g)$ *be a complete connected Riemannian manifold of dimension* $n \geq 1$ *and let* $V \in L^\infty_{\text{loc}}(\mathcal{M})$ *be a potential such that* $\lim_{|x|\to\infty} V(x) = \infty$. *Then, the spectrum of* $P = -\Delta_g + V(x)$ *is discrete. Specifically, its a set of an infinite increasing sequence of eigenvalues with finite multiplicity*

$$\inf_{x\in M} V(x) \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots + \infty.$$

Moreover, the associated eigenfunctions (e_k) *which forms a Hilbert basis of the space* $L^2(\mathcal{M})$. *The most important example in the case of* $\lim_{|x|\to\infty} V(x) = \infty$ *is Harmonic oscillator.*

Example 3.1. The one-dimensional harmonic oscillator is the Schrödinger operator $\frac{-d^2}{dx^2} + x^2$ on the manifold $\mathbb{R}$. Consider the set $Y := \{f \in H^1(\mathbb{R}), xf \in L^2(\mathbb{R})\}$ this set is subset of $H^1(\mathbb{R})$ and equipped with the scalar product

$$\langle f,g\rangle_Y = \langle f,g\rangle_{L^2} + \langle \acute{f},\acute{g}\rangle_{L^2} + \langle xf,xg\rangle_{L^2}.$$

Y is dense in $L^2(\mathbb{R})$. Moreover, by a classical argument of functional analysis, the canonical inclusion of Y in $L^2(\mathbb{R})$is compact. Consider the unbounded operator $A : D(A) \to L^2(\mathbb{R})$ with domain $D(A) = Y$.

A is defined by $Af := \acute{f} + xf$.

Using the integration by parts formula (and the fact that if $f \in Y$ then $\lim_{|x|\to\infty} x|f(x)|^2 = 0$ we get for all $f \in Y$

$$\|Af\|^2_{L^2} = -\|f\|^2_{L^2} + \|\acute{f}\|^2_{L^2} + \|xf\|^2_{L^2}.$$

For all $f \in Y$, thus we obtain

$$2\|f\|^2_{L^2} + \|Af\|^2_{L^2} = \|f\|^2_Y.$$

Consequently, the norms $\|\cdot\|_A$ and $\|\cdot\|_Y$ are equivalent on Y. Therefore, the set $Y = D(A)$ is complete for the norm $\|\cdot\|_A$. It follows that the operator $A : D(A) \to L^2(\mathbb{R})$ is closed.

$$A^* : D(A^*) \to L^2(\mathbb{R}),$$

$$A^* f = -\acute{f} + xf,$$

we called A^* the creation operator. Now we consider the operator

$$P = AA^* + I$$

with domain $D(P) = \{f \in Y : Af \in Y\}$. In fact, we have

$$D(P) = \{f \in H^2(\mathbb{R}), x^2 f \in L^2(\mathbb{R})\}$$

and for all $f \in D(P)$

$$AA^* f = -(Af)' + x(Af) = -f'' + x^2 f - f.$$

Hence

$$Pf = -f'' + x^2 f.$$

In particular, we have for all $f \in D(P)$ such that $\|f\|^2_{L^2} = 1$

$$\langle Pf, f\rangle_{L^2} = \langle AA^* f, f\rangle_{L^2} + \|f\|^2_{L^2} = \|Af\|^2_{L^2} + 1 \geq 1.$$

Let $\Theta(P)$ be the numerical range, we have $\overline{\Theta}(P) \subset [1,\infty[$. We see that the set $\mathbb{C} - \overline{\Theta(P)}$ has just one connected component and the map

$$d : \begin{cases} \mathbb{C} \to \mathbb{N} \cup \infty \\ \lambda \to \dim(\ker(P^* - \overline{\lambda} I)) \end{cases}$$

is constant on $\mathbb{C} - \overline{\Theta(P)}$ i.e. for all $\lambda \in \mathbb{C} - \overline{\Theta(P)}$

$$d(\lambda) = d(i) = d(-i)$$

(because $i, -i \in \mathbb{C} - \overline{\Theta(P)}$). But since P is self-adjoint

$$(\ker(P^* - \overline{\lambda} I)) = 0$$

thus $d(i) = d(-i) = 0$, and so $d(\lambda) = 0$ for all $\lambda \in \mathbb{C} - \overline{\Theta(P)}$. In other words, $\mathbb{C} - \overline{\Theta(P)} \subset \rho(P)$ so we get $Spec(P) \subset \rho(P) \subset \overline{\Theta(P)}$ because the spectrum of P is discrete. We conclude that

$$\text{spec}(P) \subset [1,\infty[.$$

By a simple computation

$$[A, A^*]f = AA^* f - A^* Af = 2f.$$

For all $f \in D(P)$. Next, if a vector $\varphi = D(P)$ satisfies $P\varphi = \lambda\varphi$ (where λ is a scalar), then $A\varphi \in D(P), A^*\varphi \in D(P)$, and we have

$$\begin{aligned} P(A\varphi) &= A^* AA\varphi + A\varphi \\ &= AA^* A\varphi - 2A\varphi + A \\ &= A(P\varphi - \varphi) - A\varphi \\ &= A(\lambda\varphi - \varphi) - A\varphi \\ &= (\lambda - 2)A\varphi. \end{aligned}$$

On the other hand,

$$\|A\varphi\|^2_{L^2} = \langle A\varphi, A\varphi\rangle_{L^2}$$

$$
\begin{aligned}
&= \langle A^* A\varphi, \varphi\rangle_{L^2} \\
&= \langle P\varphi - \varphi, \varphi\rangle_{L^2} \\
&= \lambda \|\varphi\|^2_{L^2} - \|\varphi\|^2_{L^2} \\
&= (\lambda - 1)\|\varphi\|^2_{L^2}.
\end{aligned}
$$

Similarly,

$$P(A^*\varphi) = (\lambda+2)A^*\varphi, \quad \|A^*\varphi\|^2_{L^2} = (\lambda+1)\|\varphi\|^2_{L^2}.$$

Now, we want to show that $\text{spec}(P) \subset \{(2n+1), n \in \mathbb{N}\}$. For the moment we only know that $\text{spec}(P) \subset [1, \infty[$. If λ is an eigenvalue of P in the interval $]1,3[$ there exist $\varphi \neq 0$ in $D(P)$ such that $P\varphi = \lambda\varphi$ thus we have $P(A\varphi) = (\lambda-2)A\varphi$ and

$$\|A\varphi\|^2_{L^2} = (\lambda-1)\|\varphi\|^2_{L^2} > 0$$

hence $A\varphi \neq 0$. Consequently $\lambda - 2 \in \text{spec}(P)$ therefore $\lambda - 2 \in]-1,1[$ which is a contradiction. So

$$\text{spec}(P) \subset \{1\} \cup [3, \infty[.$$

Using the same argument by induction we get easily that

$$\text{spec}(P) \subset \{(2n+1), n \in \mathbb{N}\}$$

as claimed. To finish, using the Hermite family $\{e_n\}_{n\in\mathbb{N}}$

$$e_n(x) = (2^n n) = (2^n n\sqrt{\pi})^{\frac{-1}{2}} e^{\frac{-x^2}{2}} H_n(x),$$

where $P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$ which form a Hilbert basis of the space $L^2(\mathbb{R})$, we have

$$A^* e_n = \sqrt{2n+2}\, e_{n+1}$$

and for any n

$$A e_n = \sqrt{2n}\, e_{n-1}$$

so for any n

$$P e_n = (2n+1)e_n.$$

Consequently, the spectrum of the operator P is

$$\text{spec}(P) = \{(2n+1), n \in \mathbb{N}\}$$

and the associated eigenvectors are given the Hermite's family.

4. Application in Spectral Geometry

In the past decade, there has been a flurry of work at intersection of spectral theory and Riemannian geometry. In this section, we will briefly present some of recent results in abstract spectral theory depending on Laplace-Beltrami operator on compact Riemannian manifold. Also, we will emphasize the interplay between spectrum of operator and geometry of manifolds by discussing two main problems (direct and inverse problems) with an eye towards recent developments.

Definition 4.1. The relationship between geometric structure of manifolds and spectrum of differential operators created a new concept which is spectral geometry. In the case of Laplace-Beltrami operator on closed Riemannian manifold this field sets two problems:

(i) Direct problem.

(ii) Inverse problem.

Direct problem. Given a compact Riemannian manifold $(\mathcal{M}, g)$ can we find the spectrum $\{\lambda_k(\mathcal{M})\}_{\{k\geq 0\}}$ of $\mathcal{M}$, this question comes under Direct problem. In fact, we can discern that the explicit computation of spectrum is not easy task, there are few examples where the spectrum of manifold is known, like (sphere, flat tori, balls), for this reason some of estimates of spectrum is introduced (we refer to Cheng [7]).

Inverse problem. Inverse problem seeks to identify features of geometry from information about Laplace's spectrum, some results are appeared when Milnor [20] answer of the question that Kac posted (see [15]), the analogy of this question is "Is the spectrum of associated on smooth function Laplacian determine the shape of manifold?". Sunada rise to give examples which clarifies iso-spectral manifolds (see [25]). In general, the data of spectrum does not determine the shape of manifold however, some of positive results as the geometric effect that we can take out from spectral invariant is shown in [14].

5. Conclusion

We covered most important analytic theorems of Schrödinger operators on manifold. The geometric aspects of spectrum also play a very significant role in mathematical physics and still a very active field of research till now.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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