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Research Article

Analytical and Numerical Study of Steady Flow of Thermo-Viscous Fluid Between Two Horizontal Parallel Plates in Relative Motion

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Abstract. In the present paper we examined the Analytical and Numerical study of steady flow of thermo viscous fluid between two parallel horizontal plates in relative motion. The numerical solutions of governing equations of flow involving velocity and temperature have been obtained using 6^{th} order R-K methods via MATHEMATICA ND solver. The governing equations of the flow also have been solved analytically. The numerical solutions for different values of the physical parameters like thermal stress interaction material coefficient, thermo conductivity parameter and the porosity coefficient have been presented in terms of tables and illustrated graphically. The results for linearized steady incompressible motion fluid flow in between two non-porous parallel horizontal plates also obtained both numerically and analytically. The numerical and analytical results are compared and are verified to be in good convergence. The solutions of the present investigation will definitely give an excellent perception of nuclear, industrial and clinical applications.

Keywords. Porosity parameter, Thermal conductivity parameter, Thermal viscosity parameter, Prandtl number

Mathematics Subject Classification (2020). 35Q35

1. Introduction

The materials of non-Newtonian behaviour have been the subject of substantial study for more than one hundred and fifty years. From the last seventy or eighty years the serious investigations have been done to enlarge these attempts in the reality of non-linear study. The non success of the linear studies in anticipating to a considerable extent of mechanical nature of materials like liquids, polymers, plastics and the metals of molten nature et cetera with respect to the stresses has been the driven force behind the investigations of non linear study in the description of materials. Many of these applications for these type of fluids which includes in the areas like energy systems, petroleum industry based on technology, chemical and nuclear industries et cetera with the fatten significance of non-newtonian behaviour of fluids in latest technologies and industries, these examinations on such class of fluids are worthwhile. The development and importance of non-linearity of fluids which reflect the interaction and inter relation between viscous and thermal effects was first introduced by Koh and Eringen [6], Green and Naghdi [3, 4] introduced the new concept of theory on thermo-viscous type of fluids earlier. Kelly [5] studied some shear flows of steady and unsteady thermo-viscous fluid flows. Later, Rao and Pattabhiramacharyulu [10] investigate some steady state problems of thermo-viscous fluids dealing with certain fluid flows. Pothanna and Aparna [8] studied unsteady thermo-viscous flow in a porous slab over an oscillating flat plate. Aparna *et al*. [1, 2] studied flow generated by slow steady rotation of a permeable sphere in a micro-polar fluid and also couple on a rotating permeable sphere in a couple stress fluid. Nagaraju and Aparna [9] explore unsteady rotatory oscillations of a vertical cylinder in Jeffery fluid with ion slip currents and porous medium. Recently, Pothanna *et al*. [7] investigated a numerical study of thermo viscous fluids passing through a cylinder using R-K method of 6^{th} order with the help of MATHEMATICA package ND-Solver.

The subject of fluid flows via porous media has intensive research for both experimental and theoretical studies since long periods, i.e., almost more than one and half centuries due to the reason of application in different areas of space related systems, geo and energy related systems, petroleum based industries, bio and astrophysics, pharmaceutical based industries and so on. Some important practical concepts involving some investigations include the passage of liquids past a solids, the filtration, extraction and production of oils from wells and the seeping through muds in drains et cetera. Filtration of beds which are using in most of chemical industry processes.

From the past studies, it can be noticed that the governing flow equations in a plane, cylindrical and spherical geometry have been studied and for which the analytical solutions have also obtained using different methods. However, in real world the flow of these type of thermo-viscous nature of fluids via porous medium are non-linear type nature. Therefore, it is required and important to develop the methods and obtain the solutions of equations numerically.

In the present investigation, the results of incompressible steady thermo viscous fluid flow equations have been obtained with the help of 6^{th} order R-K method with shooting methods using MATHEMATICA package ND solver. The problem of numerical study of incompressible thermo viscous fluid flow in between two parallel horizontal plates in porous region have not been discussed in the previous literature.

Nomenclature

$\alpha_1 = -p$	Pressure of fluid
$\alpha_3 = 2\mu$	Viscosity coefficient
$\alpha_5 = 4\mu_c$	Cross-viscosity coefficient
α_6	Thermal stress interaction parameter
α_8	Thermal stress viscosity parameter
$\beta_1 = k$	Coefficient of thermal conductivity
β_3	Coefficient of strain thermal conductivity
a_6	Dimensionless thermo-stress interaction coefficient
b_3	Dimensionless coefficient of strain-thermal conductivity
p_r	Prandtl number
c	Specific heat
f_i	i^{th} component external force
q_i	i^{th} component heat flux bi-vector
u_i	i^{th} component velocity
d_{ij}	Rate of deformation tensor
b_{ij}	Thermal gradient bivector
t_{ji}	Stress tensor
T	Dimensionless Temperature
C_1	Pressure gradient-constant
C_2	Temperature gradient-constant

Greek Letters

ρ	Fluid density
γ	External energy source
η	Fluid temperature
α_i's	Coefficient of viscosity
β_i's	Coefficient of thermal conductivity

2. Basic Governing Equations

The following laws of conservation equations satisfies thermo viscous incompressible fluids past a permeable region.

Continuity equation:

$$v_{i,i} = 0 \tag{2.1}$$

Momentum equation:

$$\rho\left[\frac{\partial v_i}{\partial t} + v_k v_{i,k}\right] = \rho F_i + t_{ji,j} - \frac{\mu}{k^*} v_i \tag{2.2}$$

Energy equation:

$$\rho c\dot{\theta} = t_{ij} d_{ij} - q_{i,i} + \rho\gamma - \frac{\mu}{k^*} v_i v_i \tag{2.3}$$

As Koh and Eringen [6] primarily introduced by the stress tensor and heat flux bi-vector basic equations of second order incompressible thermo viscous nature of fluids which are coupled in terms of d and b are given by

$$t = \alpha_1 I + \alpha_3 d + \alpha_5 d^2 + \alpha_6 b^2 + \alpha_8(db - bd) \tag{2.4}$$

and

$$h = \beta_1 b + \beta_3(bd + db), \tag{2.5}$$

where the coefficients in terms of α_i's and β_i's are the polynomial expressions in $\operatorname{tr} d, \operatorname{tr} d^2, \operatorname{tr} b^2$. The explicit polynomials for these constitutive coefficients in terms of α_i's and β_i's for the second order equations may be obtained as

$$\alpha_1 = \alpha_{1000} + \alpha_{1010} \operatorname{tr} d + \alpha_{1020} \operatorname{tr} d^2 + \alpha_{1002} \operatorname{tr} b^2,$$

$$\alpha_3 = \alpha_{3010} + \alpha_{3020} \operatorname{tr} d,$$

$$\alpha_5 = \alpha_{5020},$$

$$\alpha_6 = \alpha_{6002},$$

$$\alpha_8 = \alpha_{8011},$$

$$\beta_1 = \beta_{1001} + \beta_{1011} \operatorname{tr} d,$$

$$\beta_3 = \beta_{1011}.$$

the secondary coefficients in terms of α_{isrt} and β_{isrt} are the functions of ρ and η.

The constitutive equations of (2.4) and (2.5) with some constant values of these parameters α_i's and β_i's can be called as *thermo-viscous second order fluids*. This is the most simplest mathematical model of a flow of thermo viscous fluids which evince the inter relation between thermal and mechanical responses.

3. Formulation and Analytical Solution of the Problem

Consider the steady incompressible motion of thermo viscous fluids passing through a permeable region bounded by two non permeable parallel horizontal plates. The fluid flow is assumed to be cause by a constant pressure gradient below the horizontal parallel plates. The top plate is moving with the relative to a bottom plate with a constant velocity u_0 along with the direction of the constant pressure gradient.

Consider a rectangular coordinate system with the lower plate embedded at the origin, along the x-axis direction, y-axis is perpendicular to both the plates and these plates constitute by the values $y = 0$ and $y = h$. Further, both lower and upper the plates are maintained to be fixed temperature values θ_0 and θ_1, respectively.

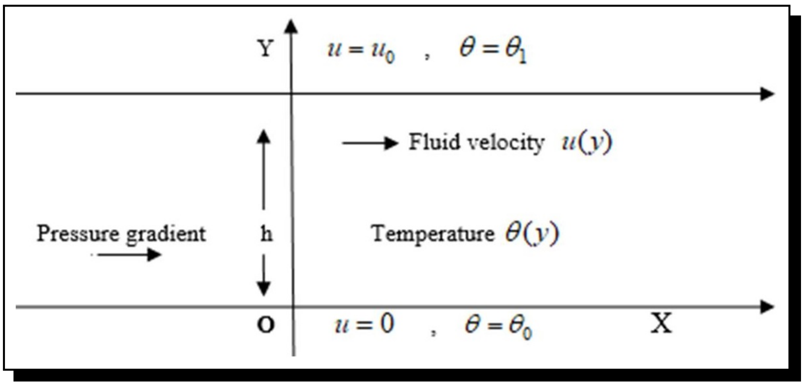

Figure 1. Flow model

Consider the incompressible steady flow in between two horizontal non permeable plates are specified with the component $[u(y),0,0]$ of velocity and $\theta(y)$ of temperature. The choice of this assumption for both velocity and temperature clearly satisfies the law of conservation equation.

3.1 Basic Equations Characterizing the Flow

In X-direction:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} - \alpha_6\frac{\partial \theta}{\partial x}\frac{\partial^2 \theta}{\partial y^2} + \rho F_x - \frac{\mu}{k^*}u \tag{3.1}$$

In Y-direction:

$$0 = \mu_c\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 + \rho F_y \tag{3.2}$$

In Z-direction:

$$0 = \alpha_8\frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y}\right) + \rho F_z \tag{3.3}$$

and the equation of energy:

$$\rho c u\frac{\partial \theta}{\partial x} = \mu\left(\frac{\partial u}{\partial y}\right)^2 - \alpha_6\frac{\partial \theta}{\partial x}\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y} + k\frac{\partial^2 \theta}{\partial y^2} + \beta_3\frac{\partial \theta}{\partial x}\frac{\partial^2 u}{\partial y^2} + \rho\gamma - \frac{\mu}{k^*}u^2 \tag{3.4}$$

together with the boundary conditions:

$$u = 0,\ \theta = \theta_0 \text{ at } y = 0$$

$$\text{and } u = u_0,\ \theta = \theta_1 \text{ at } y = h \tag{3.5}$$

The following dimensionless quantities are introduced:

$$y = hY,\ u = \left(\frac{\mu}{\rho h}\right)U,\ u_0 = \left(\frac{\mu}{\rho h}\right)U_0,\ T = \frac{\theta-\theta_0}{\theta_1-\theta_0},\ \frac{\partial\theta}{\partial x} = \frac{\theta_1-\theta_0}{h}C_2,\ \frac{\partial p}{\partial x} = \frac{\mu^2}{\rho h^3}C_1,$$

$$S = \frac{h^2}{k^*},\ p_r = \frac{\mu c}{k} \text{ and } b_3 = \frac{\beta_3}{\rho h^2 c}.$$

Here C_1 and C_2 represents fixed pressure and temperature gradients in dimensionless form respectively and S is the porosity coefficient.

The fluid flow is consider to be so slow steady flow, that the non-linearity terms in the above equations could be neglected. In terms of the above non-dimensional and by neglecting external forces and internal energy source. The equations (3.1) and (3.4) can be written as

$$0 = -C_1 + \frac{d^2U}{dY^2} - a_6C_2\frac{d^2T}{dY^2} - SU, \tag{3.6}$$

$$UC_2 = b_3C_2\frac{d^2U}{dY^2} + \frac{1}{p_r}\frac{d^2T}{dY^2} \tag{3.7}$$

together with the boundary conditions:

$$U(0) = 0,\ U(1) = U_0, \tag{3.8}$$

$$T(0) = 0,\ T(1) = 1. \tag{3.9}$$

Eliminating $\frac{d^2T}{dY^2}$ from (3.6) and (3.7), we get

$$(1 + p_ra_6b_3C_2^2)\frac{d^2U}{dY^2} - (S + p_ra_6C_2^2)U = C_1. \tag{3.10}$$

Using the equation (3.10) and employing the boundary conditions in (3.8) the velocity field is obtained as

$$U(Y) = \frac{1}{m^2\sinh m}\{m_1C_1[\sinh m(1-Y) - \sinh m] + [m_1C_1 + m^2U_0]\sinh mY\}, \tag{3.11}$$

where $m = \sqrt{\dfrac{S + p_ra_6C_2^2}{1 + p_ra_6b_3C_2^2}}$ and $m_1 = \dfrac{1}{1 + p_ra_6b_3C_2^2}$.

Using the equation (3.11) in (3.7), we get the differential equation

$$\frac{d^2T}{dY^2} = \frac{1}{m^2\sinh m}\{p_rC_2(1 + m^2b_3)[m_1C_1\sinh m(1-Y) + (m_1C_1 + m^2U_0)\sinh mY] - p_rm_1C_1C_2\sinh m\}. \tag{3.12}$$

The temperature field is the solution of (3.12) satisfies the boundary conditions in (3.9). Thus, we get

$$T(Y) = Y + \frac{p_rC_2}{2m^2}\{Y(1-Y)m_1C_1 + 2(1 + m^2b_3)[YU_0 - U(Y)]\}. \tag{3.13}$$

3.2 Physical Quantities

The flow rate:

$$Q = \int_0^1 U(Y)dY = \frac{1}{m^3}\left\{2mm_1C_1 + (2m_1C_1 + m^2U_0)\tanh\frac{m}{2}\right\}$$

The Shear stress:

$$\frac{dU}{dY} = \frac{C_1}{m\sinh m}\{(m_1C_1 + m^2U_0)\cosh mY - m_1C_1\cosh m(1-Y)]\}$$

The lower plate Shear stress:

$$\frac{dU}{dY}\Big|_{(Y=0)} = \frac{m_1C_1}{m}\tanh\frac{m}{2} + mU_0\cos echm$$

The upper plate Shear stress:

$$\frac{dU}{dY}|(Y=1) = \frac{m_1C_1}{m}\tanh\frac{m}{2} + mU_0\coth m$$

The Nussult number:

$$\frac{dT}{dY} = \frac{p_rC_2(1+b_3m^2)}{m^3\sinh m}\{m_1C_1\cosh m(1-Y) - (m_1C_1 + m^2U_0)\cosh mY + mU_0\sinh m\} + \frac{p_rC_2m_1C_1}{2m^2}[1-2Y]$$

The lower plate Nussult number:

$$\frac{dT}{dY}\Big|_{(Y=0)} = \frac{p_rC_2(1+b_3m^2)}{m^3\sinh m}\left\{m_1C_1\sec h\frac{m}{2} + mU_0(1-m\cos echm)\right\} + \frac{p_rC_2m_1C_1}{2m^2}$$

The upper plate Nussult number:

$$\frac{dT}{dY}\Big|_{(Y=1)} = \frac{p_rC_2(1+b_3m^2)}{m^3}\left\{mU_0(1-m\cos echm) - (m_1C_1 + m^2U_0)\tanh\frac{m}{2}\right\} - \frac{p_rC_2m_1C_1}{2m^2}$$

The external force obtained in the Y-direction:

$$\rho F_Y = \frac{\mu_c\mu^2}{\rho h^5 m\sin^2 hm}\{(m_1C_1 + m^2U_0)^2\sinh 2mY - m_1^2C_1^2\sinh 2m(1-Y) + 2m_1C_1(m_1C_1 + m^2U_0)\sinh m\}$$

It is observed from the above equation that external force generated in y-direction depends on Reiner-Rivilin coefficient (i.e., μ_c).

The external force obtained on the bottom plate in the Y-direction:

$$\rho F_Y\Big|_{(Y=0)} = \frac{2\mu_c\mu^2 m_1C_1}{\rho h^5 m}\left\{m^2U_0\cos echm - m_1C_1\tanh\frac{m}{2}\right\}$$

The external force obtained on the top plate in the Y-direction:

$$\rho F_Y\Big|_{(Y=1)} = \frac{2\mu_c\mu^2}{\rho h^5 m}\left\{m^2U_0\coth m + m_1C_1\coth\frac{m}{2}\right\}$$

The external force obtained in the Z-direction:

$$\rho F_Z = \frac{-\alpha_8\mu(\theta-\theta_0)p_rC_2}{2\rho h^3m^3\sin^2 hm}\{2(1+m^2b_3)\{(m_1C_1+m^2U_0)[2m_1C_1\sinh m(1-2Y)] - m_1C_1\sinh 2m(1-Y)\} + m\sinh m(m_1C_1(1-2Y)+2U_0(1+m^2b_3))\{(m_1C_1+m^2U_0)\sinh mY + m_1C_1\sinh m(1-Y)\} - 2m_1C_1\sinh m\{(m_1C_1+m^2U_0)\cosh mY - m_1C_1\cosh m(1-Y)\}\}.$$

It is noticed from the above equation that the external force generated in z-direction is depends on the thermal stress interaction coefficient (i.e., α_8).

The external force obtained on the bottom plate in the Z-direction:

$$\rho F_Z\big|_{(Y=0)} = \frac{-\alpha_8\mu(\theta-\theta_0)p_rC_2m_1C_1}{2\rho h^3m^3\sin^2 hm}\Big\{2(1+2m^2b_3)\Big\{m^2U_0-2m_1C_1\sin^2 h\frac{m}{2}\Big\} + m\sinh m\{m_1C_1+2U_0(1+m^2b_3)\Big\}\Big\}$$

4. Numerical Study of the Problem

The differential equations governed by the equations (3.6)-(3.7) by employing the boundary conditions (3.8)-(3.9) for velocity and temperature of the fluid have been obtained by using shooting methods with 6$^{\text{th}}$ order R-K methods with the package of MATHEMATICA ND solver tool. The width between two parallel plates 'h' units for both the velocity and temperature (i.e., $y \to h$) is reduced as finite value 1 after introducing the dimensional less quantities in the fluid equations. The convergence of this method was guaranteed by satisfying the *boundary conditions* (BCs) of the problem. The effect of different material coefficients such as thermo interaction stress coefficient (α_6), coefficient of thermal strain conductivity (b_3), coefficient of cross viscosity (μ_c) and Prandtl parameter (p_r) viz. fixed quantities like specific heat (c), density (ρ), fixed pressure gradient ($c1$), fixed temperature gradient ($c2$), viscosity (μ) on the velocity and temperature fields have been presented in form of tables and illustrations. Throughout this paper to carry out the calculations, the standard values for these material quantities ($c, \rho, c1, c2, \mu, \mu_c$ and p_r) are assumed to be 1.

The numerical results shown in Tables 1-2 gives the solutions of governing equations of fluid coupled in velocity and temperature fields with the different material parameters.

Table 1. Numerical solutions for the velocity with different material parameters

Y	Velocity $U(Y)$ with $c=1$, $\rho=1$, $c1=1$, $c2=1$, $\mu=1$, $\mu_c=1$, $p_r=1$, $U_0=0$								
	$b_3=0.1$, $\alpha_6=0.01$			$b_3=0.5$, $\alpha_6=0.05$			$b_3=1$, $\alpha_6=0.1$		
	$S=1$	$S=2$	$S=3$	$S=1$	$S=2$	$S=3$	$S=1$	$S=2$	$S=3$
0	0	0	0	0	0	0	0	0	0
0.1	−0.03899	−0.03988	−0.03999	−0.03999	−0.03989	−0.03999	−0.02999	−0.02889	−0.02979
0.2	−0.06644	−0.06772	−0.06882	−0.06646	−0.06778	−0.06888	−0.06788	−0.06678	−0.06556
0.3	−0.08157	−0.08222	−0.08332	−0.08167	−0.08232	−0.08342	−0.07168	−0.07232	−0.07442
0.4	−0.11069	−0.11213	−0.11313	−0.11079	−0.11233	−0.11323	−0.10079	−0.10233	−0.10323
0.5	−0.19977	−0.19998	−0.19999	−0.19882	−0.10001	−0.08899	−0.18882	−0.09801	−0.08899
0.6	−0.11069	−0.11213	−0.11313	−0.11079	−0.11233	−0.11323	−0.10079	−0.10233	−0.10323
0.7	−0.08157	−0.08222	−0.08332	−0.08167	−0.08232	−0.08342	−0.07168	−0.07232	−0.07442
0.8	−0.06644	−0.06772	−0.06882	−0.06646	−0.06778	−0.06888	−0.06788	−0.06678	−0.06556
0.9	−0.03899	0.03988	−0.03999	−0.03999	−0.03989	−0.03999	−0.02999	−0.02889	−0.02979
1	0	0	0	0	0	0	0	0	0

Table 2. Numerical solutions for the temperature with different material parameters

Y	Velocity $U(Y)$ with $c=1$, $\rho=1$, $c1=1$, $c2=1$, $\mu=1$, $\mu_c=1$, $p_r=1$, $U_0=0$								
	$b_3=0.1$, $a_6=0.01$			$b_3=0.5$, $a_6=0.05$			$b_3=1$, $a_6=0.1$		
	$S=1$	$S=2$	$S=3$	$S=1$	$S=2$	$S=3$	$S=1$	$S=2$	$S=3$
0	0	0	0	0	0	0	0	0	0
0.1	0.112112	0.101273	0.100272	0.124561	0.113222	0.112311	0.200012	0.199991	0.195555
0.2	0.300012	0.291777	0.281368	0.311199	0.299333	0.291101	0.312333	0.300211	0.300222
0.3	0.397882	0.396677	0.395774	0.399991	0.398899	0.411011	0.400233	0.399912	0.400001
0.4	0.494451	0.493355	0.492999	0.496668	0.500123	0.493332	0.500211	0.512344	0.522233
0.5	0.591212	0.590001	0.589990	0.600026	0.591144	0.600021	0.701234	0.600022	0.611122
0.6	0.691111	0.689991	0.689991	0.692222	0.699923	0.691112	0.712228	0.702333	0.700212
0.7	0.791231	0.789999	0.788889	0.798999	0.811144	0.791213	0.799991	0.822556	0.798889
0.8	0.801233	0.800021	0.799999	0.811229	0.822555	0.800144	0.833445	0.850011	0.810002
0.9	0.901341	0.901241	0.900023	0.912118	0.911199	0.918888	0.982311	0.988779	0.951133
1	1	1	1	1	1	1	1	1	1

5. Comparison of Analytical and Numerical Solutions

The analytical solutions of coupled governing linear equations (3.6) and (3.7) together with the boundary conditions (3.8)-(3.9) in terms of velocity field and temperature fields have been obtained and the numerical results for both the velocity field and temperature fields for various values of physical coefficients are presented in Table 3 and Table 4. In order to obtain the analytical solutions of the equations (3.6) and (3.7) for the velocity and the temperature for different values of physical material parameters, the algorithm code has been generated in MATLAB and executed on a PC.

Table 3. Velocity profile results comparison for $U_0=1$

Y	Numerical results for $b_3=0.1$, $a_6=0.01$		Analytical results for $b_3=0.1$, $a_6=0.01$		Numerical results for $b_3=0.5$, $a_6=0.05$		Analytical results for $b_3=0.5$, $a_6=0.05$	
	$S=1$	$S=2$	$S=1$	$S=2$	$S=1$	$S=2$	$S=1$	$S=2$
0	0	0	0	0	0	0	0	0
0.1	0.051121	0.041141	0.059999	0.041231	0.050011	0.039999	0.051122	0.040001
0.2	0.100122	0.100001	0.112223	0.104433	0.099992	0.099988	0.098888	0.099777
0.3	0.170011	0.150112	0.180011	0.153453	0.151118	0.122111	0.152113	0.122313
0.4	0.211133	0.199988	0.222211	0.199556	0.199981	0.177777	0.199333	0.178889
0.5	0.310011	0.299911	0.312223	0.299924	0.310002	0.200022	0.310678	0.200031
0.6	0.501122	0.500021	0.502114	0.500312	0.500003	0.499977	0.500112	0.499966
0.7	0.712231	0.691112	0.712333	0.691555	0.699999	0.688833	0.699989	0.688855
0.8	0.800112	0.799991	0.800122	0.799999	0.791111	0.777711	0.7911212	0.777876
0.9	0.981125	0.977714	0.981132	0.977788	0.960021	0.966663	0.960122	0.966554
1	1	1	1	1	1	1	1	1

Table 4. Temperature profile results comparison for $U_0 = 1$

Y	Numerical results for $b_3 = 0.1$, $a_6 = 0.01$		Analytical results for $b_3 = 0.1$, $a_6 = 0.01$		Numerical results for $b_3 = 0.5$, $a_6 = 0.05$		Analytical results for $b_3 = 0.5$, $a_6 = 0.05$	
	$S = 1$	$S = 2$	$S = 1$	$S = 2$	$S = 1$	$S = 2$	$S = 1$	$S = 2$
0	0	0	0	0	0	0	0	0
0.1	0.21113	0.20011	0.21133	0.20014	0.28661	0.21122	0.28677	0.24445
0.2	0.52001	0.51101	0.52011	0.49001	0.53002	0.52344	0.53033	0.52377
0.3	0.68111	0.67222	0.68211	0.68002	0.69223	0.68123	0.69243	0.68121
0.4	0.81332	0.80331	0.81444	0.80444	0.82233	0.81999	0.82288	0.81998
0.5	0.88821	0.86616	0.88822	0.86636	0.89001	0.89991	0.90021	0.89888
0.6	0.91002	0.89001	0.91014	0.89211	0.93881	0.89881	0.92111	0.89877
0.7	0.95221	0.92244	0.95317	0.92252	0.96884	0.94001	0.96823	0.94000
0.8	1.02311	1.00112	1.00113	1.10122	1.10233	1.01123	1.10235	1.01120
0.9	1.11233	1.10066	1.11344	1.10089	1.12445	1.11557	1.12466	1.11566
1	1	1	1	1	1	1	1	1

The numerical results of the governing equations (3.6) and (3.7) with respect to the boundary conditions (3.8)-(3.9) have been obtained by *Mathematica* ND solver tool. Excellent convergence of numerical results was achieved when compared with the analytical solutions. The numerical results obtained using R-K method of 6^{th} order are compared with the obtained analytical results and are verified to be in excellent agreement. The closed form results obtained are justify the viability and correctness of the present study numerical results.

6. Results and Discussion

The numerical results shown in the previous section was performed and the set of results are illustrated and represented graphically and shown in Figures 2-10. To see the physical perception of the problem, the results of velocity and temperature fields have been obtained and discussed by giving some values to different physical parameters such as thermal stress coefficient (a_6), coefficient of thermal conductivity (b_3) and the porosity coefficient (S) which characterise the flow occurrence. The effect of all these coefficients on the velocity field and temperature fields have been obtained and are illustrated graphically.

6.1 Velocity Distribution

The effect of porosity coefficient (S), coefficient of thermal stress interaction coefficient (a_6) and the coefficient of thermal conductivity (b_3) on the velocity field was discussed and presented graphically shown in Figures2-7.

Figures 2-4 depicts that, the velocity of the fluid increases as the values of porosity coefficient (S) increases. The velocity of the fluid decreases up to the middle of the channel and increases from the centre and attains the velocity of the upper plate as we move away from lower plate to the upper plate. This effect is due to the porosity of the medium bounded between the plates. It is observed from Figures 2-5 that, the velocity of the fluid flow increases slowly for increasing the value of coefficient of thermal stress interaction (a_6) and the coefficient of thermal conductivity (b_3). The rate at which increase or decrease of the fluid near the boundary is very low when compared with the middle of channel of the flow.

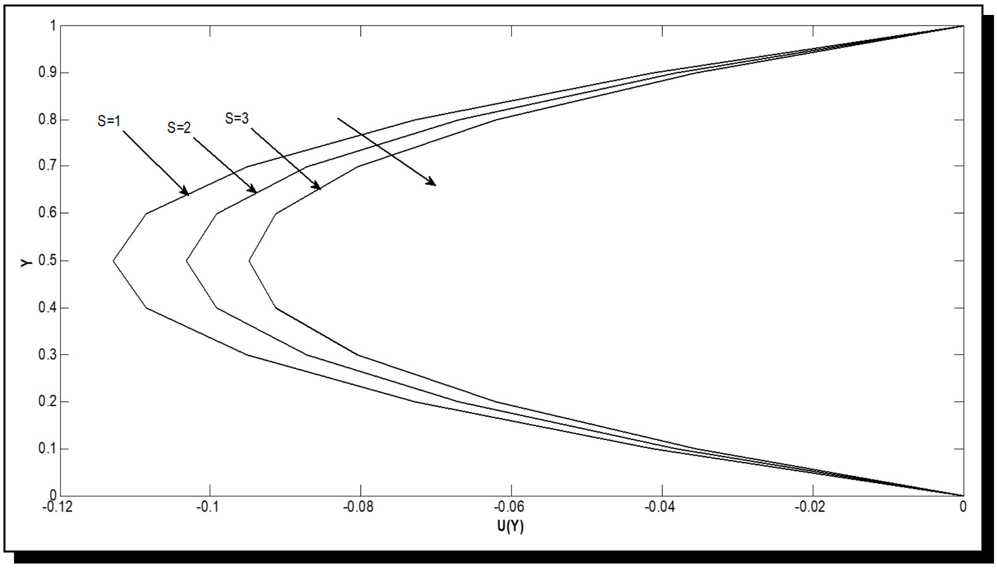

Figure 2. Velocity (U) profiles variations with $U_0 = 0$, coefficient of thermal conductivity ($b_3 = 0.1$), thermal stress coefficient ($a_6 = 0.01$) and porosity coefficient (S)

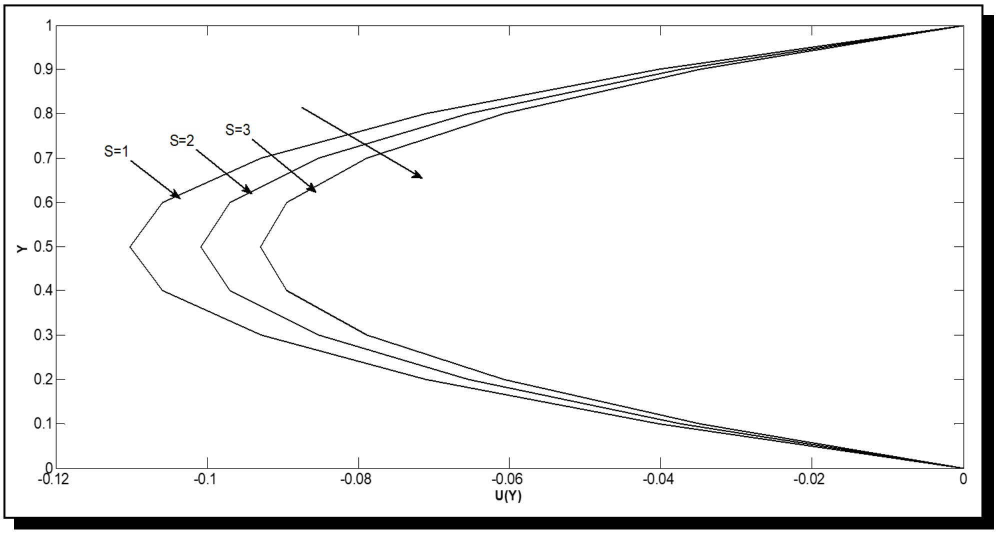

Figure 3. Velocity (U) profiles variations with $U_0 = 0$, coefficient of thermal conductivity ($b_3 = 0.5$), thermal stress coefficient ($a_6 = 0.05$) and porosity coefficient (S)

Figure 4-7 depicts that, the velocity of the fluid moving in the negative direction when the horizontal parallel plates are fixed (i.e., $U_0 = 0$) and are moving in the right direction when the horizontal parallel plates are in relative motion (i.e., $U_0 = 1$). The parabolic profile type distributions are obtained when the case of fixed parallel plates.

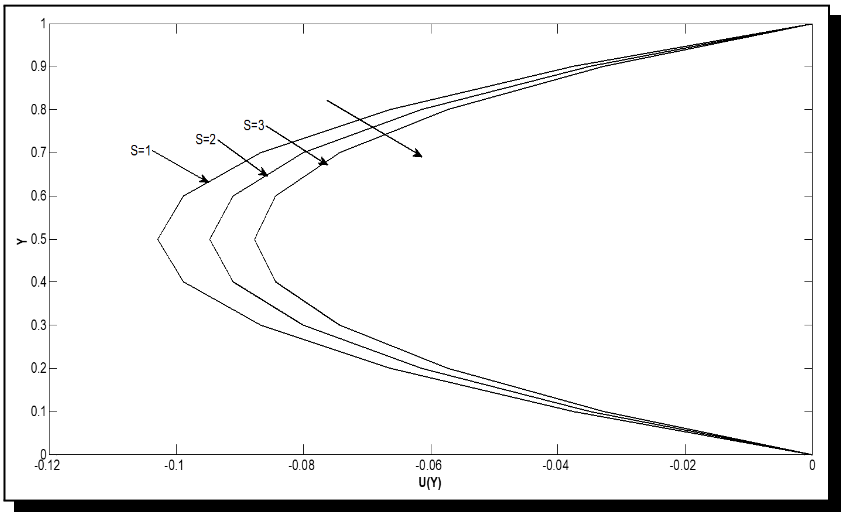

Figure 4. Velocity (U) profiles variations with $U_0 = 0$, coefficient of thermal conductivity ($b_3 = 1$), thermal stress coefficient ($a_6 = 0.1$) and porosity coefficient (S)

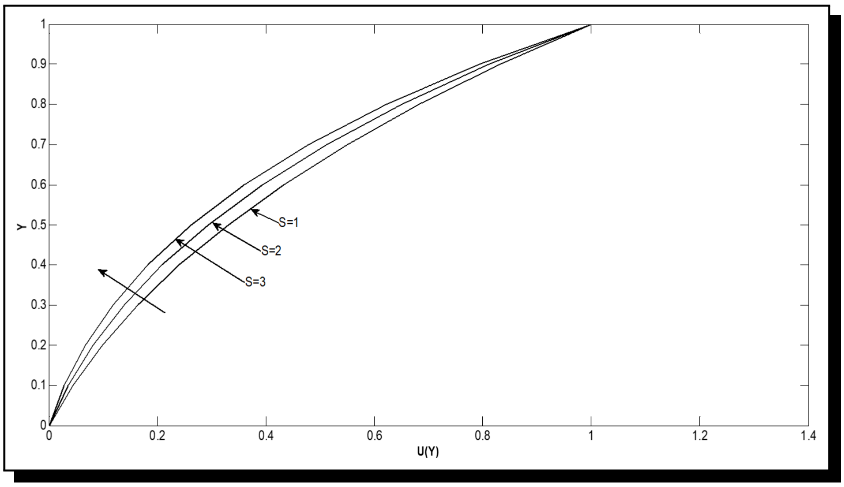

Figure 5. Velocity (U) profiles variations with $U_0 = 1$, coefficient of thermal conductivity ($b_3 = 0.1$), thermal stress coefficient ($a_6 = 0.01$) and porosity coefficient (S)

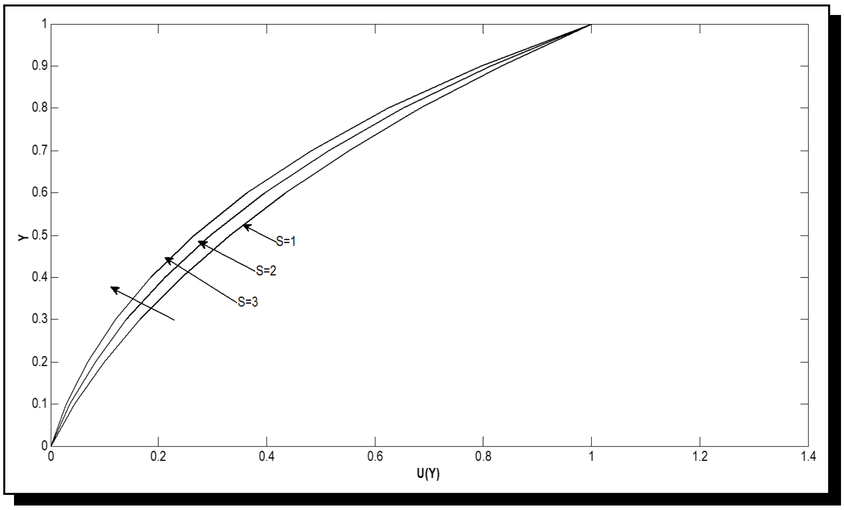

Figure 6. Velocity (U) profiles variations with $U_0 = 1$, coefficient of thermal conductivity ($b_3 = 0.5$), thermal stress coefficient ($a_6 = 0.05$) and porosity coefficient (S)

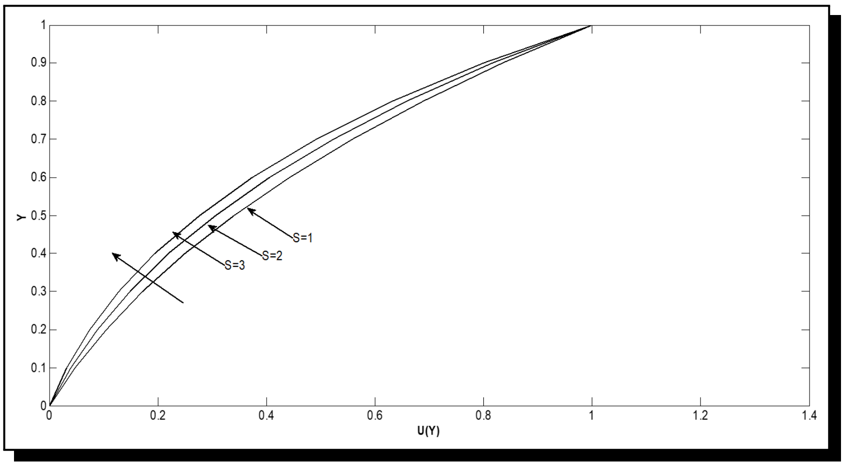

Figure 7. Velocity (U) profiles variations with $U_0 = 1$, coefficient of thermal conductivity ($b_3 = 1$), thermal stress coefficient ($a_6 = 0.1$) and porosity coefficient (S)

6.2 Temperature Distribution

The material coefficients such as thermal mechanical stress interaction coefficient (a_6), coefficient of strain thermal conductivity (b_3) and porosity coefficient (S) are influenced on the temperature of the thermo viscous fluid and which is graphically shown in Figures 8-10. It is observed that, the temperature of the fluid increases as the values of y increases from 0 to 1. Figures 8-10, it is noticed that the temperature of the fluid decreases as the values of porosity coefficient (S) increases and reach its maximum temperature at the upper plate. As thermal interaction stress coefficient (a_6) and coefficient of thermal conductivity (b_3) values increase, the temperature of the fluid slowly increases. This is effect is due to the large conversion of thermal energy sources to kinetic energy sources. From Figures 8-10, it is observed that, the temperature of the thermo-viscous fluid when the horizontal plates are fixed (i.e., $U_0 = 0$) is less when compared to the temperature when the horizontal plates are in relative motion (i.e., $U_0 = 1$). This exactly tally the physical phenomena of the problem.

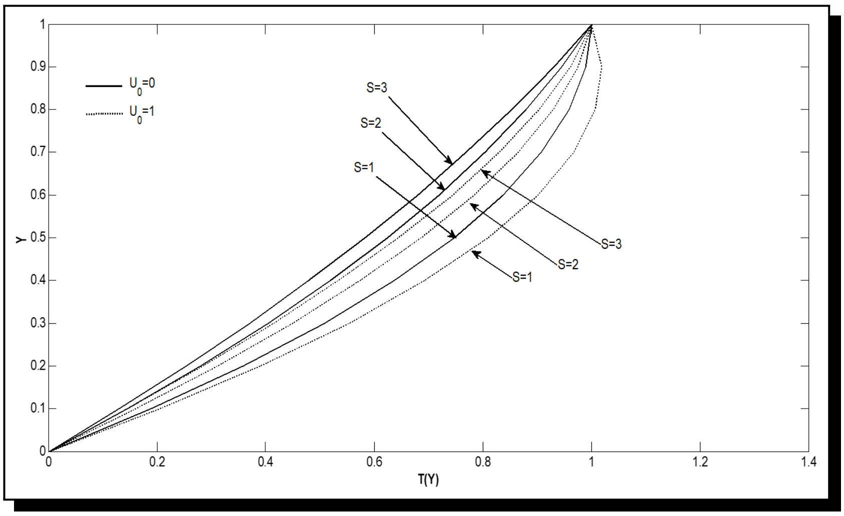

Figure 8. Temperature (T) profiles variations with coefficient of thermal conductivity ($b_3 = 0.1$), thermal stress coefficient ($a_6 = 0.01$) and porosity coefficient (S)

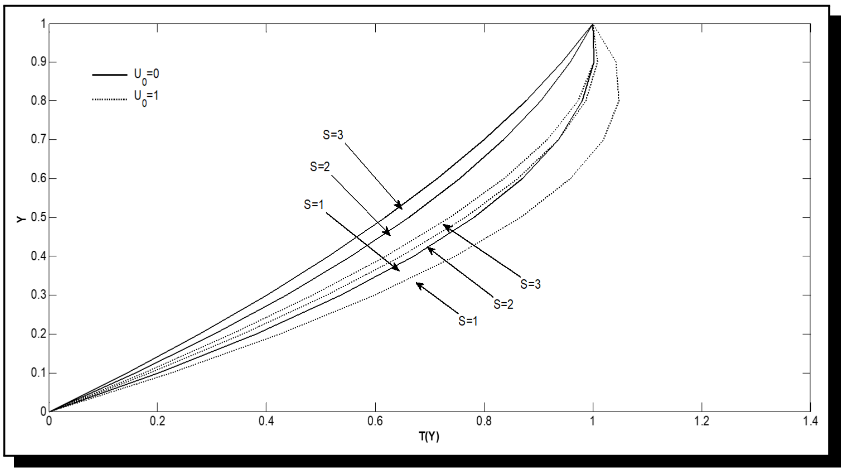

Figure 9. Temperature (T) profiles variations with coefficient of thermal conductivity ($b_3 = 0.5$), thermal stress coefficient ($a_6 = 0.05$) and porosity coefficient (S)

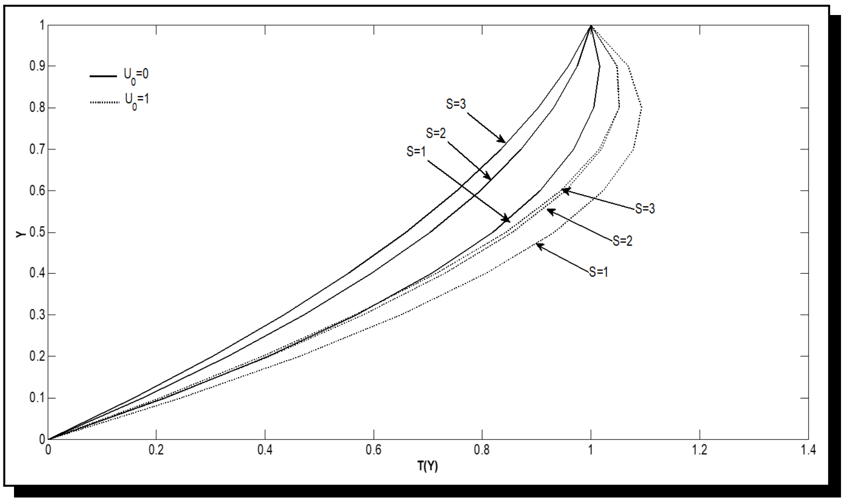

Figure 10. Temperature (T) profiles variations with coefficient of thermal conductivity ($b_3 = 1$), thermal stress coefficient ($a_6 = 0.1$) and porosity coefficient (S)

7. Conclusions

In the present problem, the analytical and numerical study of thermo viscous incompressible steady flow between two parallel horizontal plates in relative motion was examined. The analytical solution of governing equations have been obtained. The resulting governing equations are solved using 6^{th} order R-K methods by *Mathematica* ND solver tool. The solutions are carried out for various physical values of a_6, b_3, S and for some fixed values of other coefficients ($c = 1$, $\rho = 1$, $c1 = 1$, $c2 = 1$, $\mu = 1$, $\mu_c = 1$, $p_r = 1$).

(i) The velocity increases by the increase of porosity parameter (S).

(ii) The fluid velocity increases by increase of coefficient of thermal stress mechanical interaction (a_6) while increases as increase of coefficient of thermal conductivity (b_3).

(iii) The temperature of fluid decreases by increase of porosity coefficient (S).

(iv) The temperature of fluid slowly increases by increase of coefficient of thermal stress mechanical interaction (a_6) and coefficient of thermal conductivity (b_3).

(v) The numerical solutions obtained are in excellent agreement with the obtained analytical solutions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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