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Research Article

Some Results in Cone Metric Space, Using Semi-Compatible and Reciprocally Continuous Mappings

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Abstract. In this paper, we establish a result in cone metric space by generalizing the theorem proved by Jain *et al.* (Compatibility and weak compatibility for four self maps in a cone metric space, *Bulletin of Mathematical Analysis and Applications* **2**(1) (2010), 15 – 24) by employing certain weaker conditions such as semi-compatible, reciprocally continuous and sub sequentially continuous mappings. Further, our result is supported by discussing a relevant example.

Keywords. Common fixed point, Coincidence point, Cone metric space, Semi-compatible, Reciprocally continuous and sub sequentially continuous mappings

Mathematics Subject Classification (2020). 54H25, 47H10

1. Introduction

S. Banach [3], a Polish mathematician and one of the pioneers of functional analysis, proposed a contraction principle that laid foundation for many fixed point theorem. Further, metric space has been generalized in a number of ways. In this process, Huang and Zhang [5] developed the concept of cone metric space, replacing the Banach space with an ordered Banach space over the set of real numbers and proved several results. Following that, Abbas *et al.* [2], Abbas and Rhoades [1], Rezapour and Hamlbarani [9] investigated some in cone metric spaces. Thereafter, Song *et al.* [11] have derived similar fixed point theorems using weakly compatibility in cone metric spaces. Recently, Abbas and Jungck [2] generalized the finding within a normal cone metric space using weak compatibility. In a similar way, Vetro [13] used weak compatibility to prove some fixed point theorem for two self-maps meeting a contractive condition. Later, Jain *et al*. [7] proved certain fixed point theorems for four self-maps through compatibility and weak compatibility that met a contractive condition. Our major goal is to extend the results of [7] by using semi-compatibility, A-reciprocally continuity, and sub sequential continuity in cone metric spaces.

2. Preliminaries

Definition 2.1 ([5]). Let E be a real Banach space. $P \subset E$ is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a,b \in \mathbb{R}$, $a,b \geq 0$ and $x,y \in P$ imply $ax+by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in \text{int}P$ (interior of P). A cone $P \subset E$ is called normal if there is a number $K > 0$ such that $\forall\ x,y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The normal constant of P is the least positive value that fulfills the aforementioned inequality. It is clear that $K \geq 1$.

Proposition 2.2 ([6]). *Consider cone P is in a real Banach space. If $a \in P$ and $a \leq ka$, for some $k \in [0,1)$ then $a = 0$.*

Proposition 2.3 ([6]). *Consider cone P is in a real Banach space E. If for $a \in P$ and $a \ll c$, $\forall\ c \in P^0$ (interior point) then $a = 0$.*

Definition 2.4 ([5]). Let X be a non-empty set and E a real Banach space with cone P. A vector-valued function $d : X \times X \to P$ is said to be a cone metric space on X with the constant $K \geq 1$ if the following conditions are satisfied:

(d_1) $d(x,y) > 0$ and $d(x,y) = 0$ if and only if $x = y\ \forall\ x,y \in X$;

(d_2) $(x,y) = d(y,x)\ \forall\ x,y \in X$;

(d_3) $(x,y) \leq K(d(x,z)+d(y,z))\ \forall\ x,y,z \in X$.

Then d is called a cone metric in X and (X,d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Definition 2.5 ([5])**.** Let (X,d) be a cone metric space. We say that a sequence

(i) $(x_\eta)_{\eta\in N} \subseteq X$ is a cauchy sequence if for all $\epsilon > 0$ there exists an $N_\epsilon \in N$ such that for all $\eta, m > N$, $d(x_\eta, x_m) < \epsilon$, if $\lim_{\eta,m\to+\infty} d(x_\eta, x_m) = 0$.

(ii) Convergent sequence if for every $c \in E$ with $0 \ll c$, there is an N such that for all $\eta > N$, $d(x_\eta, x) \ll c$ for some fixed $x \in X$.

(iii) A cone metric space X is said to be complete if every cauchy sequence in X is convergent in X.

Definition 2.6. Cone metric space (X,d) is In a real Banach space E, let P be a cone. If $u \le v$, $v \ll w$ then $u \ll w$.

Lemma 2.7. *Cone metric space (X,d) is a real Banach space E, let P be a cone and k_1,k_2,k_3, $k_4,k > 0$. If $x_\eta \to x$, $y_\eta \to y$, $z_\eta \to z$ and $P_\eta \to p$ in X and $ka \le k_1 d(x_\eta,x) + k_2 d(y_\eta,y) + k_3 d(z_\eta,z) + k_3 d(P_\eta,p)$, then $a = 0$.*

Definition 2.8. Two self-maps F and G of a set X are *occasionally weakly compatible* (OWC) if and only if there is a point x in X which is a coincidence point of F and G at which F and G commutes.

Example 2.9. Define cone metric space (X,d) a with partial ordering $\le$ and $E = \mathbb{R}^2$,

$$P = \{(y,z) \in E \mid y,z > 0\} \subset \mathbb{R}^2, \quad X = [0,\infty), \; d : X \times X \to E,$$

in order for $d(y,z) = (|y-z|, \alpha|y-z|)$, where $\alpha \ge 0$ is some constant.
Define the self-mappings A,B and S,T. On $X = [0,\infty)$ as

$$A(x) = x^3 \quad \forall\, x \in [0,\infty)$$

and

$$S(x) = 3x^2, \text{ if } x \in [0,\infty).$$

We see that the pair has coincidence points (A,S) at $0,3$.
At $x = 3$

$$A(3) = 27 = S(3)$$

but

$$AS(3) = A(27) = 19683 \ne SA(3) = S(27) = 2187.$$

At $x = 0$, $A(0) = 0 = S(0)$ and

$$AS(0) = A(0) = 0 = SA(0) = S(0).$$

This indicates that pair (A,S) is OWC, but not weakly compatible.

Definition 2.10 ([7])**.** The self-mappings pair (A,S) on a cone metric space (X,d) is claimed to be semi-compatible, if $\lim_{\eta\to\infty} ASx_\eta = St$, whenever $\{x_\eta\}$ is a sequence in X such that $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Sx_\eta = t$, for some $t \in X$.

Example 2.11 ([7]). In a cone metric space (X,d), define self-mappings A,B and S,T.
On $X=[1,\infty)$ as

$$A(x)=\begin{cases} x, & \text{if } x\le 1,\\ 3x+1, & \text{if } x>1,\end{cases} \quad \text{and} \quad S(x)=\begin{cases} 2x-1, & \text{if } x\le 1,\\ 2x+2, & \text{if } x\in(1,4)\cup(4,\infty),\\ 13, & \text{if } x=4.\end{cases}$$

Consider a sequence $x_\eta=1+\frac{1}{\eta}$ for $\eta>1$.
Then

$$Ax_\eta=A\left(1+\frac{1}{\eta}\right)=3\left(1+\frac{1}{\eta}\right)+1\to\ 4$$

and

$$Sx_\eta=S\left(1+\frac{1}{\eta}\right)=2\left(1+\frac{1}{\eta}\right)+2\to 4, \quad \text{as } \eta\to\infty.$$

Now

$$ASx_\eta=AS\left(1+\frac{1}{\eta}\right)=A\left(4+\frac{2}{\eta}\right)=3\left(4+\frac{2}{\eta}\right)+1\to 13=S(4),$$

and

$$SAx_\eta=SA\left(1+\frac{1}{\eta}\right)=S\left(4+\frac{3}{\eta}\right)=2\left(4+\frac{3}{\eta}\right)+2=10, \quad \text{as } \eta\to\infty.$$

Then $ASx_\eta\to 13$ and $SAx_\eta\to 10$, as $\eta\to\infty$.
Hence the pair self-mappings (A,S) is semi-compatible but not compatible.

Definition 2.12 ([4]). The self-mappings pair (A,S) on a cone metric space (X,d) is mentioned to be sub-sequentially continuous if a sequence $\{x_\eta\}$ exists in X as well as $\lim_{\eta\to\infty} Ax_\eta=\lim_{\eta\to\infty} Sx_\eta=t$, in some cases $t\in X$ as well as $\lim_{\eta\to\infty} ASx_\eta=At$ and $\lim_{\eta\to\infty} SAx_\eta=St$.

If the self-mappings A,S is continuous, hence reciprocally continuous mappings but not sub-sequentially continuous as discussed below.

Example 2.13. In a cone metric space (X,d), define self-mappings A,B and S,T.
On $X\ =[1,\infty)$ as

$$A(x)=\begin{cases} x, & \text{if } x\le 1,\\ 3x-1, & \text{if } x\in(1,8)\cup(8,\infty),\\ 47, & \text{if } x=8\end{cases}$$

and

$$S(x)=\begin{cases} 2x-1, & \text{if } x\le 1,\\ x^2-1, & \text{if } x>1.\end{cases}$$

Consider a sequence $x_\eta=3-\frac{1}{\eta}$ for $\eta>1$.
Then

$$Ax_\eta=A\left(3-\frac{1}{\eta}\right)=3\left(3-\frac{1}{\eta}\right)-1=8-\frac{3}{\eta}\to 8$$

and

$$Sx_\eta = S\left(3-\frac{1}{\eta}\right) = \left(3-\frac{1}{\eta}\right)^2 - 1 \to 8, \quad \text{as } \eta \to \infty.$$

Now

$$ASx_\eta = AS\left(3-\frac{1}{\eta}\right) = A\left(\left(3-\frac{1}{\eta}\right)^2 - 1\right) = 3\left(\left(3-\frac{1}{\eta}\right)^2 - 1\right) - 1 = 23 \neq 47 = A(8)$$

and

$$SAx_\eta = SA\left(3-\frac{1}{\eta}\right) = S\left(8-\frac{3}{\eta}\right) = \left(8-\frac{3}{\eta}\right)^2 - 1 = 63 = S(8), \quad \text{as } \eta \to \infty.$$

However, for a sequence $x_\eta = 1-\frac{1}{\eta}$ for $\eta \geq 1$, then

$$Ax_\eta = 1-\frac{1}{\eta} \to 1$$

and

$$Sx_\eta = 1-\frac{1}{\eta} \to 1 \text{ as } \eta \to \infty.$$

Now

$$ASx_\eta = AS\left(1-\frac{1}{\eta}\right) = 2\left(1-\frac{1}{\eta}\right) - 1 = A(1) = 1 = A(1)$$

and

$$SAx_\eta = SA\left(1-\frac{1}{\eta}\right) = 2\left(1-\frac{1}{\eta}\right) - 1 = 1 = S(1).$$

Therefore, the mappings A, S are sub-sequentially continuous but not continuous.

Definition 2.14 ([8])**.** The self-mappings pair (A, S) on a cone metric space (X, d) is called reciprocally continuous if for each sequence $\{x_\eta\}$ in X, $\lim_{\eta\to\infty} ASx_\eta = At$ and $\lim_{\eta\to\infty} SAx_\eta = St$, whenever $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Sx_\eta = t$ for some $t \in X$.

Further reciprocally continuous mappings can be divided into A-reciprocally continuous and S-reciprocally continuous mappings.

Definition 2.15. The self-mappings pair (A, S) on a cone metric space (X, d) is called A-reciprocally continuous if for each sequence $\{x_\eta\}$ in X, $\lim_{\eta\to\infty} ASx_\eta = At$, whenever $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Sx_\eta = t$, for some $t \in X$.

Definition 2.16. The self-mappings pair (A, S) on a cone metric space (X, d) is called S-reciprocally continuous if for each sequence $\{x_\eta\}$ in X, $\lim_{\eta\to\infty} SAx_\eta = St$, whenever $\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} Sx_\eta = t$, for some $t \in X$.

Reciprocally continuous implies A-reciprocally continuous and S-reciprocally continuous but not conversely. We present a counter example as following.

Example 2.17. In a cone metric space (X,d), define the self-mappings A,B and S,T, on $X = [1,\infty)$ as

$$Ax = Bx = \begin{cases} \frac{x}{3}, & \text{if } x \in (-\infty,1), \\ 4x-3, & \text{if } x \in [1,\infty), \end{cases} \quad \text{and} \quad Sx = Tx = \begin{cases} x+2, & \text{if } x \in (-\infty,1), \\ 3x-2, & \text{if } x \in [1,\infty). \end{cases}$$

Consider a sequence $\{x_\eta\} = \left\{1+\frac{1}{\eta}\right\}$, $\eta \in N$ in X.

Then

$$\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} A\left(1+\frac{1}{\eta}\right) = 4\left(1+\frac{1}{\eta}\right)-3 = \left(1+\frac{4}{\eta}\right) = 1,$$

$$\lim_{\eta\to\infty} Sx_\eta = \lim_{\eta\to\infty} S\left(1+\frac{1}{\eta}\right) = 3\left(1+\frac{1}{\eta}\right)-2 = \left(1+\frac{3}{\eta}\right) = 1, \quad \text{as } \eta\to\infty.$$

Also,

$$\lim_{\eta\to\infty} ASx_\eta = \lim_{\eta\to\infty} AS\left(1+\frac{1}{\eta}\right) = \lim_{\eta\to\infty} A\left(1+\frac{3}{\eta}\right) = 4\left(1+\frac{3}{\eta}\right)-3 \to 1 = A(1),$$

$$\lim_{\eta\to\infty} SAx_\eta = \lim_{\eta\to\infty} SA\left(1+\frac{1}{\eta}\right) = \lim_{\eta\to\infty} S\left(1+\frac{4}{\eta}\right) = 3\left(1+\frac{4}{\eta}\right)-2 \to 1 = S(1), \quad \text{as } \eta\to\infty.$$

Consider another sequence $\{x_\eta\} = \left\{\frac{1}{\eta}-3\right\}$, $\eta \in N$ in X.

Then

$$\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty}\left(\frac{1}{3\eta}-1\right) = -1 \ \text{ as } \ \eta\to\infty.$$

$$\lim_{\eta\to\infty} Sx_\eta = \lim_{\eta\to\infty}\left(\frac{1}{\eta}-3\right) = \frac{1}{\eta}-3+2 = \frac{1}{\eta}-1 \to -1, \ \text{ as } \eta\to\infty.$$

Next,

$$\lim_{\eta\to\infty} ASx_\eta = \lim_{\eta\to\infty} AS\left(\frac{1}{\eta}-3\right) = \lim_{\eta\to\infty} A\left(\left(\frac{1}{\eta}-3\right)+2\right) = \lim_{\eta\to\infty} A\left(\frac{1}{\eta}-1\right) = \frac{1}{3}\left(\frac{1}{\eta}-1\right) \to -\frac{1}{3}$$
$$= \frac{-1}{3} = A(-1),$$

$$\lim_{\eta\to\infty} SAx_\eta = \lim_{\eta\to\infty} SA\left(\frac{1}{\eta}-3\right) = \lim_{\eta\to\infty} S\left(-1+\frac{1}{3\eta}\right) = \left(-1+\frac{1}{3\eta}\right)+2 \to 1 = 1 = S(-1), \text{ as } \eta\to\infty.$$

Thus, the self-mappings pair (A,S) is A-reciprocally continuous but neither continuous nor reciprocally continuous.

Now we present a theorem by Jain *et al*. [7].

Theorem (α). *Let (X,d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let A,B and S,T be self-mappings on X satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(ii) *the pair (A,S) is compatible and the pair (B,T) is weakly compatible;*

(iii) *one of A or S is continuous;*

(iv) $d(Ax,By) \le \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty) + \gamma[d(Ax,Ty)+d(Sx,By)]$.

for some $\lambda,\gamma,\delta,\mu \in [0\ 1)$ with $\lambda+\mu+\delta+2\gamma < 1$, $\forall\ x,y \in X$.

Then A,B,S and T will be having a single common fixed point in X.

The aforementioned result can be generalized in the following way.

3. Main Result

Theorem 3.1. *Complete cone metric space (X,d) is with respect to a cone P contained in a real Banach space E. Let A,B and S,T be self-mappings on X satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(ii) *the pair (A,S) is semi-compatible and A-reciprocally continuous and the pair (B,T) is weakly compatible;*

(iii) $d(Ax,By) \le \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty) + \gamma[d(Ax,Ty) + d(Sx,By)]$

for some $\lambda,\gamma,\delta,\mu \in [0,1)$ with $\lambda+\mu+\delta+2\gamma < 1$ with $\forall\ x,y \in X$.
Then A,B and S,T having a single common fixed point in X.

Proof. Consider $x_0 \in X$ be any arbitrary point. Using (3.3) assemble sequences $\{x_\eta\}$, and $\{y_\eta\}$ in X in order for

$$Ax_{2\eta} = Tx_{2\eta+1} = y_{2\eta} \text{ and } Bx_{2\eta+1} = Sx_{2\eta+2} = y_{2\eta+1}, \quad \eta \ge 0. \tag{3.1}$$

We show that $\{y_\eta\}$ is a cauchy sequence.
Substitute $x = x_{2\eta}$, $y = x_{2\eta+1}$ in (3.3) we get

$$\begin{aligned} d(Ax_{2\eta},Bx_{2\eta+1}) &= \lambda d(Ax_{2\eta},Sx_{2\eta}) + \mu d(Bx_{2\eta+1},Tx_{2\eta+1}) + \delta d(Sx_{2\eta},Tx_{2\eta+1}) \\ &\quad + \gamma[d(Ax_{2\eta},Tx_{2\eta+1}) + d(Sx_{2\eta},Bx_{2\eta+1})]. \end{aligned}$$

Using (3.1), we get

$$\begin{aligned} d(y_{2\eta},y_{2\eta+1}) &= \lambda d(y_{2\eta},y_{2\eta-1}) + \mu d(y_{2\eta+1},y_{2\eta}) + \delta d(y_{2\eta-1},y_{2\eta}) + \gamma[d(y_{2\eta},y_{2\eta}) + d(y_{2\eta-1},y_{2\eta+1})] \\ &= \lambda d(y_{2\eta},y_{2\eta-1}) + \mu d(y_{2\eta+1},y_{2\eta}) + \delta d(y_{2\eta-1},y_{2\eta}) + \gamma[d(y_{2\eta-1},y_{2\eta}) + d(y_{2\eta},y_{2\eta+1})]. \end{aligned}$$

Writing $d(y_\eta,y_{\eta+1}) = d_\eta$, we have

$$d_{2\eta} < \lambda d_{2\eta-1} + \mu d_{2\eta} + \delta d_{2\eta-1} + \gamma[d_{2\eta} + d_{2\eta-1}].$$

That is $(1-\gamma-\mu)\, d_{2\eta} = (\lambda+\gamma+\delta)d_{2\eta-1}$ which implies

$$d_{2\eta} = hd_{2\eta-1}, \tag{3.2}$$

where $h = \frac{(\lambda+\gamma+\delta)}{(1-\gamma-\mu)}$.
In view of (3.3), $h < 1$.
Now substitute $x = x_{2\eta+2}$, $y = x_{2\eta+1}$ in (3.3) we get

$$\begin{aligned} d(Ax_{2\eta+2},Bx_{2\eta+1}) &= \lambda d(Ax_{2\eta+2},Sx_{2\eta+2}) + \mu d(Bx_{2\eta+1},Tx_{2\eta+1}) + \delta d(Sx_{2\eta+2},Tx_{2\eta+1}) \\ &\quad + \gamma[d(Ax_{2\eta+2},Tx_{2\eta+1}) + d(Sx_{2\eta+2},Bx_{2\eta+1})]. \end{aligned}$$

Using (3.1), we get

$$d(y_{2\eta+2},y_{2\eta+1}) = \lambda d(y_{2\eta+2},y_{2\eta+1}) + \mu d(y_{2\eta+1},y_{2\eta}) + \delta d(y_{2\eta+1},y_{2\eta})$$

$$
\begin{aligned}
&+\gamma[d(y_{2\eta+2},y_{2\eta})+d(y_{2\eta+1},y_{2\eta+1})]\\
&=\lambda d(y_{2\eta+2},y_{2\eta+1})+\mu d(y_{2\eta+1},y_{2\eta})+\delta d(y_{2\eta+1},y_{2\eta})\\
&\quad+\gamma[d(y_{2\eta+2},y_{2\eta+1})+d(y_{2\eta+1},y_{2\eta})].
\end{aligned}
$$

So, we have

$$d_{2\eta+1}<\lambda d_{2\eta+1}+\mu d_{2\eta}+\delta d_{2\eta}+\gamma[d_{2\eta+1}+d_{2\eta}].$$

That is $(1-\gamma-\mu)\, d_{2\eta+1}=(\lambda+\gamma+\delta)d_{2\eta}$ which implies

$$d_{2\eta+1}=kd_{2\eta}, \tag{3.3}$$

where $k=\frac{(\lambda+\gamma+\delta)}{(1-\gamma-\mu)}$.

By condition (3.3), $k<1$.

In view of (3.2) and (3.3) we have,

$$d_{2\eta+1}=kd_{2\eta}=khd_{2\eta-1}=k^2hd_{2\eta-2}=\cdots=k^{\eta+1}h^{\eta}d_0, \quad \text{where } d_0=d(y_0,y_1)$$

and

$$d_{2\eta}=hd_{2\eta-1}=hkd_{2\eta-2}=h^2kd_{2\eta-3}=\cdots=h^{\eta}k^{\eta}d_0, \quad \text{where } d_0=d(y_0,y_1).$$

Therefore,

$$d_{2\eta+1}=k^{\eta+1}h^{\eta}d_0 \quad \text{and} \quad d_{2\eta}=h^{\eta}k^{\eta}d_0.$$

Also,

$$d(y_{\eta+l},y_{\eta})=d(y_{\eta+l},y_{\eta+l-1})+d(y_{\eta+l-1},y_{\eta+l-2})+\cdots+d(y_{\eta+1},y_{\eta}).$$

That is,

$$d(y_{\eta+l},y_{\eta})=d_{\eta+l-1}+d_{\eta+l-2}+\cdots+d_{\eta}. \tag{3.4}$$

If $\eta+l-1$ is even then by (3.4) we have

$$
\begin{aligned}
d(y_{\eta+l},y_{\eta})&=(h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}+h^{(\eta+l-1)/2}k^{(\eta+l)/2}+\cdots+)d_0\\
&=h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}[1+k+hk+hk^2+h^2k^2+\dots]d_0\\
&=h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}[(1+hk+h^2k^2+\dots)+(k+hk^2+h^2k^3+\dots)]d_0\\
&=h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}[(1+hk+h^2k^2+\dots)+k(1+hk+h^2k^2+\dots)]d_0\\
&=h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}(1+k)(1+hk+h^2k^2+\dots)d_0\\
&=h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}(1+k)(1-hk)d_0.
\end{aligned}
$$

As $hk<1$, P is closed, then

$$d(y_{\eta+l},y_{\eta})=h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}(1+k)(1-hk)d_0. \tag{3.5}$$

Now for $c\in P^0$, there exists $r>0$ such that $c-y\in P^0$ if $\|y\|<r$.

Choose a positive integer N_c then $\forall\ \eta=N_c$, then

$$\|h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}(1+k)(1-hk)d_0\|<r$$

which implies $c - h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}(1+k)(1-hk)d_0 \in P^0$ and

$$h^{(\eta+l-1)/2}k^{(\eta+l-1)/2}(1+k)(1-hk)d_0 - d(y_{\eta+l}, y_\eta) \in P \text{ on using (3.5)}$$

So, we have $c - d(y_{\eta+l}, y_\eta) \in P^0$, $\forall\ \eta = N_c$ and $\forall\ p$ by Proposition 2.6.
The same thing is true if $\eta + l - 1$ is odd.
This implies $d(y_{\eta+l}, y_\eta) \ll c$, $\forall\ \eta > N_c$, $\forall\ p$.
Hence $\{y_\eta\}$ is a cauchy sequence in X, which is complete.
As $\{y_\eta\} \to u \in X$ implies as

$$\{Ax_{2\eta}\} \to u \text{ and } \{Bx_{2\eta+1}\} \to u, \tag{3.6}$$

$$\{Sx_{2\eta}\} \to u \text{ and } \{Tx_{2\eta+1}\} \to u, \tag{3.7}$$

$$\lim_{\eta\to\infty} Ax_{2\eta} = \lim_{\eta\to\infty} Sx_{2\eta} = u. \tag{3.8}$$

Because the self-mappings pair (A, S) is semi-compatible

$$\lim_{\eta\to\infty} ASx_{2\eta} = Su \text{ and } \lim_{\eta\to\infty} Ax_{2\eta} = \lim_{\eta\to\infty} Sx_{2\eta} = u \text{ for some } u \in X. \tag{3.9}$$

Also the self-mappings pair (A, S) is A-reciprocally continuous

$$\lim_{\eta\to\infty} ASx_{2\eta} = Au. \tag{3.10}$$

From (3.9) and (3.10) we get

$$Au = Su. \tag{3.11}$$

Now

$$\begin{aligned} d(Su, u) &\le d(Su, Ax_{2\eta}) + d(Ax_{2\eta}, Bx_{2\eta+1}) + d(Bx_{2\eta+1}, u) \\ &= d(Su, ASx_{2\eta}) + d(y_{2\eta+1}, u) + d(ASx_{2\eta}, Bx_{2\eta+1}). \end{aligned}$$

Using (3.3) with $x = x_{2\eta}$ and $y = x_{2\eta+1}$ we have

$$\begin{aligned} d(Su, u) &\le d(Su, ASx_{2\eta}) + d(y_{2\eta+1}, u) + \lambda d(ASx_{2\eta}, Sx_{2\eta}) + \mu d(Bx_{2\eta+1}, Tx_{2\eta+1}) \\ &\quad + \delta d(Sx_{2\eta}, Tx_{2\eta+1}) + \gamma[d(ASx_{2\eta}, Tx_{2\eta+1}) + d(Sx_{2\eta}, Bx_{2\eta+1}) \\ &\le d(Su, ASx_{2\eta}) + d(y_{2\eta+1}, u) + \lambda d(ASx_{2\eta}, u) + \mu d(y_{2\eta+1}, y_{2\eta}) \\ &\quad + \delta d(u, y_{2\eta}) + \gamma[d(ASx_{2\eta}, y_{2\eta}) + d(u, y_{2\eta+1})] \\ &\le d(Su, ASx_{2\eta}) + d(y_{2\eta+1}, u) + \lambda[d(ASx_{2\eta}, Su) + d(Su, u)] \\ &\quad + \mu[d(y_{2\eta+1}, u) + d(u, y_{2\eta})] + \delta[d(u, Su) + d(Su, u) + d(y_{2\eta}, u)] \\ &\quad + \gamma[d(ASx_{2\eta}, Su) + d(Su, u) + d(u, y_{2\eta}) + d(u, y_{2\eta+1})] \end{aligned}$$

this implies

$$(1-\lambda-2\delta-\gamma)d(Su, u) \le (1+\lambda+\gamma)d(ASx_{2\eta}, Su) + (1+\mu+\gamma)d(y_{2\eta+1}, u) + (\mu+\delta+\gamma)d(u, y_{2\eta})$$

as $ASx_{2\eta} \to Su$, $\{y_{2\eta}\} \to u$ and $\{y_{2\eta+1}\} \to u$.
Then by Lemma 2.7 we have

$$d(Su, u) = 0 \text{ and hence } Su = u. \tag{3.12}$$

Now

$$d(Au,Su) \le d(Au,Bx_{2\eta+1}) + d(Bx_{2\eta+1},Su)$$
$$= d(y_{2\eta+1},Su) + d(Au,Bx_{2\eta+1}).$$

Using (3.3) with $x = u$ and $y = x_{2\eta+1}$, we have

$$\begin{aligned} d(Au,Su) &\le d(y_{2\eta+1},Su) + \lambda d(Au,Su) + \mu d(Bx_{2\eta+1},Tx_{2\eta+1}) + \delta d(Su,Tx_{2\eta+1}) \\ &\quad + \gamma[d(Au,Tx_{2\eta+1}) + d(Bx_{2\eta+1},Su)] \\ &\le d(y_{2\eta+1},Su) + \lambda d(Au,Su) + \mu d(y_{2\eta+1},y_{2\eta}) + \delta d(Su,y_{2\eta}) \\ &\quad + \gamma[d(Au,y_{2\eta}) + d(y_{2\eta+1},Su)] \\ &\le d(y_{2\eta+1},Su) + \lambda d(Au,Su) + \mu[d(y_{2\eta+1},Su) + d(Su,y_{2\eta})] + \delta d(Su,y_{2\eta}) \\ &\quad + \gamma[d(Au,Su) + d(Su,y_{2\eta}) + d(y_{2\eta+1},Su)]. \end{aligned}$$

So

$$(1-\lambda-\gamma)d(Au,Su) \le (\mu+\delta+\gamma)d(y_{2\eta},Su) + (1+\mu+\gamma)d(y_{2\eta+1},Su).$$

Using (3.12) $Su = u$, we have

$$(1-\lambda-\gamma)d(Au,u) \le (\mu+\delta+\gamma)d(u,u) + (1+\mu+\gamma)d(u,u).$$

As $\{y_{2\eta}\} \to u$ and $\{y_{2\eta+1}\} \to u$.
By Lemma 2.7 we get

$$d(Au,u) = 0$$

and we get

$$Au = u \tag{3.13}$$

From (3.11) and (3.12), we get

$$Au = Su = u.$$

Thus u is a coincidence point of intersection (A,S).
As $A(X) \subseteq T(X)$, $\exists\ v \in X$ with $u = Au = Tv$, then

$$u = Au = Su = Tv. \tag{3.14}$$

Substitute $x = u$ and $y = v$ in (3.3) we have

$$d(Au,Bv) \le \lambda d(Au,Su) + \mu d(Bv,Tv) + \delta d(Su,Tv) + \gamma[d(Au,Tv) + d(Su,Bv)].$$

Using (3.10) we have

$$d(u,Bv) \le (\mu+\gamma)d(u,Bv)$$

As $\mu+\gamma < 1$, by Proposition 2.2, it gives and hence $d(Bv,u) = 0$ and we get

$$Bv = u$$

thus $Bv = Tv = u$ as the self-mappings (B,T) is weakly compatible we get

$$Bu = Tu.$$

Substitute $x = u$, $y = u$ in (3.4) and using $Au = Su$, $Bu = Tu$ we get

$$d(Au, Bu) \le (\delta + 2\gamma)d(Au, Bu).$$

Hence $Au = Bu$, by Proposition 2.2 as $\delta + 2\gamma < 1$ and we have

$$u = Au = Su = Bu = Tu.$$

In this case, Thus u is common fixed point between the four self-maps A, B and S, T.

Uniqueness:

w is another common fixed point.

Let $w = Aw = Bw = Sw = Tw$.

Taking $x = u$ and $y = w$ in (3.3) we get

$$d(Au, Bw) = \lambda d(Au, Su) + \mu d(Bw, Tw) + \delta d(Su, Tw) + \gamma[d(Au, Tw) + d(Su, Bw)].$$

Hence $d(u, w) = (\delta + 2\gamma)d(u, w)$ by Proposition 2.2, as $\delta + 2\gamma < 1$,

$$d(u, w) = 0, \quad u = w.$$

Thus u is such required common fixed point for four self-maps A, B and S, T. □

Our theorem validated by discussing a relevant example.

Example 3.2. In cone metric space (X, d), the self-mappings A, B and S, T.

On $X = [0, \infty)$ define

$$Ax = Bx = \begin{cases} x^2, & \text{if } x \in [0,2], \\ 0, & \text{if } x \in (2,4] \end{cases} \quad \text{and} \quad Sx = Tx = \begin{cases} (\sqrt{2})x, & \text{if } x \in [0,2), \\ 4, & \text{if } x \in [2,4), \\ 0, & \text{if } x = 4. \end{cases}$$

Consider a sequence $x_\eta = \frac{\sqrt{2}}{\eta}$, $\eta \in N$ in X. Then $x = 0$, $\sqrt{2}$ are coincidence points of B, T.

At $x = 0$,

$$B(0) = T(0) = 0 \quad \text{and} \quad BT(0) = TB(0)$$

At $x = \sqrt{2}$,

$$B(\sqrt{2}) = 2 = T(\sqrt{2}),$$

$$BT(\sqrt{2}) = B(2) = 4, \; TB(\sqrt{2}) = T(2) = 4,$$

$$B(\sqrt{2}) = T(\sqrt{2}) \Rightarrow BT(\sqrt{2}) = TB(\sqrt{2}).$$

That implies the pair (B, T) is weakly compatibility. Now

$$\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} A\left(\frac{\sqrt{2}}{\eta}\right) = \lim_{\eta\to\infty}\left(\frac{\sqrt{2}}{\eta}\right)^2 = 0,$$

$$\lim_{\eta\to\infty} Sx_\eta = \lim_{\eta\to\infty} S\left(\frac{\sqrt{2}}{\eta}\right) = (\sqrt{2})\left(\frac{\sqrt{2}}{\eta}\right) = 0, \quad \text{as } \eta \to \infty$$

and

$$\lim_{\eta\to\infty} ASx_\eta = \lim_{\eta\to\infty} AS\left(\frac{\sqrt{2}}{\eta}\right) = \lim_{\eta\to\infty} A\left(\frac{2}{\eta}\right) = \lim_{\eta\to\infty}\left(\frac{2}{\eta}\right)^2 = 0 = S(0),$$

$$\lim_{\eta\to\infty} ASx_\eta = \lim_{\eta\to\infty} AS\left(\frac{\sqrt{2}}{\eta}\right) = \lim_{\eta\to\infty} A\left(\frac{2}{\eta}\right) = \lim_{\eta\to\infty}\left(\frac{2}{\eta}\right)^2 = 0 = A(0).$$

Consider another sequence $\{x_\eta\} = \{\sqrt{2} - \frac{1}{\eta}\}$, $\eta \in N$ in X. Then $x = 0$, $\sqrt{2}$ are coincidence points of A, S.

At $x = 0$,

$$\lim_{\eta\to\infty} Ax_\eta = \lim_{\eta\to\infty} A\left(\sqrt{2} - \frac{1}{\eta}\right) = \left(\sqrt{2}\right)^2 = 2 \text{ as } \eta \to \infty,$$

$$\lim_{\eta\to\infty} Sx_\eta = \lim_{\eta\to\infty} S\left(\sqrt{2} - \frac{1}{\eta}\right) = (\sqrt{2})\left(\sqrt{2} - \frac{1}{\eta}\right) = 2 \; as \; \eta \to \infty$$

and

$$\lim_{\eta\to\infty} ASx_\eta = \lim_{\eta\to\infty} AS\left(\sqrt{2} - \frac{1}{\eta}\right) = \lim_{\eta\to\infty} A(2) = (2)^2 = 4 = S(2),$$

$$\lim_{\eta\to\infty} ASx_\eta = \lim_{\eta\to\infty} AS\left(\sqrt{2} - \frac{1}{\eta}\right) = \lim_{\eta\to\infty} A(2) = (2)^2 = 4 = A(2),$$

$$\lim_{\eta\to\infty} SAx_\eta = \lim_{\eta\to\infty} SA\left(\sqrt{2} - \frac{1}{\eta}\right) = \lim_{\eta\to\infty} S(2) = 4 \neq S(0).$$

Thus, the self-mappings (A, S) is semi-compatible as well as A-reciprocally continuous. Further the pair (B, T) is weakly compatible. Moreover, at $x = 0$, $A(0) = S(0) = B(0) = T(0) = 0$.
But the pair of mappings (A, S) is neither compatible nor reciprocally continuous.
Now we proved contractive condition different cases as the following.

Case I: If $x \in [0, 2)$

$$d(Ax, By) \le \lambda d(Ax, Sx) + \mu d(By, Ty) + \delta d(Sx, Ty) + \gamma[d(Ax, Ty) + d(Sx, By)],$$
$$d(Ax, Bx) \le \lambda d(Ax, Sx) + \mu d(Bx, Tx) + \delta d(Sx, Tx) + \gamma[d(Ax, Tx) + d(Sx, Bx)],$$
$$d(x^2, x^2) \le \lambda d(x^2, \sqrt{2}x) + \mu d(x^2, \sqrt{2}x) + \delta d(\sqrt{2}x, \sqrt{2}x) + \gamma[d(x^2, \sqrt{2}x) + d(\sqrt{2}x, x^2)],$$
$$0 \le \lambda d(x^2, \sqrt{2}x) + \mu d(x^2, \sqrt{2}x) + \delta(0) + \gamma[d(x^2, \sqrt{2}x) + d(\sqrt{2}x, x^2)],$$
$$0 \le (\lambda + \mu + 2\gamma) d(x^2, \sqrt{2}x).$$

$0 \le (\lambda + \mu + 2\gamma)$ so that contractive condition is satisfied.

Case II: If $x = 2$

$$d(Ax, By) \le \lambda d(Ax, Sx) + \mu d(By, Ty) + \delta d(Sx, Ty) + \gamma[d(Ax, Ty) + d(Sx, By)],$$
$$d(Ax, Bx) \le \lambda d(Ax, Sx) + \mu d(Bx, Tx) + \delta d(Sx, Tx) + \gamma[d(Ax, Tx) + d(Sx, Bx)],$$
$$0 \le 0.$$

So that inequalities satisfied.

Case III: If $x \in (2,4)$

$$d(Ax,By) \leq \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty) + \gamma[d(Ax,Ty) + d(Sx,By)],$$

$$d(Ax,Bx) \leq \lambda d(Ax,Sx) + \mu d(Bx,Tx) + \delta d(Sx,Tx) + \gamma[d(Ax,Tx) + d(Sx,Bx)],$$

$$d(0,0) \leq \lambda d(0,4) + \mu d(0,4) + \delta d(4,4) + \gamma[d(0,4) + d(4,0)],$$

$$0 \leq 4\lambda + 4\mu + 0\delta + 8\gamma,$$

$$0 \leq 4(\lambda + \mu + 2\gamma),$$

$$0 \leq (\lambda + \mu + 2\gamma).$$

Case IV: If $x = 4$

$$d(Ax,By) \leq \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty) + \gamma[d(Ax,Ty) + d(Sx,By)],$$

$$d(Ax,Bx) \leq \lambda d(Ax,Sx) + \mu d(Bx,Tx) + \delta d(Sx,Tx) + \gamma[d(Ax,Tx) + d(Sx,Bx)],$$

$$0 \leq 0.$$

If we take $\lambda = \frac{1}{4}$, $\mu = \frac{1}{8}$, $\gamma = \frac{1}{16}$ and $\delta = \frac{1}{3}$.
The contractive condition (3.3) of above said Theorem 3.1 holds true and 0 is the only common fixed point for the four maps A, B and S,T.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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