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Research Article

Common Fixed Point Theorems for OWC Maps Satisfying Property (E.A) in *S*-Metric Spaces Using an Inequality Involving Quadratic Terms

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Abstract. In this study, using a quadratic inequality, we prove certain fixed point theorems for four pair wise occasionally weakly compatible maps. In fact, we slightly modify the inequality used by G. V. R. Babu *et al*. [3, 4] and apply it to S-metric spaces. We also give an example to justify the relevance and reliability of our results.

Keywords. Coincidence points, Common fixed points, Property (E.A), Occasional weak compatibility, Common property (E.A)

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

Several authors have introduced various conditions (known as compatible conditions) in order to establish the presence of common fixed points. If the two mappings commute (G. Jungck [5]), it is the simplest technique to acquire common fixed points. However, because this is the strongest condition, it is quite natural to look for weaker conditions. In 1986, Jungck [6] established the idea of compatibility between two mappings. The idea of weak compatibility came into light by the work of Jungck and Rhoades [7]. Thagafi and Shahzad [2] presented occasional weak compatibility between two mappings in 2008, which is a weaker condition than weak compatibility. Aamri and Moutawakil [1] proposed the idea of property (E.A) in 2002, which is now extensively used by authors to verify common fixed points.

In recent years, several significant generalisations of conventional metric spaces have been established. One of these is S-metric space. S-metric space was first proposed by Sedghi *et al*. [9] in 2012. In fact, they introduced this new class of metric spaces as a generalisation of a G-metric (Mustafa and Sims [8]) and D^*-metric (Sedghi *et al*. [10]). We can easily see that many theorems in metric spaces hold good in S-metric spaces.

2. Preliminaries

Definition 2.1 ([9])**.** A function $S : X \times X \times X \to [0,\infty)$, where X is a nonempty set is said to be an S-metric if,
for each $p,q,r,a \in X$,

(i) $S(p,q,r) = 0$ if and only if $p = q = r$,

(ii) $S(p,q,r) \leq S(p,p,a) + S(q,q,a) + S(r,r,a)$.

The pair (X,S) is called an S-metric space.

Example 2.2 ([11])**.** The function $S : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ defined by $S(p,q,r) = |p-r| + |q-r|$ for all $p,q,r \in \mathbb{R}$ is an S-metric.

Lemma 2.3 ([9])**.** *In an S-metric space X, $S(p,p,q) = S(q,q,p)$ for every $p,q \in X$.*

Lemma 2.4 ([9])**.** *Let $\{p_n\}$ and $\{q_n\}$ are two sequences in an S-metric space X and let $p_n \to a$ and $q_n \to b$. Then $S(p_n,p_n,q_n) \to S(a,a,b)$.*

Definition 2.5 ([9])**.** A sequence $\{p_n\}$ in an S-metric space X is said to converge to some $a \in X$ iff $\lim_{n\to\infty} S(p_n,p_n,a) = 0$. In this case, we write $\lim_{n\to\infty} p_n = a$.

Definition 2.6. Let M,N be two self maps of an S- metric space X. Then we say that the pair (M,N)

(i) is weakly compatible [7], if $MNp = NMp$ for every $p \in X$ such that $Mp = Np$.

(ii) is occasionally weakly compatible (owc) [2], if $MNp = NMp$ for some $p \in X$ such that $Mp = Np$.

(iii) satisfy property (E.A) [1], if there is a sequence $\{p_n\}$ in X such that $\lim_{n\to\infty} Mp_n = \lim_{n\to\infty} Np_n = r$, $r \in X$.

Example 2.7. Let $X = \mathbb{R}$ and the mappings M and N on X be defined by $M(p) = 4p - 1$ and $N(p) = p + \frac{1}{2}$.

Let the S-metric be defined as in Example 2.2.
For the sequence $\{p_n\}$ given by

$$p_n = \frac{1}{2} + \frac{1}{n^2}, \quad n = 1,2,\dots,$$

$$Mp_n = 1 + \frac{4}{n^2} \text{ and } Np_n = 1 + \frac{1}{n^2},$$

$$S(Mp_n, Mp_n, 1) = S\left(1 + \frac{4}{n^2}, 1 + \frac{4}{n^2}, 1\right) = \frac{8}{n^2} \to 0, \quad \text{as } n \to \infty,$$

$$S(Np_n, Np_n, 1) = S\left(1 + \frac{1}{n^2}, 1 + \frac{1}{n^2}, 1\right) = \frac{2}{n^2} \to 0, \quad \text{as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} Mp_n = \lim_{n\to\infty} Np_n = 1.$$

So the pair (M,N) satisfy property (E.A).

Definition 2.8 (Liu *et al.* [13]). Let M,N,F and G be four self maps of an S-metric space X. If there exists two sequences $\{p_n\}$ and $\{q_n\}$ in X such that $\lim_{n\to\infty} Mp_n = \lim_{n\to\infty} Fp_n = \lim_{n\to\infty} Nq_n = \lim_{n\to\infty} Gq_n = r$, $r \in X$, then we say that the pairs (M,F) and (N,G) satisfy common property (E.A).

Example 2.9. Let $X = \mathbb{R}$ and the mappings M,N,F and G on X be defined by $M(p) = 4p - 1$, $F(p) = p + \frac{1}{2}$, $N(p) = 3p - 1$, $G(p) = p + \frac{1}{3}$.
Let the S-metric on X be defined as in Example 2.2.
For the sequences $\{p_n\}$ and $\{q_n\}$ given by

$$p_n = \frac{1}{2} + \frac{1}{n^2},$$

$$q_n = \frac{2}{3} + \frac{1}{\sqrt{n}}, \quad n = 1,2,3,\dots,$$

$$Mp_n = 1 + \frac{4}{n^2},\ Fp_n = 1 + \frac{1}{n^2},\ Nq_n = 1 + \frac{3}{\sqrt{n}} \text{ and } Gq_n = 1 + \frac{1}{\sqrt{n}},$$

$$S(Mp_n, Mp_n, 1) = S\left(1 + \frac{4}{n^2}, 1 + \frac{4}{n^2}, 1\right) = \frac{8}{n^2} \to 0, \quad \text{as } n \to \infty,$$

$$S(Fp_n, Fp_n, 1) = S\left(1 + \frac{1}{n^2}, 1 + \frac{1}{n^2}, 1\right) = \frac{2}{n^2} \to 0, \quad \text{as } n \to \infty,$$

$$S(Nq_n, Nq_n, 1) = S\left(1 + \frac{3}{\sqrt{n}}, 1 + \frac{3}{\sqrt{n}}, 1\right) = \frac{6}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty,$$

$$S(Gq_n, Gq_n, 1) = S\left(1 + \frac{1}{\sqrt{n}}, 1 + \frac{1}{\sqrt{n}}, 1\right) = \frac{2}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} Mp_n = \lim_{n\to\infty} Fp_n = \lim_{n\to\infty} Nq_n = \lim_{n\to\infty} Gq_n = 1.$$

So the pairs (M,F) and (N,G) satisfy common property (E.A).

Definition 2.10. A point $p \in X$ is said to be a coincidence point of two self maps M and N of X, if $Mp = Np$. The set of all coincidence points is denoted by $C(M,N)$.

Many authors, Tas *et al.* [12], Babu and Kameshwari [4], obtained common fixed points for four maps using quadratic inequality in metric spaces. Babu and Alemayehu [3] used property (E.A) and pair-wise occasional weak compatibility for this purpose. In our work, we slightly modify the inequality used by Babu and Alemayehu [3] and obtain analogous results in S-metric spaces. This study will improve their results. We shall give suitable examples to justify our results.

3. Main Results

Proposition 3.1. *Let X be an S-metric space and M,N,F and G be four self mappings of X satisfying the quadratic inequality*

$$\begin{aligned}[S(Mx,Mx,Ny)]^2 \le c_1 \max\{[S(Fx,Fx,Mx)]^2,[S(Gy,Gy,Ny)]^2\} \\ + c_2 \max\{S(Fx,Fx,Mx)S(Fx,Fx,Ny), \\ S(Gy,Gy,Ny)S(Gy,Gy,Mx), \\ S(Fx,Fx,Ny)S(Gy,Gy,Mx)\}\end{aligned} \tag{3.1}$$

for all $x,y \in X$, where $c_1,c_2 \in [0,1)$.
Suppose that either

(i) *the pair (N,G) satisfy property (E.A), $N(X) \subseteq F(X)$ and $G(X)$ is closed, or*

(ii) *the pair (M,F) satisfy property (E.A), $M(X) \subseteq G(X)$ and $F(X)$ is closed.*

Then $C(N,G) \neq \phi$ and $C(M,F) \neq \phi$.

Proof. Suppose that (i) holds.
The property (E.A) of (N,G) implies that there is some sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Nx_n = \lim_{n\to\infty} Gx_n = z, \quad z \in X. \tag{3.2}$$

Since $N(X) \subseteq F(X)$, $Nx_n = Fy_n$ for some sequence $\{y_n\}$ in X.
This implies

$$\lim_{n\to\infty} Fy_n = z. \tag{3.3}$$

We will now show that $\lim_{n\to\infty} My_n = z$.
We consider

$$\begin{aligned}[S(My_n,My_n,Nx_n)]^2 &\le c_1 \max\{[S(Fy_n,Fy_n,My_n)]^2,[S(Gx_n,Gx_n,Nx_n)]^2\} \\ &\quad + c_2 \max\{S(Fy_n,Fy_n,My_n)S(Fy_n,Fy_n,Nx_n), \\ &\quad\quad S(Gx_n,Gx_n,Nx_n)S(Gx_n,Gx_n,My_n), \\ &\quad\quad S(Fy_n,Fy_n,Nx_n)S(Gx_n,Gx_n,My_n)\} \\ &= c_1 \max\{[S(Nx_n,Nx_n,My_n)]^2,[S(Gx_n,Gx_n,Nx_n)]^2\} \\ &\quad + c_2 S(Gx_n,Gx_n,Nx_n)S(Gx_n,Gx_n,My_n).\end{aligned}$$

On taking limit superior in the above inequality and using (3.2) and (3.3),

$$\limsup_{n\to\infty}[S(My_n,My_n,Nx_n)]^2 \le c_1 \limsup_{n\to\infty}[S(My_n,My_n,Nx_n)]^2$$

a contradiction, if our claim is not true.
So, we must have

$$\limsup_{n\to\infty}[S(My_n,My_n,Nx_n)]^2 = 0,$$

which implies that

$$\lim_{n\to\infty}[S(My_n,My_n,Nx_n)]^2 = 0.$$

Hence,

$$\lim_{n\to\infty} My_n = \lim_{n\to\infty} Nx_n = z. \tag{3.4}$$

Since $G(X)$ is closed, by (3.2),

$$z = Gv, \quad v \in X. \tag{3.5}$$

Now, we prove that $Nv = z$.
To prove this, we consider,

$$\begin{aligned}[S(My_n,My_n,Nv)]^2 \le c_1 \max\{[S(Fy_n,Fy_n,My_n)]^2,[S(Gv,Gv,Nv)]^2\} \\ + c_2 \max\{S(Fy_n,Fy_n,My_n)S(Fy_n,Fy_n,Nv), \\ S(Gv,Gv,Nv)S(Gv,Gv,My_n), \\ S(Fy_n,Fy_n,Nv)S(Gv,Gv,My_n)\}.\end{aligned}$$

On letting $n \to \infty$ and using (3.3), (3.4) and (3.5), we have

$$[S(z,z,Nv)]^2 \le c_1[S(z,z,Nv)]^2,$$

a contradiction, if $Nv \neq z$ and hence, we must have

$$Nv = z. \tag{3.6}$$

From (3.5) and (3.6),

$$Nv = Gv = z. \tag{3.7}$$

Hence, $C(N,G) \neq \phi$.
Since $z \in N(X)$ and $N(X) \subseteq F(X)$,

$$z = Fu, \quad u \in X. \tag{3.8}$$

Now, we claim $z = Mu$.
To prove our claim, we consider

$$\begin{aligned}[S(Mu,Mu,Nv)]^2 \le c_1 \max\{[S(Fu,Fu,Mu)]^2,[S(Gv,Gv,Nv)]^2\} \\ + c_2 \max\{S(Fu,Fu,Mu)S(Fu,Fu,Nv), \\ S(Gv,Gv,Nv)S(Gv,Gv,Mu), \\ S(Fu,Fu,Nv)S(Gv,Gv,Mu)\}.\end{aligned}$$

On using (3.7) and (3.8), we get

$$[S(Mu,Mu,z)]^2 \leq c_1[S(Mu,Mu,z)]^2,$$

a contradiction if $z \neq Mu$.
Therefore, we must have

$$z = Mu. \tag{3.9}$$

From (3.8) and (3.9), $Mu = Fu = z$.
Hence $C(M,F) \neq \phi$.
In the similar way, the theorem holds under the assumption (ii). □

Theorem 3.2. *If the hypothesis of Proposition* 3.1 *holds and in addition to that, if the pairs* (M,G) *and* (N,F) *are occasionally weakly compatible, then the mappings* M,N,F *and* G *have a unique common fixed point.*

Proof. We can see that $C(M,F) \neq \phi$ and $C(N,G) \neq \phi$ from Proposition 3.1.
Since the pair (M,F) is owc,

$$MFp = FMp \quad \text{for some } p \in X$$

such that

$$Mp = Fp = r, \quad r \in X. \tag{3.10}$$

$MFp = FMp$ implies

$$Mr = Fr. \tag{3.11}$$

Since the pair (N,G) is owc,

$$NGq = GNq \quad \text{for some } q \in X$$

such that

$$Nq = Gq = s, \quad s \in X. \tag{3.12}$$

$NGq = GNq$ implies

$$Ns = Gs. \tag{3.13}$$

Now let

$$Mr = Fr = r' \text{ and } Ns = Gs = s' \quad \text{for some } r', s' \in X. \tag{3.14}$$

Now, we prove that $r' = s'$.
For this, we consider

$$\begin{aligned}[S(r',r',s')]^2 &= [S(Mr,Mr,Ns)]^2 \\ &\leq c_1 \max\{[S(Fr,Fr,Mr)]^2,[S(Gs,Gs,Ns)]^2\} \\ &\quad + c_2 \max\{S(Fr,Fr,Mr)S(Fr,Fr,Ns),S(Gs,Gs,Ns)S(Gs,Gs,Mr), \\ &\quad S(Fr,Fr,Ns)S(Gs,Gs,Mr)\}.\end{aligned}$$

On using (3.14), we will have

$$[S(r',r',s')]^2 \le c_2[S(r',r',s')]^2,$$

which implies

$$r' = s'. \tag{3.15}$$

Now we prove that $r = s'$.
For this,we take

$$\begin{aligned}[S(r,r,s')]^2 &= [S(Mp,Mp,Ns)]^2 \\ &\le c_1 \max\{[S(Fp,Fp,Mp)]^2,[S(Gs,Gs,Ns)]^2\} \\ &\quad + c_2 \max\{S(Fp,Fp,Mp)S(Fp,Fp,Ns), \\ &\qquad S(Gs,Gs,Ns)S(Gs,Gs,Mp), \\ &\qquad S(Fp,Fp,Ns)S(Gs,Gs,Mp)\}.\end{aligned}$$

This implies $[S(r,r,s')]^2 \le c_2 S(r,r,s')^2$ on using (3.10) and (3.14).
Hence,

$$r = s'. \tag{3.16}$$

Finally, we prove that $r = s$.
For this purpose, We take

$$\begin{aligned}[S(r,r,s)]^2 &= [S(Mp,Mp,Nq)]^2 \\ &\le c_1 \max\{[S(Fp,Fp,Mp)]^2,[S(Gq,Gq,Nq)]^2\} \\ &\quad + c_2 \max\{S(Fp,Fp,Mp)S(Fp,Fp,Nq), \\ &\qquad S(Gq,Gq,Nq)S(Gq,Gq,Mp), \\ &\qquad S(Fp,Fp,Nq)S(Gq,Gq,Mp)\}.\end{aligned}$$

On using (3.10) and (3.12), we get $[S(r,r,s)]^2 \le c_2[S(r,r,s)]^2$, which implies that

$$r = s. \tag{3.17}$$

From (3.15),(3.16) and (3.17), we have $r' = s' = r = s$.
From (3.14),

$$Mr = Fr = Nr = Gr = r. \tag{3.18}$$

To prove that r is unique, we suppose that r^* be a common fixed point of M,N,F and G such that $r \ne r^*$.
Therefore,

$$Mr^* = Fr^* = Nr^* = Gr^* = r^*. \tag{3.19}$$

Then from the inequality (3.10),

$$\begin{aligned}[S(r,r,r^*)]^2 &= [S(Mr,Mr,Nr^*)]^2 \\ &\le c_1 \max\{[S(Fr,Fr,Mr)]^2,[S(Gr^*,Gr^*,Nr^*)]^2\}\end{aligned}$$

$$+\, c_2 \max\{S(Fr,Fr,Mr)S(Fr,Fr,Nr^*),\; S(Gr^*,Gr^*,Nr^*)S(Gr^*,Gr^*,Mr),\; S(Fr,Fr,Nr^*)S(Gr^*,Gr^*,Mr)\}.$$

On using (3.18) and (3.19), we get

$$[S(r,r,r^*)]^2 \le c_2[S(r,r,r^*)]^2,$$

which implies

$$r = r^*.$$

□

Example 3.3. Let $X = [0,1]$ and the S-metric be given as in Example 2.2.
Then, the inequality (3.1) will be

$$|Mx - Ny|^2 \le c_1 \max\{|Fx - Mx|^2, |Gy - Ny|^2\} + c_2 \max\{|Fx - Mx|\,|Fx - Ny|, |Gy - Ny|\,|Gy - Mx|, |Fx - Ny|\,|Gy - Mx|\}. \tag{3.20}$$

Let the mappings M,N,F and G on X be defined by

$$M(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ \frac{1}{10}, & \text{if } x = 1, \end{cases}$$

$$N(x) = 0,$$

$$F(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ \frac{9}{10}, & \text{if } x = 1, \end{cases}$$

$$G(x) = \frac{x}{20}.$$

Here it is clear that $G(X)$ is closed and $N(X) \subseteq F(X)$.
We can observe that $F(X)$ is not closed and $M(X) \nsubseteq G(X)$.

Case I: Let $x \in [0,1)$. Then for every $y \in [0,1]$,

$$Mx = Ny = 0,\; Fx = x \;\text{ and }\; Gy = \frac{y}{20}.$$

Therefore, $|Mx - Ny| = 0$.
Hence inequality (3.20) is true for every $c_1, c_2 \in [0,1)$.

Case II: Let $x = 1$. Then for every $y \in [0,1]$,

$$Mx = \frac{1}{10},\; Fx = \frac{9}{10},\; Ny = 0 \;\text{ and }\; Gy = \frac{y}{20},$$

$$|Mx - Ny| = \frac{1}{10},\; |Fx - Mx| = \frac{4}{5},$$

$$\begin{aligned} |Mx - Ny|^2 = \frac{1}{100} < \frac{8}{25} = \frac{1}{2}|Fx - Mx|^2 \\ &\le \frac{1}{2}\max\{|Fx - Mx|^2, |Gy - Ny|^2\} \\ &\quad + c_2 \max\{|Fx - Mx|\,|Fx - Ny|, |Gy - Ny|\,|Gy - Mx|, |Fx - Ny|\,|Gy - Mx|\}. \end{aligned}$$

Then, the inequality (3.20) is true for $c_1 = \frac{1}{2}$ and $c_2 \in [0,1)$.
Then, the inequality (3.20) holds in both the cases for $c_1 = c_2 = \frac{1}{2}$.
Also, for the sequence $\{x_n\}$ in X given by

$$x_n = \frac{1}{n^3+1}, \quad n = 1,2,3,\dots,$$
$$S(Nx_n, Nx_n, 0) = 0,$$
$$S(Gx_n, Gx_n, 0) = S\left(\frac{1}{20(n^3+1)}, \frac{1}{20(n^3+1)}, 0\right) = \frac{1}{10(n^3+1)} \to 0 \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} Nx_n = \lim_{n\to\infty} Gx_n = 0.$$

Hence, it is obvious that (N,G) satisfy property (E.A).
Furthermore, (M,F) and (N,G) are owc.
Also, 0 is the only common fixed point of M,N,F and G.
Thus, Theorem 3.2 is justified with this example.

Proposition 3.4. *Let X be an S-metric space and M,N,F and G be four self mappings of X satisfying the quadratic inequality*

$$\begin{aligned}[S(Mx,Mx,Ny)]^2 \le c_1 \max\{[S(Fx,Fx,Mx)]^2, [S(Gy,Gy,Ny)]^2\} \\ + c_2 \max\{S(Fx,Fx,Mx)S(Fx,Fx,Ny), \\ S(Gy,Gy,Ny)S(Gy,Gy,Mx), \\ S(Fx,Fx,Ny)S(Gy,Gy,Mx)\} \end{aligned} \tag{3.21}$$

for all $x,y \in X$, where $c_1,c_2 \in [0,1)$.
Suppose that

(i) *$F(X)$ and $G(X)$ are closed,*

(ii) *the pairs (N,G) and (M,F) satisfy a common property (E.A).*

Then $C(N,G) \neq \phi$ and $C(M,F) \neq \phi$.

Proof. Since (M,F) and (N,G) satisfy a common property (E.A), there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Nx_n = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} My_n = \lim_{n\to\infty} Fy_n = z. \tag{3.22}$$

The closedness of $G(X)$ and $F(X)$ implies that

$$z = Gv = Fu, \quad \text{for some } u,v \in X. \tag{3.23}$$

Now, we consider

$$\begin{aligned}[S(My_n,My_n,Nv)]^2 \le c_1 \max\{[S(Fy_n,Fy_n,My_n)]^2, [S(Gv,Gv,Nv)]^2\} \\ + c_2 \max\{S(Fy_n,Fy_n,My_n)S(Fy_n,Fy_n,Nv), \\ S(Gv,Gv,Nv)S(Gv,Gv,My_n), \\ S(Fy_n,Fy_n,Nv)S(Gv,Gv,My_n)\}. \end{aligned}$$

On letting $n \to \infty$ and using (3.22) and (3.23), we have

$$[S(z,z,Nv)]^2 \le c_1[S(z,z,Nv)]^2.$$

This implies

$$z = Nv. \tag{3.24}$$

From (3.23) and (3.24), $Nv = Gv = z$. Hence $C(N,G) \neq \phi$.
Now, we consider

$$\begin{aligned}[S(Mu,Mu,z)]^2 &= [S(Mu,Mu,Nv)]^2 \\ &\le c_1 \max\{[S(Fu,Fu,Mu)]^2,[S(Gv,Gv,Nv)]^2\} \\ &\quad + c_2 \max\{S(Fu,Fu,Mu)S(Fu,Fu,Nv), \\ &\quad S(Gv,Gv,Nv)S(Gv,Gv,Mu), \\ &\quad S(Fu,Fu,Nv)S(Gv,Gv,Mu)\}.\end{aligned}$$

On letting $n \to \infty$ and using (3.23) and (3.24), we get

$$[S(Mu,Mu,z)]^2 \le c_1[S(Mu,Mu,z)]^2.$$

This implies,

$$Mu = z. \tag{3.25}$$

From (3.23) and (3.25), we get $Fu = Mu = z$.
This implies $C(M,F) \neq \phi$. □

Theorem 3.5. *If the hypothesis of Proposition* 3.4 *holds and in addition to that, if the pairs* (M,F) *and* (N,G) *are occasionally weakly compatible, then the mappings* M,N,F *and* G *have a unique common fixed point.*

Proof. We have $C(M,F) \neq \phi$ and $C(N,G) \neq \phi$ from Proposition 3.4.
The remaining proof of the theorem runs in the same lines of that of Theorem 3.2. □

Example 3.6. Let $X = [0,1]$ and the S-metric be given as in Example 2.2.
Then, the inequality (3.21) will be

$$\begin{aligned}|Mx-Ny|^2 &\le c_1 \max\{|Fx-Mx|^2,|Gy-Ny|^2\} \\ &\quad + c_2 \max\{|Fx-Mx|\,|Fx-Ny|,|Gy-Ny|\,|Gy-Mx|, \\ &\quad |Fx-Ny|\,|Gy-Mx|\}.\end{aligned} \tag{3.26}$$

Let the mappings M,N,F and G on X be defined by

$$M(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ \frac{1}{10}, & \text{if } x = 1, \end{cases}$$

$$N(x) = 0,\ G(x) = \frac{x}{20} \ \text{ and } \ F(x) = x.$$

It is clear that both $F(X)$ and $G(X)$ are closed.

Case I: Let $x \in [0,1)$. Then, for every $y \in [0,1]$,

$$Mx = Ny = 0 \text{ and hence } |Mx - Ny| = 0.$$

Therefore, the inequality (3.26) holds for every $c_1, c_2 \in [0,1)$.

Case II: Let $x = 1$. Then for every $y \in [0,1]$,

$$Mx = \frac{1}{10},\ Fx = 1,\ Ny = 0,$$

$$|Mx - Ny| = \frac{1}{10},\ |Fx - Mx| = \frac{9}{10}.$$

Hence,

$$\begin{aligned}|Mx - Ny|^2 = \frac{1}{100} < \frac{81}{200} = \frac{1}{2}|Fx - Mx|^2 \le \frac{1}{2}\max\{|Fx - Mx|^2, |Gy - Ny|^2\} \\ + c_2 \max\{|Fx - Mx||Fx - Ny|, \\ |Gy - Ny|\,|Gy - Mx|, \\ |Fx - Ny|\,|Gy - Mx|\}.\end{aligned}$$

Then, the inequality (3.26) holds for $c_1 = \frac{1}{2}$ and $c_2 \in [0,1)$.
Thus, the inequality (3.26) holds with $c_1 = c_2 = \frac{1}{2}$ in both the cases.
Also, if $\{x_n\}$ and $\{y_n\}$ are two sequences in X given by

$$x_n = \frac{1}{n} \text{ and } y_n = \frac{1}{n^2}, \quad n = 1,2,3,\dots$$

then

$$S(My_n, My_n, 0) = 0,$$

$$S(Fy_n, Fy_n, 0) = S\left(\frac{1}{n^2}, \frac{1}{n^2}, 0\right) = \frac{2}{n^2} \to 0 \text{ as } n \to \infty,$$

$$S(Nx_n, Nx_n, 0) = 0,$$

$$S(Gx_n, Gx_n, 0) = S\left(\frac{1}{20n}, \frac{1}{20n}, 0\right) = \frac{1}{10n} \to 0 \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} My_n = \lim_{n\to\infty} Fy_n = \lim_{n\to\infty} Nx_n = \lim_{n\to\infty} Gx_n = 0.$$

Hence it is clear that (N,G) and (M,F) satisfy common property (E.A).
Furthermore, (N,G) and (M,F) are occasionally weakly compatible.
We can also see that 0 is the only common fixed point of M, N, F and G.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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