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Research Article

On Nonlinear Fractional Relaxation Differential Equations

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Abstract. In this paper, by using the Krasnoselskii fixed point theorem and the Banach fixed point theorem, we prove the existence and uniqueness of solutions for a class of nonlocal Cauchy problem for nonlinear Caputo fractional relaxation differential equations. Finally, one illustrative example is given to demonstrate our results.

Keywords. Fixed points, Fractional relaxation differential equations, Mittag-Leffler, Existence, Uniqueness

Mathematics Subject Classification (2020). 34A08, 34A12

1. Introduction

It is now well known that the fractional calculus is an extension of the ordinary differentiation and integration into arbitrary non-integer order. Fractional differential equations are used in various fields of science and engineering, such as physics, chemistry, atomic energy, information theory, harmonic oscillator, applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, biology, medicine, nonlinear oscillations, conservative systems, and so on (see [3, 4, 12, 16–18] and the references therein).

Ardjouni and Djoudi [1] investigated the positivity of solutions of a nonlinear Caputo fractional differential equation of the type:

$$\begin{cases} {}^C D^{\alpha}(\xi(t) - g(t,\xi(t))) = f(t,\xi(t)), & 0 < t \le 1, \\ \xi(0) = \xi_0 > g(0,\xi_0) > 0, \end{cases}$$

where $0 < \alpha \le 1$, g and $f : [0,1] \times [0,\infty) \to [0,\infty)$ are given continuous functions.

Chidouh *et al*. [8] studied the fractional relaxation equation involving the Caputo derivatives of order $\alpha \in (0,1)$ of the type:

$$\begin{cases} {}^C D^{\alpha}\xi(t) + \omega\xi(t) = \varepsilon(t,\xi(t)), & 0 < t \le 1, \omega > 0, \\ \xi(0) = \xi_0 > 0, \end{cases}$$

where $\varepsilon : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Peng and Wang [15] obtained existence results for nonlinear fractional differential equations with constant coefficient $\lambda > 0$ of the type:

$$\begin{cases} {}^C D^{\alpha}\xi(t) = \lambda\xi(t) + f(t,\xi(t)), & t \in [0,1], \\ \xi(0) = \xi_0 \in \mathbb{R}, \end{cases}$$

where ${}^c D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Recently, many authors have studied various properties of fractional differential equations (see [5, 10, 11, 13, 14, 19, 21–23] and the references therein).

Inspired and motivated by the above-mentioned works, we study the existence and uniqueness of solutions for the following nonlinear Caputo fractional relaxation differential equation with nonlocal condition

$$\begin{cases} {}^C D^{\alpha}(\xi(t) - g(t,\xi(t))) + \omega\xi(t) = f(t,\xi(t)), & t \in (0,T], \\ \xi(0) = \xi_0 - h(\xi), \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha}$ is the Caputo's fractional derivative of order $0 < \alpha \le 1$, $\omega > 0$, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, ξ_0 is a real constant and $h : C([0,T],\mathbb{R}) \to \mathbb{R}$ is a continuous function. This type of non-local Cauchy problem was introduced by Byszewski [6, 7]. The nonlocal condition can be more useful than the classical initial condition to describe some physical phenomenons [6, 7]. We take an example of non-local conditions as follows:

$$h(\xi) = \sum_{i=1}^{p} c_i \xi(t_i),$$

where c_i, $i = 1, \dots, p$ are constants and $0 < t_1 < \dots < t_p \le T$.

The aim of the present paper is to prove the existence and uniqueness of a solution for a class of nonlocal Cauchy problem for nonlinear Caputo fractional relaxation differential equations. The main tools employed in our results are based on Banach's and Krasnoselskii's fixed point theorems.

2. Preliminaries

In this section, we present some basic definitions, notations, and results of fractional calculus, which are used throughout this article.

Let $T>0$, and let $J=[0,T]$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|\xi\| = \sup\{\xi(t) : t \in J\}.$$

Definition 1 ([12])**.** The fractional integral of order $\alpha>0$ of a function $\xi : J \to \mathbb{R}$ is given by

$$I^\alpha \xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\xi(s)ds,$$

provided the right side is pointwise defined on J.

Definition 2 ([12])**.** The Caputo fractional derivative of order $\alpha>0$ of function $\xi : J \to \mathbb{R}$ is given by

$${}^C D^\alpha \xi(t) = I^{n-\alpha} D^{(n)}\xi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}\xi^{(n)}(s)ds,$$

where $n=[\alpha]+1$, provided the right side is pointwise defined on J.

Definition 3 ([2])**.** The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha>0, \beta \in \mathbb{C}, z \in \mathbb{C}.$$

For $\beta=1$, we obtain the Mittag-Leffler function in one parameter

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha>0, z \in \mathbb{C}.$$

Lemma 1 ([2])**.** *For $0<\alpha\le 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\omega t^\alpha)$ satisfies and*

$$0 \le E_{\alpha,\alpha}(-\omega t^\alpha) \le \frac{1}{\Gamma(\alpha)}, \quad t \in [0,\infty), \omega \ge 0,$$

$$\lim_{t\to 0^+} E_{\alpha,\alpha}(-\omega t^\alpha) = E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}.$$

Lemma 2 ([9])**.** *For $t \in [0,\infty)$ and $0<\alpha\le 1$, the one-parameter Mittag-Leffler function $E_{\alpha,1}(-t^\alpha)$ is a decreasing function of t and it is bounded from above by 1, that is,*

$$E_{\alpha,1}(-\omega t^\alpha) \le 1.$$

Furthermore, it is to be noted that

$$\lim_{t\to\infty} E_{\alpha,1}(-\omega t^\alpha) = 0.$$

Theorem 1 (Banach's fixed point theorem [20])**.** *Let Ω be a nonempty closed subset of a Banach space $(S,\|\cdot\|)$. Then any contraction mapping θ of Ω into itself has a unique fixed point.*

Theorem 2 (Krasnoselskii's fixed point theorem [20])**.** *Let Ω be a nonempty bounded closed convex subset of a Banach space $(S,\|\cdot\|)$. Suppose that F_1 and F_2 map Ω into S such that*

(i) *$F_1\xi + F_2\vartheta \in \Omega$ for all $\xi,\vartheta \in \Omega$,*

(ii) *F_1 is continuous and compact,*

(iii) *F_2 is a contraction.*

Then there is $\xi \in \Omega$ with $F_1\xi + F_2\xi = \xi$.

In the following section, we obtain existence and uniqueness results for the problem (1.1) by using fixed point theorems.

3. Existence and Uniqueness

Definition 4. A function $\xi \in C^1(J,\mathbb{R})$ is said to be a solution of the problem (1.1) if ξ satisfies ${}^C D^\alpha(\xi(t) - g(t,\xi(t))) + \omega\xi(t) = f(t,\xi(t))$, for any $t \in J$ and $\xi(0) = \xi_0 - h(\xi)$.

For the existence and uniqueness of solutions to the problem (1.1), we need the following lemma.

Lemma 3. *Let $\xi \in C(J,\mathbb{R})$ and let ξ' exist. Then ξ is a solution of the problem* (1.1) *if and only if it is a solution of the integral equation*

$$\xi(t) = (\xi_0 - h(\xi) - g(t,\xi_0))E_{\alpha,1}(-\omega t^\alpha) + g(t,\xi(t)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^\alpha)[f(s,\xi(s)) - \omega g(s,\xi(s))]ds. \tag{3.1}$$

Proof. It is easy to prove by the Laplace transform. □

Now we introduce the following set of conditions to prove our main results:

(H1) There exists a constant $\mathcal{M}_f \in \mathbb{R}^+$ such that
$$|f(t,\xi) - f(t,\vartheta)| \le \mathcal{M}_f|\xi - \vartheta|,$$
for $t \in J$, $\xi,\vartheta \in \mathbb{R}$.

(H2) There exists a constant $\mathcal{M}_g \in (0,1)$ such that
$$|g(t,\xi) - g(t,\vartheta)| \le \mathcal{M}_g|\xi - \vartheta|,$$
for $t \in J$, $\xi,\vartheta \in \mathbb{R}$.

(H3) There exists a constant $\mathcal{M}_h \in (0,1)$ such that
$$|h(\xi) - h(\vartheta)| \le \mathcal{M}_h|\xi - \vartheta|,$$
for each $\xi,\vartheta \in C(J,\mathbb{R})$.

Theorem 3. *Assume that the assumptions* (H1)-(H3) *are satisfied. If*

$$\mathcal{M}_h + \mathcal{M}_g + \frac{T^\alpha}{\Gamma(\alpha+1)}(\mathcal{M}_f + \mathcal{M}_g\omega) < 1 \tag{3.2}$$

then there exists a unique solution to the problem (1.1) *on J.*

Proof. We define the operator $\theta : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$(\theta\xi)(t) = (\xi_0 - h(\xi) - g(t,\xi_0))E_{\alpha,1}(-\omega t^\alpha) + g(t,\xi(t)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^\alpha)[f(s,\xi(s)) - \omega g(s,\xi(s))]ds.$$

For any $\xi,\vartheta \in C([0,T],\mathbb{R})$ and $t \in J$, we have

$$|(\theta\xi)(t) - (\theta\vartheta)(t)| \le |h(\xi) - h(\vartheta)| + |g(t,\xi(t)) - g(t,\vartheta(t))| + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^\alpha)|f(s,\xi(s)) - f(s,\vartheta(s))|ds$$

$$+\omega\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})|g(s,\xi(s))-g(s,\vartheta(s))|ds.$$

By (H1), (H2) and (H3), we have

$$|(\theta\xi)(t)-(\theta\vartheta)(t)|\le \mathcal{M}_h\|\xi-\vartheta\|+\mathcal{M}_g\|\xi-\vartheta\|+\frac{T^{\alpha}\mathcal{M}_f}{\Gamma(\alpha+1)}\|\xi-\vartheta\|+\frac{T^{\alpha}\mathcal{M}_g\omega}{\Gamma(\alpha+1)}\|\xi-\vartheta\|$$

thus

$$\|\theta\xi-\theta\vartheta\|\le\left(\mathcal{M}_h+\mathcal{M}_g+\frac{T^{\alpha}}{\Gamma(\alpha+1)}(\mathcal{M}_f+\mathcal{M}_g\omega\right))\|\xi-\vartheta\|.$$

From (3.2), θ is a contraction. As a consequence of Banach's fixed point theorem, there is a unique fixed point for θ which is a unique solution for the problem (1.1) on J. □

Our next result is based on Krasnoselskii's fixed point theorem.

Theorem 4. *Assume* (H2) *and* (H3) *with* $(\mathcal{M}_h+\mathcal{M}_g)<1$ *and the following hypotheses:*

(H4) *There exists* $r_1\in C(J,\mathbb{R}^+)$ *such that*

$$|f(t,\xi)|\le r_1(t),$$

for $t\in J$ *and each* $\xi\in\mathbb{R}$.

(H5) *There exists* $r_2\in C(J,\mathbb{R}^+)$ *such that*

$$|g(t,\xi)|\le r_2(t),$$

for $t\in J$ *and each* $\xi\in\mathbb{R}$.

(H6) *There exists* $r_3>0$ *such that*

$$|h(\xi)|\le r_3,$$

for $t\in J$ *and each* $\xi\in\mathbb{R}$.

Then problem (1.1) *has at least one solution in* Ω.

Proof. Choose

$$\rho\ge|\xi_0|+r_3+s+r_2^*+\frac{T^{\alpha}}{\Gamma(\alpha+1)}(r_1^*+\omega r_2^*), \tag{3.3}$$

where $r_1^*=\sup\limits_{t\in J}r_1(t)$, $r_2^*=\sup\limits_{t\in J}r_2(t)$, and $s=\sup\limits_{t\in J}|g(t,\xi_0)|$. Consider the non-empty closed convex subset

$$\Omega=\{\xi\in C(J,\mathbb{R}),\|\xi\|\le\rho\},$$

and define two operators $\mathfrak{F}_1$ and $\mathfrak{F}_2$ on Ω, as follows:

$$(\mathfrak{F}_1\xi)(t)=\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})[f(s,\xi(s))-\omega g(s,\xi(s))]ds$$

and

$$(\mathfrak{F}_2\xi)(t)=(\xi_0-h(\xi)-g(t,\xi_0))E_{\alpha,1}(-\omega t^{\alpha})+g(t,\xi(t)).$$

The proof will be given in several steps.

Step 1. We prove $\mathfrak{F}_1\xi + \mathfrak{F}_2\vartheta \in \Omega$, for all $\xi, \vartheta \in \Omega$.
For any $\xi, \vartheta \in \Omega$, we have

$$\begin{aligned}
|(\mathfrak{F}_1\xi)(t) + (\mathfrak{F}_2\vartheta)(t)| &= \Big|(\xi_0 - h(\xi) - g(t,\xi_0))E_{\alpha,1}(-\omega t^{\alpha}) + g(t,\vartheta(t)) \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})[f(s,\xi(s)) - \omega g(s,\xi(s))]ds\Big| \\
&\le |\xi_0| + r_3 + s + r_2^* + \frac{T^{\alpha}}{\Gamma(\alpha+1)}(r_1^* + \omega r_2^*).
\end{aligned}$$

Thus

$$\|\mathfrak{F}_1\xi + \mathfrak{F}_2\vartheta\| \le |\xi_0| + r_3 + s + r_2^* + \frac{T^{\alpha}}{\Gamma(\alpha+1)}(r_1^* + \omega r_2^*) \le \rho .$$

Hence, $\mathfrak{F}_1\xi + \mathfrak{F}_2\vartheta \in \Omega$, for all $\xi, \vartheta \in \Omega$.

Step 2. We prove that $\mathfrak{F}_1$ is compact and continuous.
For all $\xi \in \Omega$, we have

$$\begin{aligned}
|(\mathfrak{F}_1\xi)(t)| &= \left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})[f(s,\xi(s)) - \omega g(s,\xi(s))]ds\right| \\
&\le \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})[|f(s,\xi(s))| + \omega|g(s,\xi(s))|]ds \\
&\le \frac{T^{\alpha}}{\Gamma(\alpha+1)}(r_1^* + \omega r_2^*).
\end{aligned}$$

Thus

$$\|\mathfrak{F}_1\xi\| \le \frac{T^{\alpha}}{\Gamma(\alpha+1)}(r_1^* + \omega r_2^*).$$

Hence, $\mathfrak{F}_1$ is uniformly bounded on Ω.
Now let's prove that $\mathfrak{F}_1(\Omega)$ is equicontinuous. Let $x \in \Omega$. Then for any $0 < t_1 < t_2 \le T$, we have

$$\begin{aligned}
|(\mathfrak{F}_1\xi)(t_2) - (\mathfrak{F}_1\xi)(t_1)| &= \Big|\int_0^{t_2} (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t_2-s)^{\alpha})[f(s,\xi(s)) - \omega g(s,\xi(s))]ds \\
&\quad - \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t_1-s)^{\alpha})[f(s,\xi(s)) - \omega g(s,\xi(s))]ds\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})(|f(s,\xi(s))| + \omega|g(s,\xi(s))|)ds \\
&\quad + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}(|f(s,\xi(s))| + \omega|g(s,\xi(s))|)ds \\
&\le \frac{(r_1^* + \omega r_2^*)}{\Gamma(\alpha+1)}(2(t_2-t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha}) \\
&\le \frac{2(r_1^* + \omega r_2^*)}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \qquad (3.4)
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of inequality (3.4) tends to zero and the convergence is independent of ξ in Ω. So $\mathfrak{F}_1(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, $\mathfrak{F}_1$ is compact. Moreover, the continuity of f and g implies that $\mathfrak{F}_1$ is continuous.

Step 3. We prove that $\mathfrak{F}_2 : \Omega \to C(J,\mathbb{R})$ is a contraction mapping.
For all $\xi,\vartheta \in \Omega$ and $t \in J$, we have

$$\begin{aligned}|(\mathfrak{F}_2\xi)(t) - (\mathfrak{F}_2\vartheta)(t)| &= |h(\xi) - h(\vartheta)| + |g(t,\xi(t)) - g(t,\vartheta(t))| \\ &\le \mathcal{M}_h \|\xi - \vartheta\| + \mathcal{M}_g \|\xi - \vartheta\|.\end{aligned}$$

Thus

$$\|\mathfrak{F}_2\xi - \mathfrak{F}_2\vartheta\| \le (\mathcal{M}_h + \mathcal{M}_g)\|\xi - \vartheta\|.$$

Hence, the operator $\mathfrak{F}_2$ is a contraction.

As a consequence of Krasnoselskii's fixed point theorem, we deduce that there exists a fixed point $\xi \in \Omega$ such that $\xi = \mathfrak{F}_1\xi + \mathfrak{F}_2\xi$, which is a solution of the problem (1.1). □

4. Example

We consider the fractional initial value problem

$$\begin{aligned}&{}^C D^{\frac{1}{2}}\left(\xi(t) - \frac{1}{4}\xi(t)\cos(t)\right) + \frac{1}{2}\xi(t) = \frac{1}{(\exp(t)+4)(|\xi(t)|+1)}, \quad t \in J = [0,1], \\ &\xi(0) = 1 - \sum_{i=1}^{n} c_i \xi(t_i), \end{aligned} \tag{4.1}$$

where $0 < t_1 < \ldots < t_n < 1$ and c_i, $i = 1,2,\ldots,n$ are positive constants with

$$\sum_{i=1}^{n} c_i \le \frac{1}{5},$$

$T = 1$, $\xi_0 = 1 - \sum\limits_{i=1}^{n} c_i \xi(t_i)$, $\alpha = \omega = \frac{1}{2}$, $g(t,\xi) = \frac{1}{4}\xi\cos(t)$, and $f(t,\xi) = \frac{1}{(\exp(t)+4)(|\xi|+1)}$. For each $\xi,\vartheta \in \mathbb{R}$ and $t \in J$, we have

$$\begin{aligned}|f(t,\xi) - f(t,\vartheta)| &= \left|\frac{1}{(\exp(t)+4)(|\xi|+1)} - \frac{1}{(\exp(t)+4)(|\vartheta|+1)}\right| \\ &\le \frac{|\xi - \vartheta|}{(\exp(t)+4)(1+|\xi|)(1+|\vartheta|)} \\ &\le \frac{1}{5}|\xi - \vartheta|,\end{aligned} \tag{4.2}$$

$$|g(t,\xi) - g(t,\vartheta)| \le \frac{1}{4}|\xi - \vartheta| \tag{4.3}$$

and

$$|h(\xi) - h(\vartheta)| \le \left|\sum_{i=1}^{n} c_i \xi - \sum_{i=1}^{n} c_i \vartheta\right| \le \sum_{i=1}^{n} c_i |\xi - \vartheta| \le \frac{1}{5}|\xi - \vartheta|. \tag{4.4}$$

Hence, assumptions (H1), (H2) and (H3) are satisfied with $\mathcal{M}_f = \frac{1}{5}$, $\mathcal{M}_h = \frac{1}{5}$ and $\mathcal{M}_g = \frac{1}{4}$. The condition

$$\mathcal{M}_h + \mathcal{M}_g + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\left(\mathcal{M}_g \omega + \mathcal{M}_f\right) \simeq 0.82 < 1 \tag{4.5}$$

is satisfied. It follows from Theorem 3 that problem (4.1) has a unique solution on J.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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