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Research Article

Some Applications via Common Coupled Fixed Points in G-Metric Spaces With Altering Distance Functions

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Abstract. In this article, we give some applications to integral equations as well as homotopy theory via common coupled fixed point theorems in complete G-metric space which involve altering distance function is established. We also furnish an example which supports our main result. Our results extend, unify and generalize some existing results in the current literature.

Keywords. Common coupled fixed point, Altering distance function, ω-compatible and G-completeness

Mathematics Subject Classification (2020). 54H25, 47H10, 54E50

1. Introduction

One of the areas of interest in nonlinear functional analysis is the examination of the existence and uniqueness of fixed points of specific mappings within the context of metric spaces. The key accomplishment in this direction is the Banach contraction mapping principle, and numerous metric spaces have been investigated in metric spaces. The fixed point theory has applications in many areas including approximation theory, homotopy theory, integral, integro-differential, and impulsive differential equations.

Bhaskar and Lakshmikantham [8] demonstrated the existence and uniqueness of a coupled fixed point in the setting of partially ordered metric spaces, and numerous coupled fixed point and coincident point results have been published in the literature, since the publication of this article. Some examples of these works are the ones listed in [1,4–8,10–12].

Mustafa and Sims [16], were introduced the idea of a G-metric space in 2006. The authors discussed the topological characteristics of this space and demonstrated the G-metric equivalent of the Banach contraction mapping concept by using the concept of mixed monotone property. Following that, many authors created various fixed point results in G-metric spaces (see [2–6,14–17]). Khan *et al*. [9] introduced and proved fixed point results by the altering distance in metric space.

The goal of this study is to demonstrate that there is a unique common coupled fixed point for four mappings that satisfy generalised contractive criteria in G-metric space and that necessitate modifying the distance function. Examples of applications to homotopy theory and integral equations are also given. These results extend and generalise a number of recent, well-recognized, and congruent findings in the literature.

2. Preliminaries

First we will discuss the basic definitions of G-metric spaces.

Definition 2.1 ([16]). Let $\mathcal{P}$ be a non-empty set and let $G:\mathcal{P}\times\mathcal{P}\times\mathcal{P}\to[0,\infty)$ be a function satisfying the following properties:

(B_0) $G(p,q,r)=0$ if $p=q=r$;

(B_1) $0<G(p,p,q)$ for any $p,q\in\mathcal{P}$ with $p\neq q$;

(B_2) if $G(p,p,q)\le G(p,q,r)$ for all $p,q,r\in\mathcal{P}$ with $q\neq r$;

(B_3) $G(p,q,r)=G(P[p,q,r])$, where P is a permutation of p,q,r (symmetry);

(B_4) $G(p,q,r)\le G(p,x,x)+G(x,q,r)$ for all $p,q,r,x\in\mathcal{P}$ (rectangle inequality)

then G is said to be a G-metric on $\mathcal{P}$ and pair $(\mathcal{P},G)$ is said to be a G-metric space.

Definition 2.2 ([16]). A G-metric space $(\mathcal{P},G)$ is said to be symmetric if

$$G(p,q,q)=G(q,p,p),\quad \text{for all } p,q\in\mathcal{P}.$$

Definition 2.3. ([16]) Let $\mathcal{P}$ be a G-metric space. A sequence $\{p_n\}$ in $\mathcal{P}$ is called:

(a) G-Cauchy sequence if for every $\epsilon>0$, there is an integer $n_0\in\mathbb{Z}^+$ such that for all $n,m,l\ge n_0$, $G(p_n,p_m,p_l)<\epsilon$.

(b) G-convergent to a point $p\in\mathcal{P}$ if for each $\epsilon>0$, there is an integer $n_0\in\mathbb{Z}^+$ such that for all $n,m\ge n_0$, $G(p_n,p_m,p)<\epsilon$.

A G-metric space on $\mathcal{P}$ is said to be G-complete if every G-Cauchy sequence in $\mathcal{P}$ is G-convergent in $\mathcal{P}$.

For more properties of a G-metric we refer the reader to [16].

Definition 2.4 ([8])**.** Let $\mathcal{P}$ be a nonempty set and let $F:\mathcal{P}^2\to\mathcal{P}$ be a mapping. If $F(p,q)=p$, $F(q,p)=q$ for $p,q\in\mathcal{P}$ then (p,q) is called a coupled fixed point of F.

Definition 2.5 ([11])**.** Let $F:\mathcal{P}^2\to\mathcal{P}$ and $f:\mathcal{P}\to\mathcal{P}$ be two mappings. An element (p,q) is said to be a coupled coincident point of F and f if

$$F(p,q)=fp,\quad F(q,p)=fq.$$

Definition 2.6 ([10])**.** Let $F:\mathcal{P}^2\to\mathcal{P}$ and $f:\mathcal{P}\to\mathcal{P}$ be two mappings. An element (p,q) is said to be a coupled common point of F and f if

$$F(p,q)=fp=p,\quad F(q,p)=fq=q.$$

Definition 2.7 ([1])**.** Let $(\mathcal{P},G)$ be a G metric space. A pair (F,f) is called weakly compatible if $f(F(p,q))=F(fp,fq)$ whenever for all $p,q\in\mathcal{P}$ such that

$$F(p,q)=fp,\quad F(q,p)=fq.$$

A new category of contractive fixed point results was addressed by Khan *et al*. [9]. In their study they introduced the notion of an altering distance function which is a control function that alters distance between two points in a metric space.

Definition 2.8. The function $\zeta:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ζ is continuous and nondecreasing,

(ii) $\zeta(t)=0 \iff t=0$.

3. Main Results

Theorem 3.1. *Let $(\mathcal{P},G)$ be a G-metric space. Suppose $\zeta:[0,\infty)\to[0,\infty)$ is an altering distance function and $\chi:[0,\infty)\to[0,\infty)$ is a lower semi continuous function with $\chi(t)=0 \iff t=0$. Moreover, suppose that $R,T:\mathcal{P}^2\to\mathcal{P}$ and $f,g:\mathcal{P}\to\mathcal{P}$ be a four mappings satisfying the following: For all a,b,p,q in $\mathcal{P}$*

$$\zeta(G(R(a,b),R(a,b),T(p,q)))\le\zeta(\lambda M(a,b,p,q))-\chi(\lambda M(a,b,p,q))+LN(a,b,p,q). \tag{3.1}$$

(a) *$R(\mathcal{P}^2)\subseteq f(\mathcal{P})$, and $T(\mathcal{P}^2)\subseteq g(\mathcal{P})$,*

(b) *either (R,g) or (T,f) are ω-compatible,*

(c) *one of $g(\mathcal{P})$ or $f(\mathcal{P})$ is complete,*

where

$$\begin{aligned}M(a,b,p,q)=\max\{&\max\{G(ga,ga,fp),G(R(a,b),T(p,q),fp),G(R(a,b),R(a,b),fp),\\&\qquad G(R(a,b),ga,fp),G(R(a,b),fp,T(p,q)),G(fp,R(a,b),ga),\\&\qquad G(R(a,b),R(a,b),ga),G(T(p,q),T(p,q),fp)\},\\&\max\{G(gb,gb,fq),G(R(b,a),T(q,p),fq),G(R(b,a),R(b,a),fq),\\&\qquad G(R(b,a),gb,fq),G(R(b,a),fq,T(q,p)),G(fq,R(b,a),gy),\\&\qquad G(R(b,a),R(b,a),gb),G(T(q,p),T(q,p),fq)\}\}.\end{aligned}$$

and

$$\begin{aligned}N(a,b,p,q)=\min\{&\min\{G(ga,ga,fp),G(R(a,b),T(p,q),fp),G(R(a,b),R(a,b),fp),\\&G(R(a,b),ga,fp),G(R(a,b),fp,T(p,q)),G(fp,R(a,b),ga),\\&G(R(a,b),R(a,b),ga),G(T(p,q),T(p,q),fp)\},\\&\min\{G(gb,gb,fq),G(R(b,a),T(q,p),fq),G(R(b,a),R(b,a),fq),\\&G(R(b,a),gb,fq),G(R(b,a),fq,T(q,p)),G(fq,R(b,a),gb),\\&G(R(b,a),R(b,a),gy),G(T(q,p),T(q,p),fq)\}\}\end{aligned}$$

with $L\geq 0$ *and* $0<\lambda<1$. *Then there is a unique common coupled fixed point of* R,T,f *and* g *in* $\mathcal{P}$.

Proof. Let $a_0,b_0\in\mathcal{P}$ be arbitrary, and from (a), we construct the sequences $\{a_{2n}\}$, $\{b_{2n}\}$, $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$, in $\mathcal{P}$ as

$$R(a_{2n},b_{2n})=fa_{2n+1}=\alpha_{2n},\quad R(b_{2n},a_{2n})=fb_{2n+1}=\beta_{2n},$$
$$T(a_{2n+1},b_{2n+1})=ga_{2n+2}=\alpha_{2n+1},\quad T(b_{2n+1},a_{2n+1})=gb_{2n+2}=\beta_{2n+1},\quad \text{where } n=0,1,2,\ldots.$$

Then, from (3.1), we can get

$$\begin{aligned}\zeta(G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}))&=\zeta[G(R(a_{2n},b_{2n}),R(a_{2n},b_{2n}),T(a_{2n+1},b_{2n+1}))]\\&\leq\zeta(\lambda M(a_{2n},b_{2n},a_{2n+1},b_{2n+1}))-\chi(\lambda M(a_{2n},b_{2n},a_{2n+1},b_{2n+1}))\\&\quad+LN(a_{2n},b_{2n},a_{2n+1},b_{2n+1}),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}&M(a_{2n},b_{2n},a_{2n+1},b_{2n+1})\\&=\max\{\max\{G(ga_{2n},ga_{2n},fa_{2n+1}),G(R(a_{2n},b_{2n}),T(a_{2n+1},b_{2n+1}),fa_{2n+1}),\\&\qquad G(R(a_{2n},b_{2n}),R(a_{2n},b_{2n}),fa_{2n+1}),G(R(a_{2n},b_{2n}),ga_{2n},fa_{2n+1}),\\&\qquad G(R(a_{2n},b_{2n}),fa_{2n+1},T(a_{2n+1},b_{2n+1})),G(fa_{2n+1},R(a_{2n},b_{2n}),ga_{2n}),\\&\qquad G(R(a_{2n},b_{2n}),R(a_{2n},b_{2n}),ga_{2n}),G(T(a_{2n+1},b_{2n+1}),T(a_{2n+1},b_{2n+1}),fa_{2n+1})\},\\&\quad\max\{G(gb_{2n},gb_{2n},fb_{2n+1}),G(R(b_{2n},a_{2n}),T(b_{2n+1},a_{2n+1}),fb_{2n+1}),\\&\qquad G(R(b_{2n},a_{2n}),R(b_{2n},a_{2n}),fb_{2n+1}),G(R(b_{2n},a_{2n}),gb_{2n},fb_{2n+1}),\\&\qquad G(R(b_{2n},a_{2n}),fb_{2n+1},T(b_{2n+1},a_{2n+1})),G(fb_{2n+1},R(b_{2n},a_{2n}),gb_{2n}),\\&\qquad G(R(b_{2n},a_{2n}),R(b_{2n},a_{2n}),gb_{2n}),G(T(b_{2n+1},a_{2n+1}),T(b_{2n+1},a_{2n+1}),fb_{2n+1})\}\}\\&=\max\{\max\{G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}),G(\alpha_{2n},\alpha_{2n+1},\alpha_{2n}),G(\alpha_{2n},\alpha_{2n},\alpha_{2n}),G(\alpha_{2n},\alpha_{2n-1},\alpha_{2n}),\\&\qquad G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\alpha_{2n},\alpha_{2n},\alpha_{2n-1}),G(\alpha_{2n},\alpha_{2n},\alpha_{2n-1}),G(\alpha_{2n+1},\alpha_{2n+1},\alpha_{2n})\},\\&\quad\max\{G(\beta_{2n-1},\beta_{2n-1},\beta_{2n}),G(\beta_{2n},\beta_{2n+1},\beta_{2n}),G(\beta_{2n},\beta_{2n},\beta_{2n}),G(\beta_{2n},\beta_{2n-1},\beta_{2n}),\\&\qquad G(\beta_{2n},\beta_{2n},\beta_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n-1}),G(\beta_{2n},\beta_{2n},\beta_{2n-1}),G(\beta_{2n+1},\beta_{2n+1},\beta_{2n})\}\}\\&=\max\{\max\{G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}),G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1})\},\\&\quad\max\{G(\beta_{2n-1},\beta_{2n-1},\beta_{2n}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}\}.\end{aligned}$$

By similar arguments we obtain

$$N(a_{2n},b_{2n},a_{2n+1},b_{2n+1}) = \min\{\min\{G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}),0,G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1})\},\\ \min\{G(\beta_{2n-1},\beta_{2n-1},\beta_{2n}),0,G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}\} = 0.$$

Now, we want to show that

$$G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}) \geq G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1})$$

and

$$G(\beta_{2n-1},\beta_{2n-1},\beta_{2n}) \geq G(\beta_{2n},\beta_{2n},\beta_{2n+1})$$

for each $n \in N$.
Assume that

$$G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}) < G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1})$$

and

$$G(\beta_{2n-1},\beta_{2n-1},\beta_{2n}) < G(\beta_{2n},\beta_{2n},\beta_{2n+1})$$

for some $n \in N$, then we have

$$M(a_{2n},b_{2n},a_{2n+1},b_{2n+1}) = \max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}.$$

Then from (3.2), we can get

$$\begin{aligned}\zeta(G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1})) &\leq \zeta(\lambda\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\})\\ &\quad - \chi(\lambda\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\})\\ &\leq \zeta(\lambda\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}).\end{aligned}$$

Since ζ is increasing, we get

$$G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}) \leq \lambda\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}. \tag{3.3}$$

By similar arguments we obtain

$$G(\beta_{2n},\beta_{2n},\beta_{2n+1}) \leq \lambda\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}. \tag{3.4}$$

Combining (3.3) and (3.4), we can get

$$\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\} \leq \lambda\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}$$

which is a contradiction, because $0 < \lambda < 1$. Thus, $G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}) \geq G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1})$ and $G(\beta_{2n-1},\beta_{2n-1},\beta_{2n}) \geq G(\beta_{2n},\beta_{2n},\beta_{2n+1})$.
Therefore by above inequality we get

$$\begin{aligned}&\max\{G(\alpha_{2n},\alpha_{2n},\alpha_{2n+1}),G(\beta_{2n},\beta_{2n},\beta_{2n+1})\}\\ &\quad \leq \lambda\max\{G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}),G(\beta_{2n-1},\beta_{2n-1},\beta_{2n})\}.\end{aligned} \tag{3.5}$$

By similar arguments we obtain

$$\begin{aligned}&\max\{G(\alpha_{2n-1},\alpha_{2n-1},\alpha_{2n}),G(\beta_{2n-1},\beta_{2n-1},\beta_{2n})\}\\ &\quad \leq \lambda\max\{G(\alpha_{2n-2},\alpha_{2n-2},\alpha_{2n-1}),G(\beta_{2n-2},\beta_{2n-2},\beta_{2n-1})\}.\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}&\max\{G(\alpha_n,\alpha_n,\alpha_{n-1}),G(\beta_n,\beta_n,\beta_{n-1})\}\\ &\quad \leq \lambda\max\{G(\alpha_{n-1},\alpha_{n-1},\alpha_{n-2}),G(\beta_{n-1},\beta_{n-1},\beta_{n-2})\} \quad \forall\ n \geq 2,\end{aligned}$$

where $0 < \lambda < 1$. Hence, for $\forall\ n \geq 2$, it follows that

$$\max\{G(\alpha_n,\alpha_n,\alpha_{n-1}),G(\beta_n,\beta_n,\beta_{n-1})\} \leq \lambda^{n-1}\max\{G(\alpha_1,\alpha_1,\alpha_0),G(\beta_1,\beta_1,\beta_0)\}.$$

Thus, we have

$$G(\alpha_n,\alpha_n,\alpha_{n-1}) \leq \lambda^{n-1}\max\{G(\alpha_1,\alpha_1,\alpha_0),G(\beta_1,\beta_1,\beta_0)\}$$

and

$$G(\beta_n,\beta_n,\beta_{n-1}) \leq \lambda^{n-1}\max\{G(\alpha_1,\alpha_1,\alpha_0),G(\beta_1,\beta_1,\beta_0)\}.$$

By use of the rectangle inequality, for $n > m$, we get

$$\begin{aligned} G(\alpha_n,\alpha_n,\alpha_m) &\leq G(\alpha_m,\alpha_{m+1},\alpha_{m+1}) + G(\alpha_{m+1},\alpha_n,\alpha_n) \\ &\leq G(\alpha_m,\alpha_{m+1},\alpha_{m+1}) + G(\alpha_{m+2},\alpha_{m+2},\alpha_{m+1}) + G(\alpha_{m+2},\alpha_n,\alpha_n) \\ &\leq G(\alpha_m,\alpha_{m+1},\alpha_{m+1}) + G(\alpha_{m+2},\alpha_{m+2},\alpha_{m+1}) + \cdots + G(\alpha_{n-1},\alpha_n,\alpha_n) \\ &\leq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1})\max\{G(\alpha_1,\alpha_1,\alpha_0),G(\beta_1,\beta_1,\beta_0)\} \\ &\leq (\lambda^m + \lambda^{m+1} + \lambda^{m+2} + \cdots)\max\{G(\alpha_1,\alpha_1,\alpha_0),G(\beta_1,\beta_1,\beta_0)\} \\ &\leq \frac{\lambda^m}{1-\lambda}\max\{G(\alpha_1,\alpha_1,\alpha_0),G(\beta_1,\beta_1,\beta_0)\} \to 0 \text{ as } m \to \infty. \end{aligned}$$

By similar arguments, we obtain $G(\beta_n,\beta_n,\beta_m) \to 0$ as $n,m \to \infty$. This shows that $\{\alpha_n\},\{\beta_n\}$ are Cauchy sequences in the G-metric space $(\mathcal{P},G)$. Suppose $g(\mathcal{P})$ is complete subspace of $(\mathcal{P},G)$, then the sequences $\{\alpha_n\},\{\beta_n\}$ are convergence to τ,κ respectively in $g(\mathcal{P})$. Thus, there exist $a,b \in g(\mathcal{P})$ such that

$$\lim_{n\to\infty} \alpha_{2n} = \tau = ga, \quad \lim_{n\to\infty} \beta_{2n} = \kappa = gb.$$

We claim that $R(a,b) = \tau$ and $R(b,a) = \kappa$.By using (3.1), we have

$$\begin{aligned} \zeta(G(R(a,b),R(a,b),\tau)) &= \lim_{n\to\infty} \zeta(G(R(a,b),R(a,b),\alpha_{2n+1})) \\ &\leq \zeta\left(\lambda \lim_{n\to\infty} M(a,b,a_{2n+1},b_{2n+1})\right) - \chi\left(\lambda \lim_{n\to\infty} M(a,b,a_{2n+1},b_{2n+1})\right) \\ &\quad + L\lim_{n\to\infty} N(a,b,a_{2n+1},b_{2n+1}) \\ &\leq \zeta(\lambda\max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}) \\ &\quad - \chi(\lambda\max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}) + L(0) \\ &\leq \zeta(\lambda\max\{G(\tau,R(a,b),R(a,b)),G(\kappa,R(b,a),R(b,a))\}). \end{aligned}$$

Because of

$$\begin{aligned} &\lim_{n\to\infty} M(a,b,a_{2n+1},b_{2n+1}) \\ &\quad = \lim_{n\to\infty} \max\{\max\{G(ga,ga,\alpha_{2n}),G(R(a,b),\alpha_{2n+1},\alpha_{2n}),G(R(a,b),R(a,b),\alpha_{2n}), \\ &\qquad G(R(a,b),ga,\alpha_{2n}),G(R(a,b),\alpha_{2n},\alpha_{2n+1}),G(\alpha_{2n},R(a,b),ga), \\ &\qquad G(R(a,b),R(a,b),ga),G(\alpha_{2n+1},\alpha_{2n+1},\alpha_{2n})\}, \\ &\qquad \max\{G(gb,gb,\beta_{2n}),G(R(b,a),\beta_{2n+1},\beta_{2n}),G(R(b,a),R(b,a),\beta_{2n}), \\ &\qquad G(R(b,a),gb,\beta_{2n}),G(R(b,a),\beta_{2n},\beta_{2n+1}),G(\beta_{2n},R(b,a),gb), \\ &\qquad G(R(b,a),R(b,a),gb),G(\beta_{2n+1},\beta_{2n+1},\beta_{2n})\}\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{\max\{G(ga,ga,\tau),G(R(a,b),\tau,\tau),G(R(a,b),R(a,b),\tau),G(R(a,b),ga,\tau),\\
&\qquad G(R(a,b),\tau,\tau),G(\tau,R(a,b),\tau),G(\tau,\tau,\tau)\},\\
&\qquad \max\{G(gb,gb,\kappa),G(R(b,a),\kappa,\kappa),G(R(b,a),R(b,a),\kappa),G(R(b,a),gb,\kappa),\\
&\qquad G(R(b,a),\kappa,\kappa),G(\kappa,R(b,a),\kappa),G(\kappa,\kappa,\kappa)\}\}\\
&= \max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty} N(a,b,a_{2n+1},b_{2n+1})\\
&\quad = \lim_{n\to\infty}\min\{\min\{G(ga,ga,a_{2n}),G(R(a,b),a_{2n+1},a_{2n}),G(R(a,b),R(a,b),a_{2n}),\\
&\qquad G(R(a,b),ga,a_{2n}),G(R(a,b),a_{2n},a_{2n+1}),G(a_{2n},R(a,b),ga),\\
&\qquad G(R(a,b),R(a,b),ga),G(a_{2n+1},a_{2n+1},a_{2n})\},\\
&\qquad \min\{G(gb,gb,\beta_{2n}),G(R(b,a),\beta_{2n+1},\beta_{2n}),G(R(b,a),R(b,a),\beta_{2n}),\\
&\qquad G(R(b,a),gb,\beta_{2n}),G(R(b,a),\beta_{2n},\beta_{2n+1}),G(\beta_{2n},R(b,a),gb),\\
&\qquad G(R(b,a),R(b,a),gb),G(\beta_{2n+1},\beta_{2n+1},\beta_{2n})\}\}\\
&\quad = \min\{\min\{0,G(R(a,b),\tau,\tau)\},\min\{0,G(R(b,a),\kappa,\kappa)\}\}\\
&\quad = 0
\end{aligned}
$$

which follows that $G(R(a,b),\tau,\tau)\le\lambda\max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}$. Similarly, we can prove that $G(R(b,a),\kappa,\kappa)\le\lambda\max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}$.
Therefore, we have

$$\max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}\le\lambda\max\{G(R(a,b),\tau,\tau),G(R(b,a),\kappa,\kappa)\}$$

which is impossible. Hence $G(R(a,b),\tau,\tau)=0$ and $G(R(b,a),\kappa,\kappa)=0$ which implies that $R(a,b)=\tau$ and $R(b,a)=\kappa$. It follows that $R(a,b)=\tau=ga$ and $R(b,a)=\kappa=gb$. Since $\{R,g\}$ is weakly compatible pair, we have $R(\tau,\kappa)=g\tau$, $R(\kappa,\tau)=g\kappa$. Now we prove that $g\tau=\tau$ and $g\kappa=\kappa$.
By using (3.1) and taking the limit as $n\to\infty$, we have

$$
\begin{aligned}
\zeta(G(g\tau,g\tau,\tau)) &= \lim_{n\to\infty}\zeta(G(g\tau,g\tau,a_{2n+1}))\\
&= \lim_{n\to\infty}\zeta(G(R(\tau,\kappa),R(\tau,\kappa),T(a_{2n+1},b_{2n+1})))\\
&\le \zeta\left(\lambda\lim_{n\to\infty}M(\tau,\kappa,a_{2n+1},b_{2n+1})\right)-\chi\left(\lambda\lim_{n\to\infty}M(\tau,\kappa,a_{2n+1},b_{2n+1})\right)\\
&\qquad +L\lim_{n\to\infty}N(\tau,\kappa,a_{2n+1},b_{2n+1}),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\lim_{n\to\infty}M(\tau,\kappa,a_{2n+1},b_{2n+1}) &= \max\{\max\{G(g\tau,g\tau,\tau),G(R(\tau,\kappa),\tau,\tau),G(R(\tau,\kappa),R(\tau,\kappa),\tau),\\
&\qquad G(R(\tau,\kappa),g\tau,\tau),G(R(\tau,\kappa),\tau,\tau),\\
&\qquad G(\tau,R(\tau,\kappa),\tau),G(\tau,\tau,\tau)\},\\
&\qquad \max\{G(g\kappa,g\kappa,\kappa),G(R(\kappa,\tau),\kappa,\kappa),G(R(\kappa,\tau),R(\kappa,\tau),\kappa),\\
&\qquad G(R(\kappa,\tau),g\kappa,\kappa),G(R(\kappa,\tau),\kappa,\kappa),\\
&\qquad G(\kappa,R(\kappa,\tau),\kappa),G(\kappa,\kappa,\kappa)\}\}
\end{aligned}
$$

$$= \max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\}$$

and

$$\begin{aligned}\lim_{n\to\infty} N(a,b,a_{2n+1},b_{2n+1}) &= \min\{\min\{G(g\tau,g\tau,\tau),G(R(\tau,\kappa),\tau,\tau),G(R(\tau,\kappa),R(\tau,\kappa),\tau),\\ &\qquad G(R(\tau,\kappa),g\tau,\tau),G(R(\tau,\kappa),\tau,\tau),\\ &\qquad G(\tau,R(\tau,\kappa),\tau),G(\tau,\tau,\tau)\},\\ &\quad \min\{G(g\kappa,g\kappa,\kappa),G(R(\kappa,\tau),\kappa,\kappa),G(R(\kappa,\tau),R(\kappa,\tau),\kappa),\\ &\qquad G(R(\kappa,\tau),g\kappa,\kappa),G(R(\kappa,\tau),\kappa,\kappa),\\ &\qquad G(\kappa,R(\kappa,\tau),\kappa),G(\kappa,\kappa,\kappa)\}\}\\ &= \min\{\min\{0,G(g\tau,\tau,\tau)\},\min\{0,G(g\kappa,\kappa,\kappa)\}\}\\ &= 0.\end{aligned}$$

From (3.7), we have

$$\begin{aligned}\zeta(G(g\tau,\tau,\tau)) &\le \zeta(\lambda\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\}) - \chi(\lambda\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\}) + L(0)\\ &\le \zeta(\lambda\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\})\end{aligned}$$

which implies that $G(g\tau,\tau,\tau) \le \lambda\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\}$.
Similarly, we can prove that $G(g\kappa,\kappa,\kappa) \le \lambda\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\}$.
Therefore, we have

$$\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\} \le \lambda\max\{G(g\tau,\tau,\tau),G(g\kappa,\kappa,\kappa)\}$$

which is impossible. Hence $G(g\tau,\tau,\tau)=0$ and $G(g\kappa,\kappa,\kappa)=0$ which implies that $g\tau=\tau$ and $g\kappa=\kappa$. It follows that $R(\tau,\kappa)=\tau=g\tau$ and $R(\kappa,\tau)=\kappa=g\kappa$. Thus (τ,κ) is coupled fixed point of R and g. Since $R(\mathcal{P}^2)\subseteq f(\mathcal{P})$, so there exist $p,q\in\mathcal{P}$ such that $R(\tau,\kappa)=\tau=fp$, $R(\kappa,\tau)=\kappa=fq$.

By using (3.1) and taking the upper limit when $n\to\infty$, we have

$$\begin{aligned}\zeta(G(\tau,\tau,T(p,q))) &= \lim_{n\to\infty}\zeta(G(a_{2n},a_{2n},T(p,q)))\\ &= \lim_{n\to\infty}\zeta(G(R(a_{2n},b_{2n}),R(a_{2n},b_{2n}),T(p,q),))\\ &\le \zeta\left(\lambda\lim_{n\to\infty}M(a_{2n},b_{2n},p,q)\right) - \chi\left(\lambda\lim_{n\to\infty}M(a_{2n},b_{2n},p,q)\right)\\ &\quad + L\lim_{n\to\infty}N(a_{2n},b_{2n},p,q)\\ &\le \zeta(\lambda\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\})\\ &\quad - \chi(\lambda\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\}) + L(0)\\ &\le \zeta(\lambda\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\}).\end{aligned}$$

Because of

$$\begin{aligned}\lim_{n\to\infty}M(a_{2n},b_{2n},p,q) &= \max\{\max\{G(\tau,\tau,fp),G(\tau,T(p,q),fp),G(\tau,\tau,fp),G(\tau,\tau,fp),\\ &\qquad G(\tau,fp,T(p,q)),G(fp,\tau,\tau),G(\tau,\tau,\tau),G(T(p,q),T(p,q),fp)\},\\ &\quad \max\{G(\kappa,\kappa,fq),G(\kappa,T(q,p),fq),G(\kappa,\kappa,fq),G(\kappa,\kappa,fq),\\ &\qquad G(\kappa,fq,T(q,p)),G(fq,\kappa,\kappa),G(\kappa,\kappa,\kappa),G(T(q,p),T(q,p),fq)\}\}\\ &= \max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\}\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}N(a_{2n},b_{2n},p,q) &= \min\{\min\{G(\tau,\tau,fp),G(\tau,T(p,q),fp),G(\tau,\tau,fp),G(\tau,\tau,fp),\\ &\qquad G(\tau,fp,T(p,q)),G(fp,\tau,\tau),G(\tau,\tau,\tau),G(T(p,q),T(p,q),fp)\},\end{aligned}$$

$$\begin{aligned}
&\min\{G(\kappa,\kappa,fq),G(\kappa,T(q,p),fq),G(\kappa,\kappa,fq),G(\kappa,\kappa,fq),\\
&\qquad G(\kappa,fq,T(q,p)),G(fq,\kappa,\kappa),G(\kappa,\kappa,\kappa),G(T(q,p),T(q,p),fq)\}\}\\
&=\min\{\min\{0,G(\tau,\tau,T(p,q))\},\min\{0,G(\kappa,\kappa,T(q,p))\}\}\\
&=0
\end{aligned}$$

which follows that $G(\tau,\tau,T(p,q)) \le \lambda\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\}$. Similarly, we can prove that $G(\kappa,\kappa,T(q,p)) \le \lambda\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\}$.
Therefore, we have

$$\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\} \le \lambda\max\{G(\tau,\tau,T(p,q)),G(\kappa,\kappa,T(q,p))\}$$

which is impossible. Hence $G(\tau,\tau,T(p,q)) = 0$ and $G(\kappa,\kappa,T(q,p)) = 0$ which implies that $T(p,q) = \tau$ and $T(q,p) = \kappa$. It follows that $T(p,q) = \tau = fp$ and $T(q,p) = \kappa = fq$. Since $\{T,f\}$ weakly compatible pair, we have $T(\tau,\kappa) = f\tau$ and $T(\kappa,\tau) = f\kappa$. Now we prove that $f\tau = \tau, f\kappa = \kappa$,

$$\begin{aligned}
\zeta(G(\tau,\tau,f\tau)) &= \lim_{n\to\infty}\zeta(G(\alpha_{2n},\alpha_{2n},f\tau))\\
&= \lim_{n\to\infty}\zeta(G(R(a_{2n},b_{2n}),R(a_{2n},b_{2n}),T(\tau,\kappa)))\\
&\le \zeta\left(\lambda\lim_{n\to\infty}M(a_{2n},b_{2n},\tau,\kappa)\right)-\chi\left(\lambda\lim_{n\to\infty}M(a_{2n},b_{2n},\tau,\kappa)\right)+L\lim_{n\to\infty}N(a_{2n},b_{2n},\tau,\kappa)\\
&\le \zeta(\lambda\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\})-\chi(\lambda\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\})+L(0)\\
&\le \zeta(\lambda\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\}).
\end{aligned}$$

Because of

$$\begin{aligned}
\lim_{n\to\infty}M(a_{2n},b_{2n},\tau,\kappa) &= \max\{\max\{G(\tau,\tau,f\tau),G(\tau,T(\tau,\kappa),f\tau),G(\tau,\tau,f\tau),G(\tau,\tau,f\tau),\\
&\qquad G(\tau,f\tau,T(\tau,\kappa)),G(f\tau,\tau,\tau),G(\tau,\tau,\tau)\},\\
&\qquad \max\{G(\kappa,\kappa,f\kappa),G(\kappa,T(\kappa,\tau),f\kappa),G(\kappa,\kappa,f\kappa),G(\kappa,\kappa,f\kappa),\\
&\qquad G(\kappa,f\kappa,T(\kappa,\tau)),G(f\kappa,\kappa,\kappa),G(\kappa,\kappa,\kappa)\}\}\\
&= \max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}N(a_{2n},b_{2n},\tau,\kappa) &= \min\{\min\{G(\tau,\tau,f\tau),G(\tau,T(\tau,\kappa),f\tau),G(\tau,\tau,f\tau),G(\tau,\tau,f\tau),\\
&\qquad G(\tau,f\tau,T(\tau,\kappa)),G(f\tau,\tau,\tau),G(\tau,\tau,\tau)\},\\
&\qquad \min\{G(\kappa,\kappa,f\kappa),G(\kappa,T(\kappa,\tau),f\kappa),G(\kappa,\kappa,f\kappa),G(\kappa,\kappa,f\kappa),\\
&\qquad G(\kappa,f\kappa,T(\kappa,\tau)),G(f\kappa,\kappa,\kappa),G(\kappa,\kappa,\kappa)\}\}\\
&= 0
\end{aligned}$$

which deduce that $G(\tau,\tau,f\tau) \le \lambda\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\}$.
Similarly, we can prove that $G(\kappa,\kappa,f\kappa) \le \lambda\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\}$.
Therefore, we have $\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\} \le \lambda\max\{G(\tau,\tau,f\tau),G(\kappa,\kappa,f\kappa)\}$ which is impossible. Hence $G(\tau,\tau,f\tau) = 0$ and $G(\kappa,\kappa,f\kappa) = 0$ which implies that $f\tau = \tau$ and $f\kappa = \kappa$. It follows that $T(\tau,\kappa) = \tau = f\tau$ and $T(\kappa,\tau) = \kappa = f\kappa$. Thus (τ,κ) is common coupled fixed point of R,T,f and g. In the following we will show the uniqueness of common coupled fixed point in $\mathcal{P}$. For this purpose, assume that there is another coupled fixed point (τ',κ') of R,T,g,f. Then,

from (3.1), we have

$$\begin{aligned}\zeta(G(\tau,\tau,\tau')) &= \zeta(G(R(\tau,\kappa),R(\tau,\kappa),T(\tau',\kappa')))\\ &\le \zeta(\lambda M(\tau,\kappa,\tau',\kappa')) - \chi(\lambda M(\tau,\kappa,\tau',\kappa')) + LN(\tau,\kappa,\tau',\kappa')\\ &\le \zeta(\lambda \max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}) - \chi(\lambda\max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}) + L(0)\\ &\le \zeta(\lambda\max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}).\end{aligned}$$

Because of

$$\begin{aligned}M(\tau,\kappa,\tau',\kappa') = \max\{&\max\{G(g\tau,g\tau,f\tau'),G(R(\tau,\kappa),T(\tau',\kappa'),f\tau'),G(R(\tau,\kappa),R(\tau,\kappa),f\tau'),\\ &G(R(\tau,\kappa),g\tau,f\tau'),G(R(\tau,\kappa),f\tau',T(\tau',\kappa')),G(T(\tau',\kappa'),f\tau',T(\tau',\kappa')),\\ &G(f\tau',R(\tau,\kappa),g\tau),G(R(\tau,\kappa),R(\tau,\kappa),g\tau)\},\\ &\max\{G(g\kappa,g\kappa,f\kappa'),G(R(\kappa,\tau),T(\kappa',\tau'),f\kappa'),G(R(\kappa,\tau),R(\kappa,\tau),f\kappa'),\\ &G(R(\kappa,\tau),g\kappa,f\kappa'),G(R(\kappa,\tau),f\kappa',T(\kappa',\tau')),G(T(\kappa',\tau'),f\kappa',T(\kappa',\tau')),\\ &G(f\kappa',R(\kappa,\tau),g\kappa),G(R(\kappa,\tau),R(\kappa,\tau),g\kappa)\}\}\\ =\max\{&G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}\end{aligned}$$

and

$$N(\tau,\kappa,\tau',\kappa') = \min\{\min\{0,Gt(\tau,\tau,\tau')\},\min\{0,G(\kappa,\kappa,\kappa')\}\} = 0$$

which deduce that $G(\tau,\tau,\tau') \le \lambda\max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}$.
Similarly, we can prove that $G(\kappa,\kappa,\kappa') \le \lambda\max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}$.
Therefore, we have $\max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\} \le \lambda\max\{G(\tau,\tau,\tau'),G(\kappa,\kappa,\kappa')\}$ which is impossible. Hence $G(\tau,\tau,\tau') = 0$ and $G(\kappa,\kappa,\kappa') = 0$, which implies that $\tau = \tau'$ and $\kappa = \kappa'$. Therefore, (τ,κ) is uniqueness of common coupled fixed point of R,T,f and g. In the similar lines follows we get the unique common fixed point in $\mathcal{P}$. □

From Theorem 3.1, assuming $L = 0$ we deduce the following result:

Corollary 3.2. *Let $(\mathcal{P},G)$ be a G-metric space. Suppose $\zeta : [0,\infty) \to [0,\infty)$ is an altering distance function and $\chi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\chi(t) = 0 \iff t = 0$. Moreover, suppose that $R,T : \mathcal{P}^2 \to \mathcal{P}$ and $f,g : \mathcal{P} \to \mathcal{P}$ be a four mappings satisfying the following:*

$$\zeta(G(R(a,b),R(a,b),T(p,q))) \le \zeta(\lambda M(a,b,p,q)) - \chi(\lambda M(a,b,p,q))$$

(a) *$R(\mathcal{P}^2) \subseteq f(\mathcal{P})$, and $T(\mathcal{P}^2) \subseteq g(\mathcal{P})$,*

(b) *either (R,g) or (T,f) are ω-compatible,*

(c) *one of $g(\mathcal{P})$ or $f(\mathcal{P})$ is complete,*

where $M(a,b,p,q)$ be defined in Theorem 3.1 and $0 < \lambda < 1$. Then there is a unique common coupled fixed point of R,T,f and g in $\mathcal{P}$.

Corollary 3.3. *Let $(\mathcal{P},G)$ be a complete G-metric space. Suppose that $R : \mathcal{P}^2 \to \mathcal{P}$ be a mapping such that $G(R(a,b),R(a,b),R(p,q)) \le \lambda\max\{G(a,a,p),G(b,b,q)\}$ for all $a,b,p,q \in \mathcal{P}$ with $0 < \lambda < 1$, then there is a unique coupled fixed point of R in $\mathcal{P}$.*

Example 3.4. Let $\mathcal{P} = [0,\infty)$ and $G(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}$, $(\mathcal{P},G)$ is a complete G-metric spaces. Let $R,T : \mathcal{P}^2 \to \mathcal{P}$ and $g,f : \mathcal{P} \to \mathcal{P}$ be given by $g(x) = 4x$, $f(x) = x$ and $R(x,y) = \frac{x+y}{2}$, $T(x,y) = \frac{x+y}{8}$.
The terms $\zeta(t)$ and $\chi(t) : [0,\infty) \to [0,\infty)$ defined as, $\zeta(t) = \frac{2t}{5}$ and $\chi(t) = \frac{t}{5}$ for all $t \in [0,\infty)$. Then obviously, $R(\mathcal{P}^2) \subseteq f(\mathcal{P})$, $T(\mathcal{P}^2) \subseteq g(\mathcal{P})$ and the pairs $(R,g),(T,f)$ are ω-compatible. Now we have

$$\begin{aligned}
\zeta(G(R(a,b),R(a,b),T(x,y))) &= \frac{2}{5}G(R(a,b),R(a,b),T(x,y)) \\
&\le \frac{1}{2}\max\{|R(a,b)-T(x,y)|\} \\
&= \frac{1}{2}\max\left\{\left|\frac{a+b}{2}-\frac{x+y}{8}\right|\right\} \\
&= \frac{1}{16}\max\{|4a+4b-x-y|\} \\
&\le \frac{1}{10}\max\{\max\{|4a-x|,0\},\max\{|4b-y|,0\}\} \\
&\le \frac{1}{5}\left(\frac{1}{2}\max\{\max\{G(ga,ga,fx),0\},\max\{G(gb,gb,fy),0\}\}\right) \\
&\le \frac{2}{5}\left(\frac{1}{2}\max\{\max\{G(ga,ga,fx),0\},\max\{G(gb,gb,fy),0\}\}\right) \\
&\quad -\frac{1}{5}\left(\frac{1}{2}\max\{\max\{G(ga,ga,fx),0\},\max\{G(gb,gb,fy),0\}\}\right) \\
&\le \zeta(\lambda M(a,b,x,y)) - \chi(\lambda M(a,b,x,y)) + LN(a,b,x,y).
\end{aligned}$$

Thus all the conditions of the Theorem 3.1 are satisfied and $(0,0)$ is unique common coupled fixed point of T and f.

4. Application to Integral Equations

In this section, we study the existence of an unique solution to an initial value problem, as an application to Corollary 3.2.

Theorem 4.1. *Consider the initial value problem*

$$\frac{dx}{dt} = S(t,x,x), \quad t \in I = [0,1],\ x(0) = x_0, \tag{4.1}$$

where $S : I \times \mathbb{R}^2 \to \mathbb{R}$ *and* $x_0 \in \mathbb{R}$ *with* $\int_0^t S(s,x,y)ds = \max\left\{\begin{array}{l}\int_0^t S(s,x,x)ds \\ \int_0^t S(s,y,y)ds\end{array}\right\}$. *Then there exists unique solution in* $C(I,\mathbb{R})$ *for the initial value problem* (4.1).

Proof. The integral equation corresponding to initial value problem (4.1) is

$$x = x_0 + \int_0^t S(s,x,x)ds.$$

Let $\mathcal{P} = C(I,\mathbb{R})$ and $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for $x,y,z \in \mathcal{P}$.
Define $\zeta,\chi : [0,\infty) \to [0,\infty)$ by $\zeta = \frac{2t}{3}$, $\chi = \frac{t}{4}$. Define $R,T : \mathcal{P}^2 \to \mathcal{P}$ and $g,f : \mathcal{P} \to \mathcal{P}$ by

$$R(x,y) = \frac{x_0}{8} + \int_0^t S(s,x,y)ds, \quad T(x,y) = \frac{x_0}{16} + \int_0^t S(s,x,y)ds,$$

$$f = x_0 + 16\int_0^t S(s,x,x)ds, \qquad g = 2x_0 + 16\int_0^t S(s,x,x)ds.$$

Now

$$\begin{aligned}
&\zeta(G(R(a,b),R(x,y),T(p,q))) \\
&\quad = \frac{2}{3}G(R(a,b),R(x,y),T(p,q)) \\
&\quad = \frac{2}{3}(|R(a,b)-R(x,y)|+|R(x,y)-R(p,q)|+|R(p,q)-R(a,b)|) \\
&\quad = \frac{2}{3}\left\{\left|\frac{a_0}{8}+\int_0^t S(s,(a,b))ds - \frac{x_0}{8}+\int_0^t S(s,(x,y))ds\right|\right. \\
&\qquad + \left|\frac{x_0}{8}+\int_0^t S(s,(x,y))ds - \frac{p_0}{16}+\int_0^t S(s,(p,q))ds\right| \\
&\qquad \left. + \left|\frac{p_0}{16}+\int_0^t S(s,(p,q))ds - \frac{a_0}{8}+\int_0^t S(s,(a,b))ds\right|\right\} \\
&\quad \le \frac{1}{24}\max\{|g(a)-g|+|g-f|+|f-g(a)|, |g-g|+|g-f(q)|+|f(q)-g|\} \\
&\quad \le \frac{1}{8}\max\{G(ga,gx,fp),G(gb,gy,fq)\} \\
&\quad \le \frac{5}{12}\left(\frac{1}{2}\max\{G(ga,gx,fp),G(gb,gy,fq)\}\right) \\
&\quad \le \frac{2}{3}\left(\frac{1}{2}\max\{G(ga,gx,fp),G(gb,gy,fq)\}\right) - \frac{1}{4}\left(\frac{1}{2}\max\{G(ga,gx,fp),G(gb,gy,fq)\}\right) \\
&\quad \le \frac{2}{3}\left(\frac{1}{2}\max\left\{\max\{G(ga,gx,fp),0\},\max\{G(gb,gy,fq),0,\}\right\}\right) \\
&\quad - \frac{1}{4}\left(\frac{1}{2}\max\{\max\{G(ga,gx,fp),0\},\max\{G(gb,gy,fq),0\}\}\right) \\
&\le \zeta(\lambda M(a,b,x,y,p,q)) - \chi(\lambda M(a,b,x,y,p,q)).
\end{aligned}$$

It follows from Corollary 3.2, we conclude that the equation (4.1) has a unique solution in $C(I,\mathbb{R})$. □

5. Application to Homotopy Theory

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 5.1. *Let $(\mathcal{P},G)$ be complete G-metric space, U and $\overline{U}$ be an open and closed subset of $\mathcal{P}$ such that $U \subseteq \overline{U}$. Suppose $H : \overline{U}^2 \times [0,1] \to \mathcal{P}$ be an operator with following conditions are satisfying,*

(τ_0) *$x \ne H(x,y,\kappa)$, $y \ne H(y,x,\kappa)$, for each $x,y \in \partial U$ and $\kappa \in [0,1]$ (here ∂U is boundary of U in $\mathcal{P}$),*

(τ_1) *$G(H(x,y,\kappa),H(x,y,\kappa),H(a,b,\kappa)) \le \lambda \max\{G(x,x,a),G(y,y,b)\}$ for all $x,y \in \overline{U}$ and $\kappa \in [0,1]$,*

(τ_2) *$\exists\, M \ge 0 \ni G(H(x,y,\kappa),H(x,y,\kappa),H(x,y,\zeta)) \le M|\kappa-\zeta|$ for every $x,y,a,b \in \overline{U}$ and $\kappa,\zeta \in [0,1]$.*

Then $H(\cdot,0)$ has a coupled fixed point $\iff$ $H(\cdot,1)$ has a coupled fixed point.

Proof. Let the set

$$X = \{\kappa \in [0,1] : H(x,y,\kappa) = x, H(y,x,\kappa) = y \text{ for some } x,y,z,w, \in U\}.$$

Since $H(\cdot,0)$ has a coupled fixed point in U^2, we have that $(0,0,) \in X^2$. So that X is non-empty set. Now we show that X is both closed and open in $[0,1]$ and hence by the connectedness $X = [0,1]$. As a result, $H(\cdot,1)$ has a coupled fixed point in U^2. First, we show that X closed in $[0,1]$. To see this, Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq X$ with $\kappa_n \to \kappa \in [0,1]$ as $n \to \infty$. We must show that $\kappa \in X$. Since $\kappa_n \in X$ for $n = 0,1,2,3,\cdots$, there exists sequences $\{x_n\},\{y_n\}$ with $x_{n+1} = H(x_n,y_n,\kappa_n)$, $y_{n+1} = H(y_n,x_n,\kappa_n)$. Consider

$$\begin{aligned} G(x_{n+1},x_{n+1},x_{n+2}) &= G(H(x_n,y_n,\kappa_n),H(x_n,y_n,\kappa_n),H(x_{n+1},y_{n+1},\kappa_{n+1})) \\ &\le G(H(x_n,y_n,\kappa_n),H(x_n,y_n,\kappa_n), \\ &\quad H(x_{n+1},y_{n+1},\kappa_n)) + G(H(x_{n+1},y_{n+1},\kappa_n),H(x_{n+1},y_{n+1},\kappa_n), \\ &\quad H(x_{n+1},y_{n+1},\kappa_{n+1})) \\ &\le G(H(x_n,y_n,\kappa_n),H(x_n,y_n,\kappa_n),H(x_{n+1},y_{n+1},\kappa_n)) + M|\kappa_n - \kappa_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} \lim_{n\to\infty} G(x_{n+1},x_{n+1},x_{n+2}) &\le \lim_{n\to\infty} G(H(x_n,y_n,\kappa_n),H(x_n,y_n,\kappa_n),H(x_{n+1},y_{n+1},\kappa_n)) \\ &\le \lim_{n\to\infty} \lambda \max\{G(x_n,x_n,x_{n+1}),G(y_n,y_n,y_{n+1})\}. \end{aligned} \tag{5.1}$$

Similarly, we can prove

$$\lim_{n\to\infty} G(y_{n+1},y_{n+1},y_{n+2}) \le \lim_{n\to\infty} \lambda \max\{G(x_n,x_n,x_{n+1}),G(y_n,y_n,y_{n+1})\} \tag{5.2}$$

Combining both (5.1) and (5.2), we have

$$\begin{aligned} \lim_{n\to\infty} \max\{G(x_{n+1},x_{n+1},x_{n+2}),G(y_{n+1},y_{n+1},y_{n+2})\} &\le \lim_{n\to\infty} \lambda \max\{G(x_n,x_n,x_{n+1}),G(y_n,y_n,y_{n+1})\} \\ &\vdots \\ &\le \lim_{n\to\infty} \lambda^n \max\{G(x_0,x_0,x_1),G(y_0,y_0,y_1)\} \\ &= 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} G(x_{n+1},x_{n+1},x_{n+2}) = 0$ and $\lim_{n\to\infty} G(y_{n+1},y_{n+1},y_{n+2}) = 0$. By use of the rectangle inequality, for $n > m$, we get

$$\begin{aligned} \lim_{m\to\infty} G(x_n,x_n,x_m) &\le \lim_{m\to\infty} G(x_m,x_{m+1},x_{m+1}) + \lim_{m\to\infty} G(x_{m+1},x_n,x_n), \\ &\le \lim_{m\to\infty} G(x_m,x_{m+1},x_{m+1}) + \lim_{m\to\infty} G(x_{m+2},x_{m+2},x_{m+1}) + \cdots + \lim_{m\to\infty} G(x_{n-1},x_n,x_n) \\ &= 0. \end{aligned}$$

By similar arguments, we obtain $G(y_n,y_n,y_m) \to 0$ as $m \to \infty$. This shows that $\{x_n\}$, $\{y_n\}$ are Cauchy sequences in the G-metric space $(\mathcal{P},G)$ and by completeness of $(\mathcal{P},G)$, there exist $a,b \in \mathcal{P}$ with $\lim_{n\to\infty} x_{n+1} = a$, $\lim_{n\to\infty} y_{n+1} = b$. By using (τ_1), we have

$$\begin{aligned} G(H(a,b,\kappa),H(a,b,\kappa),a) &= \lim_{n\to\infty} G(H(a,b,\kappa),H(a,b,\kappa),H(x_{n+1},y_{n+1},\kappa)) \\ &\le \lim_{n\to\infty} \lambda \max\{G(a,a,x_{n+1}),G(b,b,y_{n+1})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned}&\max\{G(H(a,b,\kappa),H(a,b,\kappa),a),G(H(b,a,\kappa),H(b,a,\kappa),b)\}\\&\quad\le \lim_{n\to\infty}\lambda\max\{G(a,a,x_{n+1}),G(b,b,y_{n+1})\}=0.\end{aligned}$$

It follows that $H(a,b,\kappa)=a, H(b,a,\kappa)=b$. Thus $\kappa\in X$. Hence X is closed in $[0,1]$. Let $\kappa_0\in X$, then there exist $x_0,y_0\in U$ with $x_0=H(x_0,y_0,\kappa_0)$, $y_0=H(y_0,x_0,\kappa_0)$. Since U is open, then there exist $r>0$ such that $B_G(x_0,x_0,r)\subseteq U$. Choose $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$ such that $|\kappa-\kappa_0|\le\frac{1}{M^n}<\frac{\epsilon}{2}$, then for $x\in\overline{B_G(x_0,x_0,r)}=\{x\in X/G(x,x,x_0)\le r+G(x_0,x_0,x_0)\}$. Also

$$\begin{aligned}&G(H(x,y,\kappa),H(x,y,\kappa),x_0)\\&\quad=G(H(x,y,\kappa),H(x,y,\kappa),H(x_0,y_0,\kappa_0))\\&\quad\le G(H(x,y,\kappa),H(x,y,\kappa),H(x,y,\kappa_0))+G(H(x,y,\kappa_0),H(x,y,\kappa_0),H(x_0,y_0,\kappa_0))\\&\quad\le M|\kappa-\kappa_0|+G(H(x,y,\kappa_0),H(x,y,\kappa_0),H(x_0,y_0,\kappa_0))\\&\quad\le\frac{1}{M^{n-1}}+G(H(x,y,\kappa_0),H(x,y,\kappa_0),H(x_0,y_0,\kappa_0)).\end{aligned}$$

Letting $n\to\infty$, we obtain

$$\begin{aligned}G(H(x,y,\kappa),H(x,y,\kappa),x_0)&\le G(H(x,y,\kappa_0),H(x,y,\kappa_0),H(x_0,y_0,\kappa_0))\\&\le\lambda\max\{G(x,x,x_0),G(y,y,y_0)\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\max\{G(H(x,y,\kappa),H(x,y,\kappa),x_0)G(H(y,x,\kappa),H(y,x,\kappa),y_0)\}\\&\quad\le\lambda\max\{G(x,x,x_0),G(y,y,y_0)\}\\&\quad\le\lambda\max\{r+G(x_0,x_0,x_0),r+G(y_0,y_0,y_0)\}.\end{aligned}$$

Thus, for each fixed $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$, $H(\cdot,\kappa):\overline{B_G(x_0,x_0,r)}\to\overline{B_G(x_0,x_0,r)}$, $H(\cdot,\kappa):\overline{B_G(y_0,y_0,r)}\to\overline{B_G(y_0,y_0,r)}$. Then all conditions of Theorem 5.1 are satisfied. Thus, we conclude that $H(\cdot,\kappa)$ has a coupled fixed point in $\overline{U}^2$. But this must be in U^2. Since (τ_0) holds. Thus, $\kappa\in X$ for any $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$. Hence $(\kappa_0-\epsilon,\kappa_0+\epsilon)\subseteq X$. Clearly, X is open in $[0,1]$.
For the reverse implication, we use the same strategy. □

6. Conclusion

We ensured the existence and uniqueness of a common fixed point for four mappings via generalized contractive condition in G-metric space which involve altering distance function. Two illustrated applications have been provided.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas, M. A. Khan and S. N. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, *Applied Mathematics and Computation* **217**(1) (2010), 195 – 202, DOI: 10.1016/j.amc.2010.05.042.

[2] M. Abbas, T. Nazir, W. Shatanawi and Z. Mustafa, Fixed and related fixed point theorems for three maps in G-metric spaces, *Hacettepe Journal of Mathematics and Statistics* **41**(2) (2012), 291 – 306, URL: https://dergipark.org.tr/tr/download/article-file/86447.

[3] M. Abbas, T. Nazir and P. Vetro, Common fixed point results for three maps in G-metric spaces, *Filomat* **25**(4) (2011), 1 – 17, DOI: 10.2298/FIL1104001A.

[4] R. P. Agarwal and E. Karapınar, Remarks on some coupled fixed point theorems in G-metric spaces, *Fixed Point Theory and Applications* **2013** (2013), Article number: 2, DOI: 10.1186/1687-1812-2013-2.

[5] H. Aydi, B. Damjanović, B. Samet and W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, *Mathematical and Computer Modelling* **54**(9-10) (2011), 2443 – 2450, DOI: 10.1016/j.mcm.2011.05.059.

[6] H. Aydi, M. Postolache and W. Shatanawi, Coupled fixed point results for (ψ,ϕ)-weakly contractive mappings in ordered G-metric spaces, *Computers & Mathematics with Applications* **63**(1) (2012), 298 – 309, DOI: 10.1016/j.camwa.2011.11.022.

[7] L. Ćirić and V. Lakshmikantham, Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Stochastic Analysis and Applications* **27**(6) (2009), 1246 – 1259, DOI: 10.1080/07362990903259967.

[8] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: Theory, Methods & Applications* **65**(7) (2006), 1379 – 1393, DOI: 10.1016/j.na.2005.10.017.

[9] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bulletin of the Australian Mathematical Society* **30** (1984), 1 – 9, DOI: 10.1017/S0004972700001659.

[10] J. K. Kim and S. Chandok, Coupled common fixed point theorems for generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces, *Fixed Point Theory and Applications* **2013** (2013), Article number: 307, DOI: 10.1186/1687-1812-2013-307.

[11] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications* **70**(12) (2009), 4341 – 4349, DOI: 10.1016/j.na.2008.09.020.

[12] N. V. Luong and N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Analysis: Theory, Methods & Applications* **74**(3) (2011), 983 – 992, DOI: 10.1016/j.na.2010.09.055.

[13] Z. Mustafa, M. Khandagji and W. Shatanawi, Fixed point results on complete G-metric spaces, *Studia Scientiarum Mathematicarum Hungarica* **48**(3) (2011), 304 – 319, DOI: 10.1556/sscmath.48.2011.3.1170.

[14] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory and Applications* **2008** (2008), Article number: 189870, DOI: 10.1155/2008/189870.

[15] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces, *International Journal of Mathematics and Mathematical Sciences* **2009** (2009), Article ID: 283028, DOI: 10.1155/2009/283028.

[16] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis* **7**(2) (2006), 289 – 297, URL: http://yokohamapublishers.jp/online2/opjnca/vol7/p289.html.

[17] Z. Mustafa and B. Sims, Fixed Point theorems for contractive mappings in complete G-metric spaces, *Fixed Point Theory and Applications* **2009** (2009), Article number: 917175, DOI: 10.1155/2009/917175.