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Splines with Minimal Defect and Decomposition Matrices

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Abstract. Finite-dimensional space of twice continuously differentiable splines on a nonuniform grid are considered. We also construct a system of linear functionals biorthogonal to the splines and resolve an interpolation problem generated by this system. We derive the decomposition matrices on an interval and on a segment for the space of forth order (third degree) splines associated with infinite and finite nonuniform grids respectively.

Introduction

Splines and wavelets are used in information theory and, in particular, for creating effective algorithms of processing large information flows (cf. [1]). The most important aspects of the theory of splines are related to interpolation and approximation, as well as to the smoothness and stability of solutions of interpolation and approximation problems.

In this paper, we regard approximation relations as a system of equations which leads to (polynomial [2] or nonpolynomial [3]) minimal splines of maximal smoothness of arbitrary order [9]. For twice continuously differentiable splines of forth order (third degree) – splines with minimal defect on a nonuniform grid we construct a system of linear functionals biorthogonal to the splines and resolve an interpolation problem generated by this system. We derive the decomposition matrices on an interval and on a segment for the space of fourth order (third degree) splines associated with infinite and finite nonuniform grids respectively. Some general approach to construction of biorthogonal systems discussed in the paper [13]. Such representations yield the wavelet decomposition of signals with rapidly varying characteristics (cf. [4, 5, 14]), which essentially saves resources of computational devices. The known two-scale difference (refinement) equations (cf., for example, [6]) is a particular case of the calibration relations obtained in papers [10, 11].

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1. Preliminaries

Introduce the notation: $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+ \stackrel{\text{def}}{=} \{j \mid j \geqslant 0, j \in \mathbb{Z}\}$, $\mathbb{R}^1$ is the set of real numbers. The vector (linear) space of $(m+1)$-dimensional column vectors is denoted by $\mathbb{R}^{m+1}$. We identify vectors of this space with one-column matrices and apply the usual matrix operations, in particular, for two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{m+1}$ the expression $\mathbf{a}^T\mathbf{b}$ is the Euclidean inner product of these vectors. Components of vectors are written in the square brackets and enumerated by $0, 1, \ldots, m$, for example, $\mathbf{a} = ([\mathbf{a}]_0, [\mathbf{a}]_1, \ldots, [\mathbf{a}]_m)^T$. The quadratic matrix with columns $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^{m+1}$ (in the indicated order) is denoted by $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_m)$, and $\det(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_m)$ denotes its determinant. An ordered set $\mathbf{A} \stackrel{\text{def}}{=} \{\mathbf{a}_j\}_{j \in \mathbb{Z}}$ of vectors $\mathbf{a}_j \in \mathbb{R}^{m+1}$ is called a *chain*. A chain is *complete* if $\det(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_j) \neq 0$ for all $j \in \mathbb{Z}$. The set of all functions continuous on (α, β) is denoted by $C(\alpha, \beta)$. For any $S \in \mathbb{Z}_+$ we introduce the notation $C^S(\alpha, \beta) \stackrel{\text{def}}{=} \{u \mid u^{(i)} \in C(\alpha, \beta) \text{ for all } i = 0, 1, 2, \ldots, S\}$, setting $C^0(\alpha, \beta) = C(\alpha, \beta)$. If the components of a vector-valued function $\mathbf{u} \in \mathbb{R}^{m+1}$ are S times continuously differentiable on an interval (α, β), we write $\mathbf{u} \in \mathbf{C}^S(\alpha, \beta)$. We use similar notation $C^S[a, b]$ and $\mathbf{C}^S[a, b]$ for the corresponding spaces on a segment $[a, b]$.

2. Space of splines

On an interval $(\alpha, \beta) \subset \mathbb{R}^1$, we consider a grid $X \stackrel{\text{def}}{=} \{x_j\}_{j \in \mathbb{Z}}$,

$$X : \ldots < x_{-1} < x_0 < x_1 < \ldots, \tag{2.1}$$

where $\alpha \stackrel{\text{def}}{=} \lim_{j \to -\infty} x_j$ and $\beta \stackrel{\text{def}}{=} \lim_{j \to +\infty} x_j$ (the cases $\alpha = -\infty$ and $\beta = +\infty$ are not excluded).

We introduce the notation $M \stackrel{\text{def}}{=} \cup_{j \in \mathbb{Z}}(x_j, x_{j+1})$, $S_j \stackrel{\text{def}}{=} [x_j, x_{j+m+1}]$, $J_k \stackrel{\text{def}}{=} \{k-m, k-m+1, \ldots, k\}$, where $k, j \in \mathbb{Z}$. For $K_0 \geqslant 1$, $K_0 \in \mathbb{R}^1$, we denote by $\mathscr{X}(K_0, \alpha, \beta)$ the class of grids of the form (2.1) possessing the *local quasiuniformity* property (see [7] for more details)

$$K_0^{-1} \leqslant \frac{x_{j+1} - x_j}{x_j - x_{j-1}} \leqslant K_0 \quad \text{for all } j \in \mathbb{Z}.$$

We set $h_X \stackrel{\text{def}}{=} \sup_{j \in \mathbb{Z}} (x_{j+1} - x_j)$.

Let $\mathbb{X}(M)$ be the linear space of real-valued functions on the set M. We consider a vector-valued function $\boldsymbol{\varphi} : (\alpha, \beta) \mapsto \mathbb{R}^{m+1}$ with components in $\mathbb{X}(M)$. If a chain of vectors $\{\mathbf{a}_j\}$ is complete, then the relations

$$\begin{aligned} \sum_{j' \in J_k} \mathbf{a}_{j'}\, \omega_{j'}(t) &\equiv \boldsymbol{\varphi}(t) \quad \text{for all } t \in (x_k, x_{k+1}), \text{ for all } k \in \mathbb{Z}, \\ \omega_j(t) &\equiv 0 \qquad \text{for all } t \notin S_j \cap M, \end{aligned} \tag{2.2}$$

uniquely determine the functions $\omega_j(t)$, $t \in M$, $j \in \mathbb{Z}$. It is clear that $\operatorname{supp} \omega_j(t) \subset S_j$.

By the Cramer formula, from the system of linear algebraic equations (2.2) we find

$$\omega_j(t) = \frac{\det(\{\mathbf{a}_{j'}\}_{j' \in J_k, j' \neq j} \; \|'^j \; \varphi(t))}{\det(\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, \ldots, \mathbf{a}_k)} \quad \text{for all } t \in (x_k, x_{k+1}), \text{ for all } j \in J_k,$$

where $\|'^j$ means that the determinant in the numerator is obtained from the determinant in the denominator by replacing $\mathbf{a}_j$ with $\varphi(t)$ (preserving the column order).

The linear span of functions $\{\omega_j\}_{j\in\mathbb{Z}}$ is called the *space of minimal* $(\mathbf{A}, \varphi)$*-splines of* $(m+1)$*-th order* $(m$*-th degree*$)$ on the grid X and is denoted by

$$\mathbb{S}(X, \mathbf{A}, \varphi) \stackrel{\text{def}}{=} \Big\{ u \mid u = \sum_{j \in \mathbb{Z}} c_j \, \omega_j \text{ for all } c_j \in \mathbb{R}^1 \Big\}.$$

The conditions (2.2) are called the *approximation relations*, the vector-valued function φ is called the *generator* of $(\mathbf{A}, \varphi)$-splines, and the chain $\mathbf{A}$ is called the *defining* chain for $(\mathbf{A}, \varphi)$-splines.

For a vector-valued function $\varphi \in \mathbf{C}^S(\alpha, \beta)$ we set

$$\varphi_k \stackrel{\text{def}}{=} \varphi(x_k), \quad \varphi_k^{(i)} \stackrel{\text{def}}{=} \varphi^{(i)}(x_k), \quad i = 0, 1, \ldots, S, \quad k \in \mathbb{Z}.$$

We consider the vector-valued function $\mathbf{\Pi}(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}) \in \mathbb{R}^{m+1}$ defined by the identity

$$\mathbf{\Pi}^T(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1})\,\mathbf{z} \equiv \det\big(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}, \mathbf{z}\big)$$

for all $\mathbf{z}, \mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1} \in \mathbb{R}^{m+1}$. The vector-valued function $\mathbf{\Pi}(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1})$ is called the *m-fold vector product* (cf. details in [8]) in the space $\mathbb{R}^{m+1}$ and is denoted by $\mathbf{z}_0 \times \mathbf{z}_1 \times \ldots \times \mathbf{z}_{m-1}$.

For $\varphi \in \mathbf{C}^{m-1}(\alpha, \beta)$ we consider the vectors

$$\mathbf{d}_j \stackrel{\text{def}}{=} \varphi_j \times \varphi_j' \times \ldots \times \varphi_j^{(m-1)}. \tag{2.3}$$

Let $\varphi \in \mathbf{C}^m(\alpha, \beta)$. We introduce the Wronskian determinant

$$W(t) \stackrel{\text{def}}{=} \det\big(\varphi(t), \varphi'(t), \ldots, \varphi^{(m-1)}(t), \varphi^{(m)}(t)\big).$$

We define the following vector chain $\mathbf{A}^* \stackrel{\text{def}}{=} \{\mathbf{a}_j^*\}$:

$$\mathbf{a}_j^* \stackrel{\text{def}}{=} -\mathbf{d}_{j+1} \times \mathbf{d}_{j+2} \times \ldots \times \mathbf{d}_{j+m}. \tag{2.4}$$

Theorem 2.1. *Let* $\varphi \in \mathbf{C}^m(\alpha, \beta)$*. If* $|W(t)| \geqslant c = \text{const} > 0$ *for all* $t \in (\alpha, \beta)$ *and* $X \in \mathscr{X}(K_0, \alpha, \beta)$ *for some* $K_0 \geqslant 1$*, then for sufficiently small* h_X *the space* $\mathbb{S}(X, \mathbf{A}^*, \varphi)$ *lies in the space* $C^{m-1}(\alpha, \beta)$*.*

The proof of this theorem can be found in [9].

Corollary 2.1. *Under the assumptions of Theorem* 2.1*, the chain* $\{\mathbf{d}_j\}_{j\in\mathbb{Z}}$ *is complete and*

$$\mathbf{d}_j^T \mathbf{a}_j^* \neq 0, \quad \mathbf{d}_{j+m+1}^T \mathbf{a}_j^* \neq 0.$$

The space $\mathbb{S}(X, \mathbf{A}^*, \varphi)$ is called the *space of minimal B_φ-splines of $(m+1)$-th order (m-th degree)* on the grid X, and splines of this space are referred to as *minimal splines of maximal smoothness*. The difference between the degree of the spline and the order of the highest continuous derivative is called the *defect* of the spline. Minimal splines of maximal smoothness are referred to as *splines with minimal defect*.

Let $m = 3$. We consider a vector-valued function $\varphi : (\alpha, \beta) \mapsto \mathbb{R}^4$ with components in $\mathbb{X}(M)$. It is obvious that the equalities

$$\mathbf{d}_{j+p}^T \mathbf{a}_j^* = 0 \quad \text{for all } p = 1,2,3, \text{ for all } j \in \mathbb{Z},$$

hold for any $p = 1,2,3$ in view of the properties of the m-fold vector product.

Theorem 2.2. *If $\varphi \in \mathbf{C}^3(\alpha, \beta)$, then $\omega_j \in C^2(\alpha, \beta)$ and*

$$\omega_j(t) = \begin{cases} \dfrac{\mathbf{d}_j^T \varphi(t)}{\mathbf{d}_j^T \mathbf{a}_j^*}, & t \in [x_j, x_{j+1}), \\[2ex] \dfrac{\mathbf{d}_j^T \varphi(t)}{\mathbf{d}_j^T \mathbf{a}_j^*} - \dfrac{\mathbf{d}_j^T \mathbf{a}_{j+1}^*}{\mathbf{d}_j^T \mathbf{a}_j^*} \dfrac{\mathbf{d}_{j+1}^T \varphi(t)}{\mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^*}, & t \in [x_{j+1}, x_{j+2}), \\[2ex] \dfrac{\mathbf{d}_{j+4}^T \varphi(t)}{\mathbf{d}_{j+4}^T \mathbf{a}_j^*} - \dfrac{\mathbf{d}_{j+4}^T \mathbf{a}_{j-1}^*}{\mathbf{d}_{j+4}^T \mathbf{a}_j^*} \dfrac{\mathbf{d}_{j+3}^T \varphi(t)}{\mathbf{d}_{j+3}^T \mathbf{a}_{j-1}^*}, & t \in [x_{j+2}, x_{j+3}), \\[2ex] \dfrac{\mathbf{d}_{j+4}^T \varphi(t)}{\mathbf{d}_{j+4}^T \mathbf{a}_j^*}, & t \in [x_{j+3}, x_{j+4}). \end{cases} \tag{2.5}$$

The proof of this theorem can be found in [11].

Let $[\varphi(t)]_0 \equiv 1$ for all $t \in (\alpha, \beta)$. If a vector chain $\mathbf{A}^N \stackrel{\text{def}}{=} \{\mathbf{a}_j^N\}$ is defined by the formula $\mathbf{a}_j^N \stackrel{\text{def}}{=} [\mathbf{d}_{j+1} \times \mathbf{d}_{j+2} \times \mathbf{d}_{j+3}]_0^{-1} \mathbf{d}_{j+1} \times \mathbf{d}_{j+2} \times \mathbf{d}_{j+3}$, then $\sum_j \omega_j(t) \equiv 1$ for all $t \in (\alpha, \beta)$. The space $\mathbb{S}(X, \mathbf{A}^N, \varphi)$ is the *space of normalized B_φ-splines of third order* on the grid X.

Corollary 2.2 (cf. [2]). *For $\varphi(t) = (1, t, t^2, t^3)^T$ the functions $\omega_j(t)$ coincide with the known polynomial B-splines of fourth degree.*

We consider finite-dimensional spaces of splines. We set $a \stackrel{\text{def}}{=} x_0$, $b \stackrel{\text{def}}{=} x_n$, $J_{3,n} \stackrel{\text{def}}{=} \{-3, -2, \ldots, n-1, n\}$. From the infinite grid X we extract a finite grid X_n, $n \in \mathbb{N}$, $n \geqslant 4$,

$$X_n : x_{-3} < \ldots < a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b < \ldots < x_{n+3},$$

and from the complete infinite chain $\mathbf{A}^* \in \mathbb{A}$ we extract a finite chain $\mathbf{A}_n^* \stackrel{\text{def}}{=} \{\mathbf{a}_{-4}^*, \mathbf{a}_{-3}^*, \ldots, \mathbf{a}_n^*\}$.

We restrict all functions in the space $\mathbb{S}(X, \mathbf{A}^*, \varphi)$ onto the set $[a, b]$. The set of these restrictions is the finite-dimensional linear space

$$\mathbb{S}(X_n, \mathbf{A}_n^*, \varphi) \stackrel{\text{def}}{=} \left\{ u \mid u = \sum_{j \in J_{3,n-1}} c_j\, \omega_j \quad \text{for all} \quad c_j \in \mathbb{R}^1 \right\} \subset C^2[a, b].$$

Theorem 2.3. *The function* $u_n(t) \stackrel{\text{def}}{=} \sum_{j \in J_{3,n-1}} c_j\, \omega_j(t)$, $t \in [a, b]$, *is the trace of the function* $u(t) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} c_j\, \omega_j(t)$, $t \in (\alpha, \beta)$, *on the segment* $[a, b]$, *belongs to the space* $\mathbb{S}(X_n, \mathbf{A}_n^*, \varphi)$, *and is completely determined by the grid points* $\{x_j\}_{j \in J_{3,n+3}}$, *vectors* $\{\varphi_j^{(S)}\}_{j \in J_{3,n+3}}$, $S = 0, 1, 2$, *and coefficients* $\{c_j\}_{j \in J_{3,n-1}}$.

The proof follows from the definition of the spaces $\mathbb{S}(X, \mathbf{A}^*, \varphi)$ and $\mathbb{S}(X_n, \mathbf{A}_n^*, \varphi)$.

Corollary 2.3. *The restrictions of* ω_j *form a linearly independent system on the segment* $[a, b]$; *moreover,* $\dim \mathbb{S}(X_n, \mathbf{A}_n^*, \varphi) = n + 3$.

3. Biorthogonal system of functionals and calibration relations

We consider a linear space $\mathfrak{U}$ over the field of real numbers and the dual space $\mathfrak{U}^*$ of linear functionals f over the space $\mathfrak{U}$. The value of a functional f at an element $u \in \mathfrak{U}$ is denoted by $\langle f, u \rangle$. A system of functionals $\{f_j\}_{j \in \mathbb{Z}}$ is said to be *biorthogonal* to the system of functions $\{\omega_{j'}\}_{j' \in \mathbb{Z}}$ if $\langle f_j, \omega_{j'} \rangle = \delta_{j,j'}$ for all $j, j' \in \mathbb{Z}$, where $\delta_{j,j'}$ is the Kronecker symbol.

We consider linear functionals $\{f_j\}_{j \in \mathbb{Z}}$ defined on $C^2(\alpha, \beta)$ by the formula

$$\langle f_j, u \rangle \stackrel{\text{def}}{=} a_{j+1}^0 u(x_{j+1}) + a_{j+1}^1 u'(x_{j+1}) + a_{j+1}^2 u''(x_{j+1}),$$

where

$$a_{j+1}^0 \stackrel{\text{def}}{=} \mathbf{d}_{j+2}^T \varphi'_{j+1} \mathbf{d}_{j+3}^T \varphi''_{j+1} - \mathbf{d}_{j+3}^T \varphi'_{j+1} \mathbf{d}_{j+2}^T \varphi''_{j+1}, \tag{3.1}$$

$$a_{j+1}^1 \stackrel{\text{def}}{=} -(\mathbf{d}_{j+2}^T \varphi_{j+1} \mathbf{d}_{j+3}^T \varphi''_{j+1} - \mathbf{d}_{j+3}^T \varphi_{j+1} \mathbf{d}_{j+2}^T \varphi''_{j+1}), \tag{3.2}$$

$$a_{j+1}^2 \stackrel{\text{def}}{=} \mathbf{d}_{j+2}^T \varphi_{j+1} \mathbf{d}_{j+3}^T \varphi'_{j+1} - \mathbf{d}_{j+3}^T \varphi_{j+1} \mathbf{d}_{j+2}^T \varphi'_{j+1}. \tag{3.3}$$

The result of the action of a functional f_j on a function u is defined by the value of u and its derivatives at the point x_{j+1} which is referred to as the *support* of f_j and is written as $\operatorname{supp} f_j = x_{j+1}$.

Theorem 3.1. *The system of linear functionals* $\{f_j\}_{j \in \mathbb{Z}}$ *is biorthogonal to the system of splines* $\{\omega_{j'}\}_{j' \in \mathbb{Z}}$, *i.e.*

$$\langle f_j, \omega_{j'} \rangle = \delta_{j,j'}, \quad \text{for all } j, j' \in \mathbb{Z}. \tag{3.4}$$

Proof. Let us prove that a system of functionals $\{f_j\}_{j \in \mathbb{Z}}$ is biorthogonal to a system of splines $\{\omega_{j'}\}_{j' \in \mathbb{Z}}$ if and only if

$$\langle f_j, \varphi \rangle = \mathbf{a}_j^*. \tag{3.5}$$

Indeed, applying functional f_j to approximation relations (2.2), we obtain the equality

$$\sum_{j'=j-m}^{j} \mathbf{a}^*_{j'} \langle f_j, \omega_{j'} \rangle = \langle f_j, \boldsymbol{\varphi} \rangle. \tag{3.6}$$

By biorthogonality (3.5) holds. Conversely, if (3.5) holds, then (3.6) implies

$$\sum_{j'=j-m}^{j} \mathbf{a}^*_{j'} \langle f_j, \omega_{j'} \rangle = \mathbf{a}^*_j.$$

By the completeness of a chain $\mathbf{a}^*_{j'}$, by location of support of functional and by uniqueness of the last system we get biorthogonality.

By the definition (2.4), we can represent vector $\mathbf{a}^*_j$ as symbolic determinant

$$\mathbf{a}^*_j = -\mathbf{d}_{j+1} \times \mathbf{d}_{j+2} \times \mathbf{d}_{j+3} = \begin{vmatrix} \boldsymbol{\varphi}_{j+1} & \boldsymbol{\varphi}'_{j+1} & \boldsymbol{\varphi}''_{j+1} \\ \mathbf{d}^T_{j+2}\,\boldsymbol{\varphi}_{j+1} & \mathbf{d}^T_{j+2}\,\boldsymbol{\varphi}'_{j+1} & \mathbf{d}^T_{j+2}\,\boldsymbol{\varphi}''_{j+1} \\ \mathbf{d}^T_{j+3}\,\boldsymbol{\varphi}_{j+1} & \mathbf{d}^T_{j+3}\,\boldsymbol{\varphi}'_{j+1} & \mathbf{d}^T_{j+3}\,\boldsymbol{\varphi}''_{j+1} \end{vmatrix}.$$

Expanding the determinant along the first row we obtain

$$\mathbf{a}^*_j = a^0_{j+1}\boldsymbol{\varphi}_{j+1} + a^1_{j+1}\boldsymbol{\varphi}'_{j+1} + a^2_{j+1}\boldsymbol{\varphi}''_{j+1},$$

where coefficients a^0_{j+1}, a^1_{j+1}, a^2_{j+1} defined by formulas (3.1)-(3.3); hence (3.5) holds. Therefore (3.4) holds. □

We consider the interpolation problem

$$\langle f_j, u \rangle = v_j \quad \text{for all } j \in \mathbb{Z},\ u \in \mathbb{S}(X, \mathbf{A}^*, \boldsymbol{\varphi}), \tag{3.7}$$

where $\{v_j\}_{j\in\mathbb{Z}}$ is a given sequence (infinite towards both directions) of numbers.

Theorem 3.2. *In the space* $\mathbb{S}(X, \mathbf{A}^*, \boldsymbol{\varphi})$ *there exists a unique solution to the problem* (3.7)*, and this solution is determined by the formula*

$$u(t) = \sum_{j\in\mathbb{Z}} v_j\, \omega_j(t).$$

Proof. The assertion follows from Theorem 3.1. □

From the original grid X for fixed $k \in \mathbb{Z}$ we eliminate one grid point x_{k+1}. On the obtained *enlarged* (*sparse*) grid $\widetilde{X}$, we consider splines $\widetilde{\omega}_j(t)$, $j \in \mathbb{Z}$.

Suppose that $\xi \stackrel{\text{def}}{=} x_{k+1}$ and $\widetilde{x}_j$ are grid points of the new grid $\widetilde{X} \stackrel{\text{def}}{=} \{\widetilde{x}_j \mid j \in \mathbb{Z}\}$:

$$\widetilde{x}_j \stackrel{\text{def}}{=} \begin{cases} x_j, & j \leqslant k, \\ x_{j+1}, & j \geqslant k+1. \end{cases}$$

We use the tilde for denoting the above-introduced objects considered in the new grid $\widetilde{X}$. The functions $\widetilde{\omega}_j(t)$ can be found by formula (2.5), by replacing the grid points of the original grid x_j with $\widetilde{x}_j$, $j \in \mathbb{Z}$. It is easy to see that

$$\mathbf{d}_j = \begin{cases} \widetilde{\mathbf{d}}_j, & j \leqslant k, \\ \widetilde{\mathbf{d}}_{j-1}, & j \geqslant k+2. \end{cases} \tag{3.8}$$

From (2.3), (2.4), and (3.8) we find

$$\mathbf{a}_j^* = \begin{cases} \widetilde{\mathbf{a}}_j^*, & j \leqslant k-3, \\ \widetilde{\mathbf{a}}_{j-1}^*, & j \geqslant k+1. \end{cases} \tag{3.9}$$

It is obvious that for $t \in (\alpha, \beta)$

$$\widetilde{\omega}_j(t) \equiv \begin{cases} \omega_j(t), & j \leqslant k-4, \\ \omega_{j+1}(t), & j \geqslant k+1. \end{cases} \tag{3.10}$$

We introduce infinite-dimensional column vectors with components $\omega_j(t)$ and $\widetilde{\omega}_j(t)$, $j \in \mathbb{Z}$:

$$\boldsymbol{\omega}(t) \stackrel{\text{def}}{=} (\ldots, \omega_{-2}(t), \omega_{-1}(t), \omega_0(t), \omega_1(t), \omega_2(t), \ldots)^T,$$
$$\widetilde{\boldsymbol{\omega}}(t) \stackrel{\text{def}}{=} (\ldots, \widetilde{\omega}_{-2}(t), \widetilde{\omega}_{-1}(t), \widetilde{\omega}_0(t), \widetilde{\omega}_1(t), \widetilde{\omega}_2(t), \ldots)^T.$$

Any function $\widetilde{\omega}_i(t)$ can be represented as a finite linear combination of functions $\omega_j(t)$:

$$\widetilde{\boldsymbol{\omega}}(t) = \widetilde{\mathfrak{P}}\, \boldsymbol{\omega}(t) \quad \Leftrightarrow \quad \widetilde{\omega}_i(t) = \sum_{j \in \mathbb{Z}} \widetilde{\mathfrak{p}}_{i,j}\, \omega_j(t) \quad \text{for all } i \in \mathbb{Z}, \tag{3.11}$$

where $\widetilde{\mathfrak{P}}$ is an infinite matrix of the form $\widetilde{\mathfrak{P}} \stackrel{\text{def}}{=} (\widetilde{\mathfrak{p}}_{i,j})_{i,j \in \mathbb{Z}}$ with entries $\widetilde{\mathfrak{p}}_{i,j} \stackrel{\text{def}}{=} \langle f_j, \widetilde{\omega}_i \rangle$. The identities (3.11) are called the *knot removal calibration relations*, the matrix $\widetilde{\mathfrak{P}}$ is called the *matrix of sparse reconstruction* on (α, β) (cf. [11]).

We consider the case where the original grid X is extended by a new grid point $\overline{\xi}$ and the splines $\overline{\omega}_j(t)$, $j \in \mathbb{Z}$, are constructed on this refined grid $\overline{X}$. Let $\overline{\xi} \in (x_k, x_{k+1})$, and let $\overline{x}_j$ be grid points of the new grid $\overline{X} \stackrel{\text{def}}{=} \{\overline{x}_j \mid j \in \mathbb{Z}\}$:

$$\overline{x}_j \stackrel{\text{def}}{=} \begin{cases} x_j, & j \leqslant k, \\ \overline{\xi}, & j = k+1, \\ x_{j-1}, & j \geqslant k+2. \end{cases}$$

We use the bar for denoting the above-introduced objects considered in the new grid $\overline{X}$. The functions $\overline{\omega}_j(t)$ can be found according formula (2.5) by replacing the points of the original grid x_j with the points $\overline{x}_j$, $j \in \mathbb{Z}$.

We introduce the infinite-dimensional column vector $\overline{\boldsymbol{\omega}}(t)$ with components $\overline{\omega}_j(t)$, $j \in \mathbb{Z}$:

$$\overline{\boldsymbol{\omega}}(t) \stackrel{\text{def}}{=} (\ldots, \overline{\omega}_{-2}(t), \overline{\omega}_{-1}(t), \overline{\omega}_0(t), \overline{\omega}_1(t), \overline{\omega}_2(t), \ldots)^T.$$

Any function $\omega_i(t)$ can be represented as a finite linear combination of functions $\overline{\omega}_j(t)$:

$$\boldsymbol{\omega}(t) = \overline{\mathfrak{P}}\,\overline{\boldsymbol{\omega}}(t) \quad \Longleftrightarrow \quad \omega_i(t) = \sum_{j\in\mathbb{Z}} \overline{\mathfrak{p}}_{i,j}\,\overline{\omega}_j(t) \quad \text{for all } i \in \mathbb{Z}, \tag{3.12}$$

where $\overline{\mathfrak{P}}$ is an infinite matrix of the form $\overline{\mathfrak{P}} \stackrel{\text{def}}{=} (\overline{\mathfrak{p}}_{i,j})_{i,j\in\mathbb{Z}}$ with entries $\overline{\mathfrak{p}}_{i,j} \stackrel{\text{def}}{=} \langle \overline{f}_j, \omega_i \rangle$. The identities (3.12) are called the *knot insertion calibration relations*, the matrix $\overline{\mathfrak{P}}$ is called the *matrix of dense reconstruction* on (α, β) (cf. [10, 11]).

4. **Decomposition matrices**

We consider a system of functionals $\{\widetilde{f}_j\}_{j\in\mathbb{Z}}$ that is biorthogonal to the system of splines $\{\widetilde{\omega}_{j'}\}_{j'\in\mathbb{Z}}$. We proceed by computing the expressions

$$\widetilde{\mathfrak{q}}_{i,j} \stackrel{\text{def}}{=} \langle \widetilde{f}_i, \omega_j \rangle \quad \text{for all } i, j \in \mathbb{Z}.$$

Theorem 4.1. *For $i, j, k \in \mathbb{Z}$ the following relations hold:*

$$\widetilde{\mathfrak{q}}_{i,j} = \begin{cases} \delta_{i,j}, & \{j \leqslant k-4, \text{ for all } i \in \mathbb{Z}\} \cup \\ & \{j = k-3, \dots, k+1, i \leqslant k-3\}, \\ \mathbf{d}_{k+3}^T \widetilde{\mathbf{a}}_{k-2}^* \left(\mathbf{d}_{k+3}^T \mathbf{a}_{k-2}^* - \mathbf{d}_{k+3}^T \mathbf{a}_{k-3}^* \dfrac{\mathbf{d}_{k+2}^T \mathbf{a}_{k-2}^*}{\mathbf{d}_{k+2}^T \mathbf{a}_{k-3}^*} \right)^{-1}, & i = k-2, j = k-2, \\ \dfrac{\mathbf{d}_{k-3}^T \widetilde{\mathbf{a}}_{k-2}^*}{\mathbf{d}_{k-3}^T \mathbf{a}_{k-3}^*} - \dfrac{\mathbf{d}_{k-3}^T \mathbf{a}_{k-2}^*}{\mathbf{d}_{k-3}^T \mathbf{a}_{k-3}^*} \dfrac{\mathbf{d}_{k-2}^T \widetilde{\mathbf{a}}_{k-2}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*}, & i = k-2, j = k-3, \\ \dfrac{\mathbf{d}_{k-1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k-1}^T \mathbf{a}_{k-1}^*}, & i = k-1, j = k-1, \\ \dfrac{\mathbf{d}_{k-2}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*} - \dfrac{\mathbf{d}_{k-2}^T \mathbf{a}_{k-1}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*} \dfrac{\mathbf{d}_{k-1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k-1}^T \mathbf{a}_{k-1}^*}, & i = k-1, j = k-2, \\ \dfrac{\mathbf{d}_{k+1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k+1}^T \mathbf{a}_{k-3}^*}, & i = k-1, j = k-3, \\ \delta_{i,j-1}, & \{j = k-3, \dots, k+1, i \geqslant k\} \cup \\ & \{j \geqslant k+2, \text{ for all } i \in \mathbb{Z}\}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

Proof. 1. Let $j \leqslant k-4$. By the relations (3.10) we have $\omega_j = \widetilde{\omega}_j$. By biorthogonality we have

$$\widetilde{\mathfrak{q}}_{i,j} = \langle \widetilde{f}_i, \omega_j \rangle = \langle \widetilde{f}_i, \widetilde{\omega}_j \rangle = \delta_{i,j} \quad j \leqslant k-4, \quad \text{for all } i \in \mathbb{Z}.$$

2. Let $j \geqslant k+2$. By the relations (3.10), we have $\omega_j = \widetilde{\omega}_{j-1}$. Hence

$$\widetilde{\mathfrak{q}}_{i,j} = \langle \widetilde{f}_i, \omega_j \rangle = \langle \widetilde{f}_i, \widetilde{\omega}_{j-1} \rangle = \delta_{i,j-1} \quad j \geqslant k+2, \quad \text{for all } i \in \mathbb{Z}.$$

3. Let $j = k-3, k-2, k-1, k, k+1,\ i \leqslant k-3$. By (3.9) we have $\widetilde{\mathbf{a}}_i^* = \mathbf{a}_i^*$, therefore $\langle \widetilde{f}_i, \varphi\rangle = \langle f_i, \varphi\rangle$. Hence the application of functional $\widetilde{f}_i$ to a function ω_j is equivalent to the application of functional f_i to the same function. By biorthogonality we have

$$\widetilde{\mathfrak{q}}_{i,j} = \langle \widetilde{f}_i, \omega_j\rangle = \langle f_i, \omega_j\rangle = \delta_{i,j} \quad j = k-3, k-2, k-1, k, k+1,\ i \leqslant k-3.$$

4. Let $j = k-3, k-2, k-1, k, k+1,\ i \geqslant k$. By (3.9) we have $\widetilde{\mathbf{a}}_i^* = \mathbf{a}_{i+1}^*$, therefore $\langle \widetilde{f}_i, \varphi\rangle = \langle f_{i+1}, \varphi\rangle$. By biorthogonality we have

$$\widetilde{\mathfrak{q}}_{i,j} = \langle \widetilde{f}_i, \omega_j\rangle = \langle f_{i+1}, \omega_j\rangle = \delta_{i+1,j} = \delta_{i,j-1},$$

$$j = k-3, k-2, k-1, k, k+1, \quad i \geqslant k.$$

It remains to consider $i = k-2, j = k-3, k-2$ and $i = k-1, j = k-3, k-2, k-1$.

5. We consider $\widetilde{\mathfrak{q}}_{k-2,k-2} = \langle \widetilde{f}_{k-2}, \omega_{k-2}\rangle$. For $t \in [\widetilde{x}_{k-2}, \widetilde{x}_{k-1}] = [x_{k-2}, x_{k-1}]$ by (3.11) the following calibration relations hold $\widetilde{\omega}_{k-2}(t) = \widetilde{\mathfrak{p}}_{k-2,k-2}\omega_{k-2}(t) + \widetilde{\mathfrak{p}}_{k-2,k-1}\omega_{k-1}(t)$. By taking into account the location of supports of the functions considered there, we conclude that the expressions $\widetilde{\mathfrak{p}}_{k-2,k-2}\,\omega_{k-2}(t)$ and $\widetilde{\omega}_{k-2}(t)$ coincide. Hence the values of functionals $\widetilde{f}_{k-2}$ on these expressions coincide. By biorthogonality (3.4), we have

$$\begin{aligned}
\widetilde{\mathfrak{q}}_{k-2,k-2} &= \langle \widetilde{f}_{k-2}, \omega_{k-2}\rangle \\
&= \langle \widetilde{f}_{k-2}, \widetilde{\omega}_{k-2}\rangle / \widetilde{\mathfrak{p}}_{k-2,k-2} \\
&= \langle f_{k-2}, \widetilde{\omega}_{k-2}\rangle^{-1} \\
&= \mathbf{d}_{k+3}^T \widetilde{\mathbf{a}}_{k-2}^* \left(\mathbf{d}_{k+3}^T \mathbf{a}_{k-2}^* - \mathbf{d}_{k+3}^T \mathbf{a}_{k-3}^* \frac{\mathbf{d}_{k+2}^T \mathbf{a}_{k-2}^*}{\mathbf{d}_{k+2}^T \mathbf{a}_{k-3}^*} \right)^{-1}.
\end{aligned}$$

6. We consider $\widetilde{\mathfrak{q}}_{k-2,k-3} = \langle \widetilde{f}_{k-2}, \omega_{k-3}\rangle$. We use the representation (2.5) of function ω_j for $j = k-3$. Since $\operatorname{supp} \widetilde{f}_{k-2} \subset [\widetilde{x}_{k-2}, \widetilde{x}_{k-1}]$, it suffices to consider only the case $[\widetilde{x}_{k-2}, \widetilde{x}_{k-1}] = [x_{k-2}, x_{k-1}]$, i.e. to use formula (2.5) for $[x_{j+1}, x_{j+2}]$, where $j = k-3$:

$$\omega_{k-3}(t) = \frac{\mathbf{d}_{k-3}^T \varphi(t)}{\mathbf{d}_{k-3}^T \mathbf{a}_{k-3}^*} - \frac{\mathbf{d}_{k-3}^T \mathbf{a}_{k-2}^*}{\mathbf{d}_{k-3}^T \mathbf{a}_{k-3}^*} \frac{\mathbf{d}_{k-2}^T \varphi(t)}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*}, \quad t \in [x_{k-2}, x_{k-1}].$$

Since $\langle \widetilde{f}_{k-2}, \varphi\rangle = \widetilde{\mathbf{a}}_{k-2}^*$, from previous equality we find

$$\widetilde{\mathfrak{q}}_{k-2,k-3} = \langle \widetilde{f}_{k-2}, \omega_{k-3}\rangle = \frac{\mathbf{d}_{k-3}^T \widetilde{\mathbf{a}}_{k-2}^*}{\mathbf{d}_{k-3}^T \mathbf{a}_{k-3}^*} - \frac{\mathbf{d}_{k-3}^T \mathbf{a}_{k-2}^*}{\mathbf{d}_{k-3}^T \mathbf{a}_{k-3}^*} \frac{\mathbf{d}_{k-2}^T \widetilde{\mathbf{a}}_{k-2}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*}.$$

7. We consider $\widetilde{\mathfrak{q}}_{k-1,k-1} = \langle \widetilde{f}_{k-1}, \omega_{k-1}\rangle$. For $t \in [\widetilde{x}_{k-1}, \widetilde{x}_k] = [x_{k-1}, x_k]$ by (3.11) the following calibration relations hold $\widetilde{\omega}_{k-1}(t) = \widetilde{\mathfrak{p}}_{k-1,k-1}\omega_{k-1}(t) + \widetilde{\mathfrak{p}}_{k-1,k}\omega_k(t)$. By taking into account the location of supports of the functions considered there, we conclude that the expressions $\widetilde{\mathfrak{p}}_{k-1,k-1}\,\omega_{k-1}(t)$ and $\widetilde{\omega}_{k-1}(t)$ coincide. Hence

the values of functionals $\widetilde{f}_{k-1}$ on these expressions coincide. By biorthogonality (3.4), we have

$$\widetilde{\mathfrak{q}}_{k-1,k-1} = \langle \widetilde{f}_{k-1}, \omega_{k-1} \rangle = \langle \widetilde{f}_{k-1}, \widetilde{\omega}_{k-1} \rangle / \widetilde{\mathfrak{p}}_{k-1,k-1} = \langle f_{k-1}, \widetilde{\omega}_{k-1} \rangle^{-1} = \frac{\mathbf{d}_{k-1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k-1}^T \mathbf{a}_{k-1}^*}.$$

8. We consider $\widetilde{\mathfrak{q}}_{k-1,k-2} = \langle \widetilde{f}_{k-1}, \omega_{k-2} \rangle$. We use the representation (2.5) of function ω_j for $j = k-2$. It is clear that only following formula is required

$$\omega_j(t) = \frac{\mathbf{d}_j^T \varphi(t)}{\mathbf{d}_j^T \mathbf{a}_j^*} - \frac{\mathbf{d}_j^T \mathbf{a}_{j+1}^*}{\mathbf{d}_j^T \mathbf{a}_j^*} \frac{\mathbf{d}_{j+1}^T \varphi(t)}{\mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^*}, \qquad t \in [x_{j+1}, x_{j+2}],$$

which takes the following form for $j = k-2$:

$$\omega_{k-2}(t) = \frac{\mathbf{d}_{k-2}^T \varphi(t)}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*} - \frac{\mathbf{d}_{k-2}^T \mathbf{a}_{k-1}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*} \frac{\mathbf{d}_{k-1}^T \varphi(t)}{\mathbf{d}_{k-1}^T \mathbf{a}_{k-1}^*}, \quad t \in [x_{k-1}, x_k].$$

Since $\langle \widetilde{f}_{k-1}, \varphi \rangle = \widetilde{\mathbf{a}}_{k-1}^*$ from previous equality we find

$$\widetilde{\mathfrak{q}}_{k-1,k-2} = \langle \widetilde{f}_{k-1}, \omega_{k-2} \rangle = \frac{\mathbf{d}_{k-2}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*} - \frac{\mathbf{d}_{k-2}^T \mathbf{a}_{k-1}^*}{\mathbf{d}_{k-2}^T \mathbf{a}_{k-2}^*} \frac{\mathbf{d}_{k-1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k-1}^T \mathbf{a}_{k-1}^*}.$$

9. We consider $\widetilde{\mathfrak{q}}_{k-1,k-3} = \langle \widetilde{f}_{k-1}, \omega_{k-3} \rangle$. We take into account that $\operatorname{supp} \widetilde{f}_{k-1} \subset [\widetilde{x}_{k-1}, \widetilde{x}_k]$, therefore we need representation of the function ω_{k-3} only on segment $[\widetilde{x}_{k-1}, \widetilde{x}_k] = [x_{k-1}, x_k]$. Thus, in (2.5) we set $j = k-3$:

$$\omega_{k-3}(t) = \frac{\mathbf{d}_{k+1}^T \varphi(t)}{\mathbf{d}_{k+1}^T \mathbf{a}_{k-3}^*} - \frac{\mathbf{d}_{k+1}^T \mathbf{a}_{k-4}^*}{\mathbf{d}_{k+1}^T \mathbf{a}_{k-3}^*} \frac{\mathbf{d}_k^T \varphi(t)}{\mathbf{d}_k^T \mathbf{a}_{k-4}^*}, \quad t \in [x_{k-1}, x_k].$$

Since $\langle \widetilde{f}_{k-1}, \varphi \rangle = \widetilde{\mathbf{a}}_{k-1}^*$, from previous equality we find

$$\widetilde{\mathfrak{q}}_{k-1,k-3} = \langle \widetilde{f}_{k-1}, \omega_{k-3} \rangle = \frac{\mathbf{d}_{k+1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k+1}^T \mathbf{a}_{k-3}^*} - \frac{\mathbf{d}_{k+1}^T \mathbf{a}_{k-4}^*}{\mathbf{d}_{k+1}^T \mathbf{a}_{k-3}^*} \frac{\mathbf{d}_k^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_k^T \mathbf{a}_{k-4}^*}.$$

Since $\mathbf{d}_k^T \widetilde{\mathbf{a}}_{k-1}^* = 0$, we find

$$\widetilde{\mathfrak{q}}_{k-1,k-3} = \frac{\mathbf{d}_{k+1}^T \widetilde{\mathbf{a}}_{k-1}^*}{\mathbf{d}_{k+1}^T \mathbf{a}_{k-3}^*}. \qquad \square$$

Consider the matrix $\widetilde{\mathfrak{Q}} \stackrel{\text{def}}{=} (\widetilde{\mathfrak{q}}_{i,j})_{i,j \in \mathbb{Z}}$ with entries given by formula (4.1). The matrix $\widetilde{\mathfrak{Q}}$ is called the *matrix of sparse decomposition* on (α, β).

Remark 4.1. The matrix $\widetilde{\mathfrak{Q}}$ can be represented in the form

$$\widetilde{\mathfrak{Q}} \stackrel{\text{def}}{=} \begin{array}{c} \\ \dots \\ k-4 \\ k-3 \\ k-2 \\ k-1 \\ k \\ k+1 \\ \dots \end{array} \begin{array}{c} \begin{array}{ccccccccc} \dots & k-4 & k-3 & k-2 & k-1 & k & k+1 & k+2 & \dots \end{array} \\ \left(\begin{array}{ccccccccc} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & \widetilde{\mathfrak{q}}_{k-2,k-3} & \widetilde{\mathfrak{q}}_{k-2,k-2} & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & \widetilde{\mathfrak{q}}_{k-1,k-3} & \widetilde{\mathfrak{q}}_{k-1,k-2} & \widetilde{\mathfrak{q}}_{k-1,k-1} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}\right) \end{array}.$$

We consider a system of functionals $\{\overline{f}_j\}_{j\in\mathbb{Z}}$ that is biorthogonal to the system of splines $\{\overline{\omega}_{j'}\}_{j'\in\mathbb{Z}}$. We proceed by computing the expressions

$$\overline{\mathfrak{q}}_{i,j} \stackrel{\text{def}}{=} \langle f_i, \overline{\omega}_j\rangle \quad \text{for all } i, j \in \mathbb{Z}.$$

Theorem 4.2. *For $i, j, k \in \mathbb{Z}$ the following relations hold:*

$$\overline{\mathfrak{q}}_{i,j} = \begin{cases} \delta_{i,j}, & \{j \leqslant k-4, \text{ for all } i \in \mathbb{Z}\} \cup \\ & \{j = k-3, \dots, k+1, i \leqslant k-3\}, \\ \overline{\mathbf{d}}^T_{k+3}\mathbf{a}^*_{k-2}\left(\overline{\mathbf{d}}^T_{k+3}\overline{\mathbf{a}}^*_{k-2} - \overline{\mathbf{d}}^T_{k+3}\overline{\mathbf{a}}^*_{k-3}\dfrac{\overline{\mathbf{d}}^T_{k+2}\overline{\mathbf{a}}^*_{k-2}}{\overline{\mathbf{d}}^T_{k+2}\overline{\mathbf{a}}^*_{k-3}}\right)^{-1}, & i = k-2, j = k-2, \\ \dfrac{\overline{\mathbf{d}}^T_{k-3}\mathbf{a}^*_{k-2}}{\overline{\mathbf{d}}^T_{k-3}\overline{\mathbf{a}}^*_{k-3}} - \dfrac{\overline{\mathbf{d}}^T_{k-3}\overline{\mathbf{a}}^*_{k-2}}{\overline{\mathbf{d}}^T_{k-3}\overline{\mathbf{a}}^*_{k-3}}\dfrac{\overline{\mathbf{d}}^T_{k-2}\mathbf{a}^*_{k-2}}{\overline{\mathbf{d}}^T_{k-2}\overline{\mathbf{a}}^*_{k-2}}, & i = k-2, j = k-3, \\ \dfrac{\overline{\mathbf{d}}^T_{k-1}\mathbf{a}^*_{k-1}}{\overline{\mathbf{d}}^T_{k-1}\overline{\mathbf{a}}^*_{k-1}}, & i = k-1, j = k-1, \\ \dfrac{\overline{\mathbf{d}}^T_{k-2}\mathbf{a}^*_{k-1}}{\overline{\mathbf{d}}^T_{k-2}\overline{\mathbf{a}}^*_{k-2}} - \dfrac{\overline{\mathbf{d}}^T_{k-2}\overline{\mathbf{a}}^*_{k-1}}{\overline{\mathbf{d}}^T_{k-2}\overline{\mathbf{a}}^*_{k-2}}\dfrac{\overline{\mathbf{d}}^T_{k-1}\mathbf{a}^*_{k-1}}{\overline{\mathbf{d}}^T_{k-1}\overline{\mathbf{a}}^*_{k-1}}, & i = k-1, j = k-2, \\ \dfrac{\overline{\mathbf{d}}^T_{k+1}\mathbf{a}^*_{k-1}}{\overline{\mathbf{d}}^T_{k+1}\overline{\mathbf{a}}^*_{k-3}}, & i = k-1, j = k-3, \\ \delta_{i,j-1}, & \{j = k-3, \dots, k+1, i \geqslant k\} \cup \\ & \{j \geqslant k+2, \text{for all } i \in \mathbb{Z}\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Proof. The proof is similar to the proof of the Theorem 4.1 (cf. [12]). □

Consider the matrix $\overline{\mathfrak{Q}} \stackrel{\text{def}}{=} (\overline{\mathfrak{q}}_{i,j})_{i,j\in\mathbb{Z}}$ with entries given by formula (4.2). The matrix $\overline{\mathfrak{Q}}$ is called the *matrix of refining decomposition* on (α, β).

Theorem 4.3. *The matrices* $\widetilde{\mathfrak{Q}}$ *and* $\overline{\mathfrak{Q}}$ *are the left inverse to the matrices* $\widetilde{\mathfrak{P}}^T$ *and* $\overline{\mathfrak{P}}^T$ *correspondingly, i.e.*

$$\widetilde{\mathfrak{Q}}\widetilde{\mathfrak{P}}^T = I, \quad \overline{\mathfrak{Q}}\,\overline{\mathfrak{P}}^T = I,$$

where I *is the unit matrix.*

Proof. Transposing the relation (3.11), we obtain the following equality for vector-rows $(\widetilde{\boldsymbol{\omega}})^T(t) = (\boldsymbol{\omega})^T(t)\widetilde{\mathfrak{P}}^T$. Multiplying this equality by the vector-column $\widetilde{\mathbf{f}} \stackrel{\text{def}}{=} (\widetilde{f}_j)_{j\in\mathbb{Z}}$, and taking into account the biorthogonality property (3.5), we obtain the unit matrix I on the left-hand side, whereas the matrix $\widetilde{\mathfrak{Q}}$ is appeared on the right-hand side (cf. (4.1)). Thus, $I = \widetilde{\mathfrak{Q}}\widetilde{\mathfrak{P}}^T$. Transposing the relation (3.12), we obtain the following equality for vector-rows $(\boldsymbol{\omega})^T(t) = (\overline{\boldsymbol{\omega}})^T(t)\overline{\mathfrak{P}}^T$. Multiplying this equality by the vector-column $\mathbf{f} \stackrel{\text{def}}{=} (f_j)_{j\in\mathbb{Z}}$, we obtain $I = \overline{\mathfrak{Q}}\,\overline{\mathfrak{P}}^T$. □

We consider decomposition matrices in the finite-dimensional case, using the above-introduced restrictions of all functions to the segment $[a, b]$. Extract a finite collection of $n+3$ functionals from the set of functionals $\{f_j\}_{j\in\mathbb{Z}}$, a finite collection of $n+2$ functionals from the set of functionals $\{\widetilde{f}_j\}_{j\in\mathbb{Z}}$, a finite collection of $n+4$ functionals from the set of functionals $\{\overline{f}_j\}_{j\in\mathbb{Z}}$.

Theorem 4.4. *For the systems of functionals* $\{f_i\}_{i\in J_{3,n-1}}$, $\{\widetilde{f}_j\}_{j\in J_{3,n-2}}$ *and* $\{\overline{f}_l\}_{l\in J_{3,n}}$ *the following relations hold*

$$\langle f_i, \omega_{i'}\rangle = \delta_{i,i'}, \quad i, i' \in J_{3,n-1},$$
$$\langle \widetilde{f}_j, \widetilde{\omega}_{j'}\rangle = \delta_{j,j'}, \quad j, j' \in J_{3,n-2},$$
$$\langle \overline{f}_l, \overline{\omega}_{l'}\rangle = \delta_{l,l'}, \quad l, l' \in J_{3,n},$$

and $\operatorname{supp} f_i \subset [a, b]$, $\operatorname{supp} \widetilde{f}_j \subset [a, b]$, $\operatorname{supp} \overline{f}_l \subset [a, b]$.

Proof. The assertion follows from the biorthogonality property (3.5). □

A rectangular number $(n+2)\times(n+3)$-matrix $\widetilde{\mathfrak{Q}}_n \stackrel{\text{def}}{=} (\widetilde{\mathfrak{q}}_{i,j})$, $i \in J_{3,n-2}$, $j \in J_{3,n-1}$, is called the *matrix of sparse decomposition* on $[a, b]$.

Remark 4.2. The matrix $\widetilde{\mathfrak{Q}}_n$ can be represented as

$$\widetilde{\mathfrak{Q}}_n \stackrel{\text{def}}{=} \begin{array}{c} \\ -3 \\ \dots \\ k-4 \\ k-3 \\ k-2 \\ k-1 \\ k \\ \dots \\ n-2 \end{array} \!\!\begin{array}{c} \begin{array}{cccccccccc} -3 & \dots & k-4 & k-3 & k-2 & k-1 & k & k+1 & \dots & n-1 \end{array} \\ \left(\begin{array}{cccccccccc} 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \widetilde{\mathfrak{q}}_{k-2,k-3} & \widetilde{\mathfrak{q}}_{k-2,k-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \widetilde{\mathfrak{q}}_{k-1,k-3} & \widetilde{\mathfrak{q}}_{k-1,k-2} & \widetilde{\mathfrak{q}}_{k-1,k-1} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{array}\right) \end{array}.$$

A rectangular number $(n+3)\times(n+4)$-matrix $\overline{\mathfrak{Q}}_n \stackrel{\text{def}}{=} (\overline{\mathfrak{q}}_{i,j})$, $i \in J_{3,n-1}$, $j \in J_{3,n}$, is called the *matrix of dense decomposition* on $[a,b]$.

Theorem 4.5. *For the matrices $\widetilde{\mathfrak{P}}_n$ and $\widetilde{\mathfrak{Q}}_n$, $\overline{\mathfrak{P}}_n$ and $\overline{\mathfrak{Q}}_n$ following relations hold*

$$\widetilde{\mathfrak{Q}}_n \widetilde{\mathfrak{P}}_n^T = I_{n+2}, \quad \overline{\mathfrak{Q}}_n \overline{\mathfrak{P}}_n^T = I_{n+3},$$

where I_{n+2}, I_{n+3} are unit matrices of order $n+2$ and $n+3$ respectively.

Proof. The proof is similar to the proof of the Theorem 4.3 for finite-dimensional case. □

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