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On Frames in Banach Spaces

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Abstract. Banach frames of type ωP^*, shrinking Banach frames and retro shrinking Banach frames in Banach spaces have been introduced and studied. Necessary and sufficient conditions for a Banach frame (retro shrinking Banach frame) to be shrinking are given. Relation between various types of Banach frames are discussed.

1. Introduction

D. Gabor [12] in 1946, introduced a fundamental approach to signal decomposition in terms of elementary signals. Duffin and Schaeffer [8] in 1952, while addressing some deep problems in non-harmonic Fourier series, abstracted Gabor's method to define frames for Hilbert spaces. Later, in 1986, Daubechies, Grossmann and Meyer [7] found new applications to wavelet and Gabor transforms in which frames played an important role.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a separable Hilbert space H is called *frame* (Hilbert) for H if there exists positive constants A and B $(0 < A \leq B < \infty)$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in H.$$

The positive constants A and B are called *lower* and *upper bounds* of the frame $\{x_n\}_{n\in\mathbb{N}}$, respectively. They are not unique.

The operator $T : l^2(\mathbb{N}) \to H$ defined as:

$$T(\{c_k\}_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} c_k x_k, \quad \text{for all } \{c_k\}_{k\in\mathbb{N}} \in l^2(\mathbb{N}),$$

is called the *pre-frame operator* or the *synthesis operator* and its adjoint $T^* : H \to l^2(\mathbb{N})$ given by

$$T^*(x) = \{\langle x, x_k\rangle\}_{k\in\mathbb{N}}, \quad \text{for all } x \in H,$$

is called the *analysis operator*.

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Composing T and T^* we obtain the *frame operator* $S = TT^* : H \to H$ given by

$$S(x) = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, \quad \text{for all } x \in H.$$

The series converging unconditionally. It follows that

$$\langle Sx, x \rangle = \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2, \quad \text{for all } x \in H.$$

So, the frame operator S is a positive, self-adjoint invertible operator on H. This gives the reconstruction formula for all $x \in H$:

$$x = SS^{-1}x = \sum_{k=1}^{\infty} \langle S^{-1}x, x_k \rangle x_k = \sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle x_k.$$

Today, frames play important roles in many applications in mathematics, science and engineering. Moreover, frames provides both great liberties in the design of vector space decompositions, as well as quantitative measure on the computability and robustness of the corresponding reconstructions. In the theoretical direction, powerful tools from operator theory and Banach spaces are being employed to study frames. For a nice introduction to theory of frames an interested reader refer to [5, 15] and references therein.

Coifman and Weiss [6] introduced the notion of atomic decomposition for function spaces. Later, Feichtinger and Gröchenig [10] extended this idea to Banach spaces. This concept was further generalized by Gröchenig [13] who introduced the notion of Banach frames for Banach spaces. Casazza, Han and Larson [2] also carried out a study of atomic decompositions and Banach frames. Banach frames were further studied in [3, 4, 11, 17, 18, 19, 20, 21, 22, 23, 24]. Recently, various generalization of frames in Banach spaces have been introduced and studied. Han and Larson [14] defined a Schauder frame for a Banach space E to be an inner direct summand (i.e. a compression) of a Schauder basis of E. Schauder frames were further studied in [26, 27]. The notion of retro Banach frames in Banach spaces introduced and studied in [17].

In this paper Banach frames of type ωP^* and P^*, shrinking Banach frames and retro shrinking Banach frames in Banach spaces have been introduced and studied. Sufficient conditions for finite sum of Banach frames to be of type P^* are obtained. Necessary and sufficient conditions for a Banach frame (retro shrinking Banach frame) to be shrinking Banach frame are given. Necessary and sufficient conditions for retro shrinking Banach frames in finite dimensional Banach spaces to be of type P^* are obtained. Relation between various types of Banach frames are discussed.

2. **Preliminaries**

Throughout this paper E will be denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ (which will be $\mathbb{R}$ or $\mathbb{C}$), E^* the conjugate space of E and

$\pi : E \to E^{**}$ denotes the canonical mapping of E into E^{**}. For a sequence $\{f_n\} \subset E^*$, $[f_n]$ denotes the closure of linear span of $\{f_n\}$ in the norm topology of E^* and $\widetilde{[f_n]}$ the closure of linear span of $\{f_n\}$ in $\sigma(E^*, E)$-topology.

Definition 2.1 ([13])**.** Let E be a Banach space and let E_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d if:

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$.

(ii) There exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad \text{for all } x \in E. \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad \text{for all } x \in E\,.$$

The positive constants A and B are called the *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$, respectively. The operator $S : E_d \to E$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality (2.1) is called the *frame inequality*.

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E, then $(\{f_n\}, S)$ is called an *exact Banach frame*.

Definition 2.2 ([17])**.** Let E ba a Banach space. Let $(E^*)_d$ be a Banach space of scalar-valued sequences indexed by $\mathbb{N}$ and associated with E^*. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* for E^* with respect to $(E^*)_d$ if:

(i) $\{f(x_n)\} \in (E^*)_d$, for all $f \in E^*$.

(ii) There exist positive constants A_0 and B_0 with $0 < A_0 \leq B_0 < \infty$ such that

$$A_0\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B_0\|f\|_{E^*}, \quad \text{for all } f \in E^*. \tag{2.2}$$

(iii) T is a bounded linear operator such that $T(\{f(x_n)\}) = f$, for all $f \in E^*$.

The positive constants A_0 and B_0 are called, respectively, the *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \to E^*$ is called the *reconstruction operator* (or the *pre-frame operator*), and the inequality (2.2) is called the *retro frame inequality*.

The retro Banach frame $(\{x_n\}, T)$ is called *tight* if $A_0 = B_0$ and *normalized tight* if $A_0 = B_0 = 1$. If removal of one x_j render the collection $\{x_n\}_{n\neq j}$ no longer a retro Banach frame for E^*, then $(\{x_n\}, T)$ is called an *exact retro Banach frame*.

Definition 2.3. A Banach frame $(\{f_n\}, S)$ for a Banach space E is said to be of *type P* if $(\{f_n\}, S)$ is exact and there exists a vector $z_0 \in E$ such that $f_n(z_0) = 1$, for all $n \in \mathbb{N}$.

The vector z_0 is called the *associated vector* of $(\{f_n\}, S)$.

Definition 2.4 ([25]). A retro Banach frame $(\{x_n\}, T)$ for E^* is said to be *strong* if there exists a sequence $\{f_n\} \subset E^*$, such that

(i) $f_n(x_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.
(ii) $(x_n, f_n) \subset S_E \times S_{E^*}$, for all $n \in \mathbb{N}$.

where $S_E = \{x \in E : \|x\| = 1\}$, the unit sphere in E.

The sequence $\{f_n\} \subset E^*$ is called the *admissible sequence* of $(\{x_n\}, T)$.

Definition 2.5 ([14]). A pair (x_n, f_n) ($\{x_n\} \subset E$, $\{f_n\} \subset E^*$) is called a *Schauder frame* for E if

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \quad \text{for all } \ x \in E,$$

where the series converges in the norm topology of E.

Definition 2.6 ([27]). A Schauder frame (x_n, f_n) is called *pre-shrinking* if (f_n, x_n) is a Schauder frame for E^*.

Lemma 2.7. *Let E be a Banach space and $\{f_n\} \subset E^*$ be a sequence such that $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. Then E is linearly isometric to the Banach space $X = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_X = \|x\|_E$, $x \in E$.*

Lemma 2.8 ([19]). *Let $(\{f_n\}, S)$ (where $\{f_n\} \subset E^*$, $S : E_d \to E$) be a Banach frame for E with respect to E_d. Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin [\widetilde{f_i}]_{i \neq n}$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $(\{f_n\}, S)$ is exact. Fix $n \in \mathbb{N}$. Then, there exists no reconstruction operator S_0 such that $(\{f_i\}_{i \neq n}, S_0)$ is a Banach frame for E. Therefore, by using Lemma 2.7, $[\widetilde{f_i}]_{i \neq n} \neq E^*$. Hence $f_n \notin [\widetilde{f_i}]_{i \neq n}$, for all $n \in \mathbb{N}$.

Conversely, let $f_n \notin [\widetilde{f_i}]_{i \neq n}$ for all $n \in \mathbb{N}$ and let $(\{f_n\}, S)$ be not exact. Then, there exists a positive integer m_0 and a reconstruction operator S_0 such that $(\{f_i\}_{i \neq m_0}, S_0)$ is a Banach frame for E. So, by using frame inequality for $(\{f_i\}_{i \neq m_0}, S_0)$, we obtain $[\widetilde{f_i}]_{i \neq n} = E^*$. This gives $f_{m_0} \in [\widetilde{f_i}]_{i \neq m_0}$, a contradiction. □

Remark 2.9. Let $(\{f_n\}, S)$ be an exact Banach frame for E. Then, there exists a sequence $\{x_n\} \subset E$, called an *admissible sequence* of vector to $(\{f_n\}, S)$ such that

$$f_i(x_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad \text{for all } i, j \in \mathbb{N}.$$

Lemma 2.10. *Let $(\{f_n\}, S)$ be a Banach frame for E. If*

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i(x), \ (x \in E) \text{ exists } \Rightarrow \lim_{n \to \infty} \alpha_i^{(n)} \text{ exists } (i \in \mathbb{N}),$$

then $(\{f_n\}, S)$ is an exact Banach frame for E.

Proof. Let $f \in [f_n]$. Then $f = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i$. Let $\{z_n\} \subset E$ be a sequence such that $\Big(\sum_{i=1}^{n} \alpha_i f_i\Big)(z_j) = \alpha_j$, for all $j \in \mathbb{N}$. Now for $f \in [f_n]$, we have $f(x) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i(x)$, $x \in E$. So, by hypothesis $\lim_{n\to\infty} \alpha_i^{(n)}$ exists, $i \in \mathbb{N}$. Thus, for each j, $\pi(z_j)$ is a continuous linear functional on $[f_n]$. Furthermore, $f_i(z_j) = \pi(z_j)f_i = \delta_{i,j}$, for all $i, j \in \mathbb{N}$. Hence $(\{f_n\}, S)$ is an exact Banach frame for E. □

Definition 2.11. Let $(\{f_n\}, S)$ be a Banach frame for E and that $\{m_n\}$, $\{p_n\}$ increasing sequence of positive integers, where $m_0 = 0$ and $m_{n-1} \leq p_n \leq m_n$, $n \in \mathbb{N}$. Define a sequence $\{\psi_n\} \in E^*$ by:

$$\psi_k = \begin{cases} f_k & \text{if } k \neq p_n \\ f_{p_n} + g_n & \text{if } k = p_n. \end{cases}, \quad n \in \mathbb{N},$$

where $g_n = \sum_{i=m_{n-1}+1}^{p_n - 1} \alpha_i f_i + \sum_{i=p_n+1}^{m_n} \alpha_i f_i$, for all $n \in \mathbb{N}$. Then, $\{\psi_n\}$ is called the *block perturbation* of $\{f_n\}$.

Theorem 2.12 ([32], page-109). *Let E be a Banach space and $f_i \in E^*$ $(i = 1, \ldots, n)$. Given n scalar $\alpha_1, \alpha_2, \ldots, \alpha_n$, a necessary and sufficient condition that there exists, for each $\epsilon > 0$, an element $x_0 \in X$ such that $f_i(x_0) = \alpha_i$ $(i = 1, 2, \ldots, n)$ and $\|x_n\| \leq \gamma + \epsilon$ is that the inequality*

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \leq \gamma \left\|\sum_{i=1}^{n} \beta_i f_i\right\|$$

holds for any choice of n numbers $\beta_1, \beta_2, \ldots, \beta_n$.

Theorem 2.13 ([9], p. 609). *Let T be a compact operator in a complex Banach space X, and let λ be a fixed non-zero complex number. Then, the non-homogenous equations*

$$(\lambda I - T)x = y \tag{2.3}$$

$$(\lambda I - T^*)f = g \tag{2.4}$$

have a unique solution for any $y \in X$ or $g \in X^*$ if and only if each of the homogenous equations

$$(\lambda I - T)x = 0 \tag{2.5}$$

$$(\lambda I - T^*)f = 0 \tag{2.6}$$

has zero as the solution. Furthermore, if one of the homogenous equations has a non-zero solution, then they both have the same finite number of linearly independent solutions. In this case the equation (2.2) *and* (2.3) *have solutions if and only if y and g are orthogonal to all the solutions of* (2.4) *and* (2.5), *respectively. Moreover,* *the general solution for* (2.2) *is found by adding a particular solution of* (2.2) *to the general solution of* (2.4).

3. Banach frames of Type ωP^*

Definition 3.1. A Banach frame $(\{f_n\}, S)$ for a Banach space E is said to be of type ωP^* (*weak of type P^**) if there exists a functional $\Phi \in E^{**}$ such that

$$\Phi(f_n) = 1, \quad \text{for all } n \in \mathbb{N}.$$

If $(\{f_n\}, S)$ is exact and of type ωP^*, then $(\{f_n\}, S)$ is called a Banach frame of type *type P^**

The functional Φ is called an *associated functional* of $(\{f_n\}, S)$.

Remark 3.2. The condition $\Phi(f_n) = 1$, $n \in \mathbb{N}$, resembles dynamics of frames! Physical interpretation of this can be understood as the earth rotates about its axis. Here 1 is the axis and Φ is the action of rotation on $\{f_n\}$.
To show existence of Banach frames of type ωP^*, we have following example.

Example 3.3. Let $E = c_0$.

(a) Define $\{f_n\} \subset E^*$ by $f_n(x) = \xi_n$, for all $n \in \mathbb{N}$, $(x = \{\xi_j\} \in E)$. Then, there exists a reconstruction operator $S : E_d = \{\{f_n(x)\} : x \in E\} \to E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d and with bounds $A = B = 1$. Also, $\Phi = (1, 1, 1, \ldots) \in E^{**}$ is such that $\Phi(f_n) = 1$, for all $n \in \mathbb{N}$. Hence $(\{f_n\}, S)$ is a Banach frame of type ωP^*.

(b) Define $\{g_n\} \subset E^*$ by

$$\left.\begin{array}{l} g_1(x) = 0 \\ g_n(x) = \xi_{n-1}, \quad n = 2, 3, \ldots \end{array}\right\}, \quad x = \{\xi_j\} \in E\,.$$

Then, there exists a reconstruction operator $S_0 : E_{d_0} = \{\{g_n(x)\} : x \in E\} \to E$ such that $(\{g_n\}, S_0)$ is a Banach frame for E with respect to E_{d_0} which is not of type ωP^*.

Remark 3.4. A Banach frame of type P is always of type P^*. Indeed, let $(\{f_n\}, S)$ be a Banach frame of type P for E with associated vector z_0. Then, $\Phi = \pi(z_0) \in E^{**}$ is such that $\Phi(f_n) = 1$, for all $n \in \mathbb{N}$. Towards the converse one may observe that a Banach frame of type P^* need not be of type P. The Banach frame $(\{f_n\}, S)$ given in Example 3.3(a) is of type P^* but not of type P.

If a signal is transmitted to a receiver, then there is some kind of disturbances (in general) in the received signal. To overcome these disturbances from the receiver, frames plays an important role. An interesting discussion in this direction is given in the book by Christensen [5]. The most important signal space is $L^2(\Omega)$. If $\{x_n\}$ is a frame (Hilbert) for $L^2(\Omega)$ space, then each element of $L^2(\Omega)$ can be recovered by an infinite combinations of frame elements. On the other hand a Banach frame reconstruct the space $L^2(\Omega)$ by an operator (pre-frame operator). Let $\{\phi_n\}, W$ be a Banach frame for $L^2(\Omega)$. Then, elements of $L^2(\Omega)$ can be recovered by the pre-frame operator W. Let $f_0 \in L^2(\Omega)$. Then, in general, there is no pre-frame operator U associated with $(\{f_n + f_0\}, U)$ which recover $L^2(\Omega)$. We say that f_0 is called *doping functional*. We extend the said problem regarding recovery of concern signal space to general Banach spaces. That is, if $(\{f_n\}, S)$ is a Banach frame for a Banach space E, then E (in general) not recovered by pre-frame operator U associated with $\{f_n + f_0\}$. However, we can recover E by a pre-frame operator associated with $\{f_n + f_0\}$ provided $(\{f_n\}, S)$ is of type ωP^* with a certain action of its associated functional on f_0. This is given in the following proposition.

Proposition 3.5. *Let $(\{f_n\}, S)$ be a Banach frame of type ωP^* for a Banach space E with associated functional Φ. Then, there exists a reconstruction operator U such that $(\{f_n + f_0\}, U)$ is Banach frame for E provided $\Phi(f_0) \neq -1$, where f_0 is a doping functional.*

Proof. Assume that $\Phi(f_0) \neq -1$. Let $T = I + f_0 \otimes \Phi$ be a bounded linear operator on E^*, where I is the identity operator on E^*. Then, $Tf = 0$ gives $f + \Phi(f) f_0 = 0$. If $\Phi(f) \neq 0$, then $\Phi(f_0) = -1$, a contradiction. Thus, T is one-one. Also, $f_0 \otimes \Phi$ is compact, because it is finite dimensional. By Theorem 2.13, T is invertible. Therefore, there exists a reconstruction operator U such that $(\{f_n + f_0\}, U)$ is Banach frame for E with bounds $A_0 = B_0 = 1$. □

The following example gives an application of Proposition 3.5.

Example 3.6. Let $E = c_0$. Define $\{f_n\} \subset E^*$ by

$$\left.\begin{aligned} &f_1(x) = 2\xi_1 \\ &f_n(x) = \xi_n, \quad n = 2, 3, \dots \end{aligned}\right\}, \quad x = \{\xi_j\} \in E.$$

Then, there exists a reconstruction operator $S_0 : E_{d_0} = \{\{f_n(x)\} : x \in E\} \to E$ such that $(\{f_n\}, S_0)$ is a Banach frame of type ωP^* for E with associated functional $\Phi = (1/2, 1, 1, 1, \dots) \in E^{**}$. Consider the doping functional $f_0 = f_2$. Then, $f_0 \in E^*$ is a non-zero functional such that $\Phi(f_0) \neq -1$. Hence by Proposition 3.5 there exists a reconstruction operator U such that $(\{f_n + f_0\}, U)$ is a Banach frame for E.

4. Finite Sum of Banach frames of type P^*

Let $(\{f_{1,n}\}, S_1)$ and $(\{f_{2,n}\}, S_2)$ be Banach frames of type P^* for a Banach spaces E. Then, in general, there exists no reconstruction operator Θ_0 such that $\left(\left\{\sum_{i=1}^{2} f_{i,2}\right\}, \Theta_0\right)$ is a Banach frame of type P^* for E.

Example 4.1. Let $E = c_0$ and let $\{f_{1,n}\}$ and $\{f_{2,n}\} \subset E^*$ be sequences defined by

$$\left.\begin{aligned} &f_{1,1}(x) = \xi_1, && f_{1,n}(x) = \frac{-n}{n+1}\xi_n, \\ &f_{2,1}(x) = -2\xi_1, && f_{2,n}(x) = \xi_n, \end{aligned}\right\} \quad n = 2, 3, 4, 5, \dots, \; x = \{\xi_n\} \in E.$$

Then, there exists reconstructions operators S_1 and S_2 such that $(\{f_{1,n}\}, S_1)$ and $(\{f_{2,n}\}, S_2)$ are Banach frames of type P^* for E.

Furthermore, $\Theta_0 : E_d = \Big\{\Big\{\Big(\sum_{i=1}^{2} f_{i,n}\Big)(x)\Big\} : x \in E\Big\} \to E$ is a bounded linear operator such that $\Big(\Big\{\sum_{i=1}^{2} f_{i,2}\Big\}, \Theta_0\Big)$ is a Banach frame (exact) for E. But there is no functional $\Phi \in E^{**}$ is such that $\Phi\Big(\sum_{i=1}^{2} f_{i,n}\Big) = 1$, for all $n \in \mathbb{N}$. Hence $\Big(\Big\{\sum_{i=1}^{2} f_{i,2}\Big\}, \Theta_0\Big)$ is not of type P^*.

The following proposition gives sufficient conditions under which finite sum of Banach frames of type P^* turns out to be of type P^*.

Proposition 4.2. *Let* $(\{f_{i,n}\}, S_i)$ $(S_i : E_{d_i} = \{\{(f_{i,n})(x)\} : x \in E\} \subset E_{d_0} \to E)$*, be Banach frames of type* P^* *for a Banach space* E *with associated functionals* Φ_i $(i = 1, 2, 3, \ldots, k)$*. Assume that:*

(i) $\lim_{n\to\infty} \sum_{l=1}^{m_n} \alpha_l^{(n)} \Big(\sum_{i=1}^{k} f_{i,n}\Big)(x) = 0,\ x \in E \ \Rightarrow\ \lim_{n\to\infty} \alpha_l^{(n)} = 0$*, for all* $l \in \mathbb{N}$,

(ii) $$\|\{f_{j_0,n}\}\|_{E_{d_{j_0}}} \le \Big\|\Big\{\Big(\sum_{i=1}^{k} f_{i,n}\Big)(x)\Big\}\Big\|_{E_{d_0}},\ x \in E, \text{ for some } 1 \le j_0 \le k, \tag{4.1}$$

(iii) $\Big(\sum_{j=1}^{k} \Phi_j\Big)\Big(\sum_{i=1 (i\neq j)}^{k} f_{i,n}\Big) = 1 - k$.

Then, there exists a reconstruction operator Θ_0 *such that* $\Big(\Big\{\sum_{i=1}^{k} f_{i,n}\Big\}, \Theta_0\Big)$ *is a normalized tight Banach frame of type* P^* *for* E.

Proof. By using inequality 4.1, there exists a reconstruction operator $\Theta_0 : E_{d_1} = \Big\{\Big\{\Big(\sum_{i=k}^{k} f_{i,n}\Big)(x)\Big\} : x \in E\Big\} \to E$ such that $\Big(\Big\{\sum_{i=1}^{k} f_{i,n}\Big\}, \Theta_0\Big)$ is a Banach frame for E with respect to E_{d_1} and with bounds $A_0 = B_0 = 1$.

Let $\lim_{n\to\infty} \sum_{l=1}^{m_n} \alpha_l^{(n)} \Big(\sum_{i=1}^{k} f_{i,n}\Big)(x) = 0$, for all $x \in E$. Then, by hypothesis $\lim_{n\to\infty} \alpha_l^{(n)}$ exists for all $l \in \mathbb{N}$.

So, by using Lemma 2.10 there exists a sequence $\{z_i\}_{i=1}^{n} \subset E$ such that $\Big(\sum_{i=1}^{k} f_{i,n}\Big)(z_m) = \delta_{n,m}$, for all $n, m \in \mathbb{N}$.

Thus, $\Big(\Big\{\sum_{i=1}^{k} f_{i,n}\Big\}, \Theta_0\Big)$ is an exact normalized tight Banach frame for E.

Put $\Psi = \sum_{j=1}^{k} \Phi_j$. Then, $\Psi \in E^{**}$ and $\Psi\Big(\sum_{i=1}^{k} f_{i,n}\Big) = 1$, for all $n \in \mathbb{N}$. Hence $\Big(\Big\{\sum_{i=1}^{k} f_{i,n}\Big\}, \Theta_0\Big)$ is a normalized tight Banach frame of type P^* for E. □

5. Associated Banach Frames in Banach Spaces

Let (x_n, f_n) be a Schauder frame for E. Then, there exist reconstruction operators S and W such that $(\{f_n\}, S)$ is a Banach frame for E and $(\{x_n\}, W)$ is a retro Banach frame for E^*. We say that $(\{f_n\}, S)$ and $(\{x_n\}, W)$ are *associated Banach frame* and *associated retro Banach frame* of the Schauder frame (x_n, f_n), respectively.

Let $(\{f_n\}, S)$ be associated Banach frame of type ωP^*. Then, in general, there exists no reconstruction operator $\widetilde{S_0}$ for perturbed sequence $\{f_n - f_{n+1}\}$ such that $(\{f_n - f_{n+1}\}, \widetilde{S_0})$ is a Banach frame of type ωP^* for E. In this direction following theorem gives sufficient conditions for perturbation of associated Banach frame of type ωP^*.

Theorem 5.1. *Let (x_n, f_n) be a Schauder frame for a Banach space E and let $(\{f_n\}, S)$ be an associated Banach frame of type ωP^*. Let $\{\alpha_n\}$ be a sequence of scalars such that $\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}} = 1$, for all $n \in \mathbb{N}$ and that there is no $z \in E$ such that $\left\| \sum_{i=1}^{n} \alpha_i x_i - z \right\|_E \to 0$ as $n \to \infty$. Then, there exists a reconstruction operator $\widetilde{S_0}$ such that $\left(\{\frac{1}{\alpha_n} f_n - \frac{1}{\alpha_{n+1}} f_{n+1}\}, \widetilde{S_0}\right)$ is a Banach frame of type ωP^* for E.*

Proof. First we show that $\left(\{\frac{1}{\alpha_n} f_n - \frac{1}{\alpha_{n+1}} f_{n+1}\}, \widetilde{S_0}\right)$ is a Banach frame for E. Assume that it is not true. Then, there exists a non-zero vector x_0 such that $(\frac{1}{\alpha_n} f_n - \frac{1}{\alpha_{n+1}} f_{n+1})(x_0) = 0$, for all n $\in \mathbb{N}$.
This gives

$$\frac{1}{\alpha_n} f_n(x_0) = \frac{1}{\alpha_{n+1}} f_{n+1}(x_0), \quad \text{for all } n \in \mathbb{N}.$$

By using frame inequality of $(\{f_n\}, S)$ we obtain:

$$f_n(x_0) = \frac{\alpha_n}{\alpha_1} f_1(x_0) \neq 0, \quad \text{for all } n \in \mathbb{N}.$$

Now, (x_n, f_n) is a Schauder frame for E, we have

$$\begin{aligned} x_0 &= \sum_{n=1}^{\infty} f_n(x_0) x_n \\ &= \sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_1} f_1(x_0)) x_n. \end{aligned}$$

Thus, $\left\| \sum_{i=1}^{n} \beta_i x_i - x_0 \right\|_E \to 0$ as $n \to \infty$, where $\beta_i = \frac{\alpha_i}{\alpha_1} f_1(x_0)$. This is a contradiction. Hence there exists a reconstruction operator $\widetilde{S}$ such that $(\{\frac{1}{\alpha_n} f_n - \frac{1}{\alpha_{n+1}} f_{n+1}\}, \widetilde{S_0})$ is a Banach frame for E.
Now, $(\{f_n\}, S)$ is of type ωP^*, there exists a functional $\Phi \in E^{**}$ such that $\Phi(f_n) = 1$, for all n $\in \mathbb{N}$.

Therefore

$$\Phi\left(\frac{1}{\alpha_n}f_n - \frac{1}{\alpha_{n+1}}f_{n+1}\right) = 1, \quad \text{for all } n \in \mathbb{N}.$$

Hence $\left(\{\frac{1}{\alpha_n}f_n - \frac{1}{\alpha_{n+1}}f_{n+1}\}, \widetilde{S_0}\right)$ is a Banach frame of type ωP^*. $\square$

Now we show that an associated Banach frame of type ωP^* (of pre-shrinking Schauder frame) produces another Banach frame of type ωP^*:

Let (x_n, f_n) be a pre-shrinking Schauder frame for E with associated Banach frame $(\{f_n\}, S)$ which is of type ωP^*.

Then, $\mathscr{X} = \left\{\{\gamma_i\} \subset \mathbb{K} : \sum\limits_{i=1}^{\infty} \gamma_i f_i \text{ converges}\right\}$ is a Banach space with norm given by $\|\{\gamma_i\}\|_{\mathscr{X}} = \sup\limits_{1 \le n < \infty} \left\| \sum\limits_{i=0}^{n} \gamma_i f_i \right\|_{E^*}$.

Define $\mathscr{W} : E^* \to \mathscr{X}$ by $\mathscr{W}(f) = \{\pi(x_n)(f)\}$, $f \in E$. Then $\mathscr{W}$ is an isomorphism of E^* into $\mathscr{X}$. Also, $U : \mathscr{X} \to E^*$ defined by $U(\{\gamma_i\}) = \sum\limits_{i=1}^{\infty} \gamma_i f_i$ is a bounded linear operator from $\mathscr{X}$ onto E^*.

Let $\mathscr{Z} = \operatorname{Ker} U$. Then, $\mathscr{Z}$ is a closed subspace of $\mathscr{X}$ such that $\mathscr{W}(E^*) \cap \mathscr{Z} = \{0\}$. If $\{\gamma_i\} \in \mathscr{X}$ is any element such that $f = \sum\limits_{i=1}^{\infty} \gamma_i f_i$, then $\{\pi_(x_n)(f)\} \in \mathscr{W}(E^*)$ and

$$\begin{aligned}\sum_{i=1}^{\infty}(\gamma_i - \pi(x_i)(f))f_i &= \sum_{i=1}^{\infty}\gamma_i f_i - \sum_{i=1}^{\infty}\pi(x_i)(f)f_i \\ &= 0.\end{aligned}$$

Therefore, $\{\gamma_i - \pi(x_i)(f)\} \in \mathscr{Z}$ such that

$$\{\gamma_i\} = \{\pi(x_i)(f)\} + \{\gamma_i - \pi(x_i)(f)\}.$$

Hence $\mathscr{X} = \mathscr{W}(E^*) \oplus \mathscr{Z}$.

Let ν be a projection of $\mathscr{X}$ onto $\mathscr{W}(E^*)$.

Then

$$\nu(\{\gamma_i\}) = \left\{\pi(x_n)\left(\sum_{i=1}^{\infty}\gamma_i f_i\right)\right\}, \quad \{\gamma_i\} \in \mathscr{X}.$$

Therefore, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\nu(e_k) &= \left\{\pi(x_n)\left(\sum_{i=1}^{\infty}\delta_{i,k} f_i\right)\right\}, \quad \text{where } \delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases} \\ &= \mathscr{W}(f_k).\end{aligned}$$

Thus, $f_n = \mathscr{W}^{-1}(\nu(e_n))$ for all $n \in \mathbb{N}$, and $\{e_n\}$ be the sequence of canonical unit vectors. Hence $(\mathscr{W}^{-1}(\nu(e_n)), S)$ is a Banach frame of type ωP^* for E.

This is summarized in the following.

Theorem 5.2. *Let (x_n, f_n) be a pre-shrinking Schauder frame for a Banach space E with associated Banach frame $(\{f_n\}, S)$ which is of type ωP^*. Then, there exists a projection ν of $\mathscr{X}$ onto $\mathscr{W}(E^*)$ along $\mathscr{Z}$ such that $(\mathscr{W}^{-1}(\nu(e_n)), S)$ is a Banach frame of type ωP^* for E.*

6. Shrinking Banach Frames

Definition 6.1. A Banach frame $(\{f_n\}, S)$ for a Banach space E is said to be *shrinking* if there exists a reconstruction operator W such that $(\{f_n\}, W)$ is a retro Banach frame for E^{**}.

Definition 6.2. An exact Banach frame $(\{f_n\}, S)$ for a Banach space E with admissible sequence of vector $\{x_n\} \subset E$ is said to be *retro shrinking* if there exists a reconstruction operator T such that $(\{x_n\}, T)$ is a retro Banach frame for E^*.

Definition 6.3. A Banach frame $(\{f_n\}, S)$ for a Banach space E is said to be *bi-shrinking* if it is both shrinking and retro shrinking.

Towards the existence of shrinking Banach frame, retro shrinking Banach frame and bi-shrinking Banach frame we have following example.

Example 6.4. (a) Let $(\{g_n\}, S_0)$ be a Banach frame for $E = c_0$ given in Example 3.3(b). Then there is a reconstruction operator $W : (E^{**})_{d_1} = \{\{\psi(g_n)\} : \psi \in E^{**}\} \to E^{**}$ such that $(\{g_n\}, W)$ is a retro Banach frame for E^{**}. Hence $(\{g_n\}, S_0)$ is shrinking Banach frame for E. But $(\{g_n\}, S_0)$ is not retro shrinking.

(b) Let $E = \ell^1$ and let $\{f_n\} \subset E^*$ be a sequence given by $f_n(x) = \xi_n$, for all $n \in \mathbb{N}$, $(x = \{\xi_j\} \in E)$. Then, there exists a reconstruction operator $S : E_{d_0} = \{\{g_n(x)\} : x \in E\} \to E$ such that $(\{f_n\}, S)$ is an exact Banach frame for E with admissible sequence of vectors $\{x_n = e_n\} \subset E$ (sequence of canonical unit vectors in E). Also, there exists a reconstruction operator $T : (E^*)_d = \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that$(\{x_n\}, T)$ is retro Banach frame for E. Therefore, $(\{f_n\}, S)$ is retro shrinking Banach frame for E. But there is no reconstruction operator W such that $(\{f_n\}, W)$ is retro Banach frame for E^{**}. Hence $(\{f_n\}, S)$ is not shrinking.

(c) Let $E = c_0$. Define $\{f_n\} \subset E^*$ by

$$\left.\begin{aligned} & f_2(x) = \xi_1 + \frac{1}{2}\xi_2, \\ & f_{2n-1}(x) = \xi_{2n-1}, n \in \mathbb{N} \\ & f_{2n}(x) = -\xi_{2n-3} + \xi_{2n-1} + \frac{1}{2^n}\xi_{2n},\ n = 2, 3, \ldots. \end{aligned}\right\}, \quad x = \{\xi_j\} \in E\,.$$

Then, there exists a reconstruction operator $S : E_d = \{\{f_n(x)\} : x \in E\} \to E$ such that $(\{f_n\}, S)$ is an exact Banach frame for E with admissible sequence of vectors $\{x_n\} \subset E$ given by:

$$\left.\begin{aligned} x_{2n-1} &= e_{2n-1} - 2^n e_{2n} + 2^{n+1} e_{2n+2} \\ x_{2n} &= 2^n e_{2n}, \end{aligned}\right\}, \quad n \in \mathbb{N}.$$

By nature of construction of $\{x_n\}$ and $\{f_n\}$ there exists reconstruction operator $T : E_d = \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that such that $(\{x_n\}, T)$ is retro Banach frame for E^* and $W : (E^{**})_{d_0} = \{\{\psi(f_n)\} : \psi \in E^{**}\} \to E^{**}$ such that $(\{f_n\}, W)$ is retro Banach frame for E^{**}. Hence $(\{f_n\}, S)$ is bi-shrinking Banach frame for E.

The following proposition shows that in reflexive Banach spaces every Banach frame for a Banach space is shrinking.

Proposition 6.5. *Every Banach frame $(\{f_n\}, S)$ in reflexive Banach spaces is shrinking.*

Proof. Let $(\{f_n\}, S)$ be a Banach frame for E with respect to E_d. If there exists no reconstruction operator W such that $(\{f_n\}, W)$ is retro Banach frame for E^{**}, then there exists a non-zero functional $\phi \in E^{**}$ such that $\phi(f_n) = 0$, for all $n \in \mathbb{N}$. By reflexivity of E there is a non-zero vector $x \in E$ such that $\pi(x) = \phi$. So, $f_n(x) = \pi(x)(f_n) = \phi(f_n) = 0$, for all $n \in \mathbb{N}$. Therefore, by using frame inequality of $(\{f_n\}, S)$, we obtain $\phi = 0$, contradiction. Hence there exists a reconstruction operator W such that $(\{f_n\}, W)$ is retro Banach frame for E^{**}. Thus, $(\{f_n\}, S)$ is shrinking. □

Remark 6.6. Banach frames in reflexive Banach spaces, in general, not retro shrinking.

Example 6.7. Let $E = l^2$. Define $\{f_n\} \subset E^*$ by

$$\left.\begin{aligned} f_1(x) &= \xi_1 \\ f_n(x) &= \xi_{n-1}, \quad n = 2, 3, \dots \end{aligned}\right\}, \quad x = \{\xi_j\} \in E.$$

Then, there exists a reconstruction operator $S : E_d = \{\{f_n(x)\} : x \in E\} \to E$ such that $(\{f_n\}, S)$ is a Banach frame for E which is shrinking but not retro shrinking.

The following theorem gives necessary and sufficient condition for a Banach frame to be shrinking.

Theorem 6.8. *A Banach frame $(\{f_n\}, S)$ for a Banach space E is shrinking if and only if* $\operatorname{dist}(f, [f_1, f_2, \dots, f_n]) \to 0$ *as $n \to \infty$, for all $f \in E^*$.*

Proof. Assume that $(\{f_n\}, S)$ is shrinking. Then, there exists a reconstruction operator W such that $(\{f_n\}, W)$ is a retro Banach frame for E^{**}. So, there are positive constants A_0 and B_0 such that

$$A_0 \|\psi\|_{E^{**}} \le \|\{\psi(f_n)\}\|_{(E^{**})_d} \le B_0 \|\psi\|_{E^{**}}, \quad \text{for all } \psi \in E^{**}. \tag{6.1}$$

By using retro frame inequality of $(\{f_n\}, W)$, we obtain $\operatorname{dist}(f, [f_1, f_2, \dots, f_n]) \to 0$ as $n \to \infty$, for all $f \in E^*$.

Conversely, if $\text{dist}(f, [f_1, f_2, \dots, f_n]) \to 0$ as $n \to \infty$, for all $f \in E^*$, then by Lemma 2.7 there exists a reconstruction operator $W : (E^{**})_{d_0} = \{\{\psi(f_n)\} : \psi \in E^{**}\} \to E^{**}$ such that $(\{f_n\}, W)$ is a retro Banach frame for E^{**}. Hence $(\{f_n\}, S)$ is shrinking. $\square$

The following proposition gives an application of retro shrinking Banach frames.

Proposition 6.9. *Let $(\{f_n\}, S)$ be retro shrinking Banach frame for E with associated sequence of vectors $\{x_n\} \subset E$. If $\left\| z_j - \sum_{i=1}^{n} \gamma_{i,j} x_i \right\|_E \to 0$, as $n \to \infty$, $1 \leq j \leq m$, and $z_1, z_2, \dots, z_m$ are linearly independent vectors in E, then* $\text{rank}(\gamma_{i,j})_{i \in \mathbb{N}, j=1,2,\dots,m} = m$.

Proof. This follows from properties of determinants and frame inequality of $(\{f_n\}, S)$. $\square$

The following proposition gives necessary and sufficient condition for retro shrinking Banach frame to be shrinking.

Proposition 6.10. *Let $(\{f_n\}, S)$ be retro shrinking Banach frame for E with admissible sequence of vectors $\{x_n\} \subset E$ Then $(\{f_n\}, S)$ is shrinking if and only if there exists a reconstruction operator J such that $(\{\pi(x_n\}, J)$ is the only exact Banach frame for E^* with admissible sequence $\{f_n\}$.*

Proof. Suppose first that $(\{f_n\}, S)$ is shrinking. Then, there exists a reconstruction operator W such that $(\{f_n\}, W)$ is retro Banach frame for E^{**}. Let $(\{\psi_n\}, J_0)$ be an exact frame for E^* with admissible sequence $\{f_n\}$. Then, by using retro frame inequality of $(\{f_n\}, W)$ we obtain $\pi(x_n) = \psi_n$, for all $n \in \mathbb{N}$.

On the other hand let $(\{\pi(x_n\}, J)$ is the only exact Banach frame for E^* with admissible sequence of vectors $\{f_n\}$. If $(\{f_n\}, S)$ is not shrinking, then there exists a non-zero functional $\Psi \in E^{**}$, such that $\Psi(f_n) = 0$, for all $n \in \mathbb{N}$.

Put $\Phi_1 = \pi(x_1) - \Psi$ and $\Phi_{n+1} = \pi(x_n)$, $n \in \mathbb{N}$.
Then, there exists a reconstruction operator J_1 such that $(\{\Phi_n\}, J_1)$ is an exact Banach frame for E^* with admissible sequence $\{f_n\}$, a contradiction. Hence $(\{f_n\}, S)$ is shrinking Banach frame for E. $\square$

An exact Banach frame, in general, not retro shrinking. The following proposition gives necessary and sufficient condition under which an exact Banach frame turns out to be retro shrinking.

Proposition 6.11. *An exact Banach frame $(\{f_n\}, S)$ for E with admissible sequence of vectors $\{x_n\} \subset E$ is retro shrinking if and only if there is a reconstruction operator J such that $(\{\pi(x_n\}, J)$ is normalized tight Banach frame for E^*.*

Proof. If $(\{f_n\}, S)$ is retro shrinking, then there exists a reconstruction operator T such that $(\{x_n\}, T)$ is retro Banach frame for E^*. Then, by using retro frame inequality for $(\{x_n\}, T)$, there is a reconstruction operator $J : \Theta_d = \{\{\pi(x_n)(\psi)\} : \psi \in E^*\} \to E^*$ such that $(\{\pi(x_n\}, J)$ is normalized tight Banach frame for E^*.

On the other hand if $(\{f_n\},S)$ is not retro shrinking, then there is non-zero functional $f_0 \in E^*$ such that $f_0(x_n) = 0$ for all $n \in \mathbb{N}$. That is, $\pi(x_n)(f_0) = 0$ for all $n \in \mathbb{N}$. So, by using frame inequality for $(\{\pi(x_n\},J)$, we obtain $f_0 = 0$, a contradiction. Hence $(\{f_n\},S)$ is retro shrinking. $\square$

The following theorem provides necessary and sufficient conditions under which retro shrinking Banach frames for finite dimensional Banach spaces turns out to be of type P^*.

Theorem 6.12. *Let $(\{f_n\},S)_{n=1,2,\dots,l}$ (where $S : \mathfrak{X}_d \to \mathfrak{X}$) be a retro shrinking Banach frame for a finite dimensional Banach space $\mathfrak{X}$ with admissible sequence of vectors $\{x_n\}_{n=1,2,\dots,l} \subset \mathfrak{X}$. Then, the following conditions are equivalent:*

(a) *$(\{f_n\},S)_{n=1,2,\dots,l}$ is of type P^*.*

(b) *There exists a functional $f_0 \in \mathfrak{X}^*$ and sequence $\{\Psi_n\}_{n=1,2,\dots,l} \subset \mathfrak{X}^{**}$ such that*

$$\Psi_n(f_m + f_0) = \delta_{n,m}, \quad \textit{for all } n,m.$$

Proof. (a)$\Rightarrow$(b): Define a sequence $\{\Psi_n\} \subset \mathfrak{X}^{**}$ by

$$\Psi_1 = \pi(x_1) - \frac{1}{2}\Phi,$$
$$\Psi_n = \pi(x_n), \quad n = 2,3,\dots l,$$

where Φ is the associated functional to $(\{f_n\},S)$ and $\pi : \mathfrak{X} \to \mathfrak{X}^{**}$ is the canonical mapping.

Put $f_0 = f_1$. Then, for $f_0 \in \mathfrak{X}^*$ is a non-zero functional such that

$$\Psi_n(f_m + f_0) = \delta_{n,m}, \quad \text{for all } \ n,m\,.$$

(b)$\Rightarrow$(a): Since $(\{f_n\},S)_{n=1,2,\dots,l}$ is a retro shrinking Banach frame for $\mathfrak{X}$ with admissible sequence of vectors $\{x_n\}_{n=1,2,\dots,l} \subset \mathfrak{X}$, so $\pi(x_k)(f_0) \neq 0$ for some k.

For scalars $\alpha_1, \alpha_2, \dots, \alpha_l$, we have

$$\begin{aligned}
\left|\pi(x_k)\left(\sum_{i=1}^{l} \alpha_i(f_i + f_0)\right)\right| &= \left|\sum_{i=1}^{l}(\alpha_i \pi(x_k)f_i + \alpha_i \pi(x_k)f_0)\right| \\
&= \left|\alpha_k + \sum_{i=1}^{l} \alpha_i \pi(x_k)f_0\right| \\
&\geq \left|\sum_{i=1}^{l} \alpha_i \pi(x_k)f_0\right| - |\alpha_k| \\
&= \left|\sum_{i=1}^{l} \alpha_i\right| |\pi(x_k)f_0| - \left|\sum_{i=1}^{l} \alpha_i \Psi_k(f_i + f_0)\right|.
\end{aligned}$$

Thus

$$\left|\sum_{i=1}^{l} \alpha_i\right| \leq \frac{(\|\pi(x_k)\| + \|\Psi_k\|)}{|\pi(x_k)(f_0)|}\left\|\sum_{i=1}^{l} \alpha_i(f_i + f_0)\right\|$$

By Theorem 2.12, there exists $x_0 \in \mathfrak{X}$ such that

$$(f_n + f_0)(x_0) = 1, \quad \text{for all } n = 1, 2, \ldots, l.$$

This gives $f_n(x_0) = 1 - f_0(x_0)$ for all $n = 1, 2, \ldots, l$. If $f_0(x_0) = 1$, then $f_n(x_0) = 0$, for all $n = 1, 2, \ldots, l$. So, frame inequality for $(\{f_n\}, S)_{n=1,2,\ldots,l}$ gives $x_0 = 0$, a contradiction. Thus, $f_0(x_0) \neq 1$. Put $\Phi = \pi\left(\frac{1}{1-f_0(x_0)} x_0\right)$. Then, Φ is a functional in $\mathfrak{X}^{**}$ such that $\Phi(f_n) = 1$, for all $n = 1, 2, \ldots, l$. Hence $(\{f_n\}, S)_{n=1,2,\ldots,l}$ is of type P^*. $\square$

The following theorem show that bi-shrinking Banach frames are invariant under block perturbation.

Theorem 6.13. *Let $(\{f_n\}, S)$ be a bi-shrinking Banach frame for E and let $\{\psi_n\}$ be block perturbation of $\{f_n\}$. Then there exists a reconstruction operator $\widetilde{S}$ such that $(\{\psi_n\}, \widetilde{S})$ is bi-shrinking Banach frame for E.*

Proof. First we show that there exists a reconstruction operator $\widetilde{S}$ such that $(\{\psi_n\}, \widetilde{S})$ is Banach frame for E. Assume that this is not possible. Then, there exists a non-zero vector z_0 in E such that $\psi_n(z_0) = 0$, for all $n \in \mathbb{N}$. By definition of block perturbation, we obtain $f_n(z_0) = 0$, for all $n \in \mathbb{N}$. Thus, frame inequality of $(\{f_n\}, S)$ gives $z_0 = 0$, a contradiction. Hence, there exists a reconstruction operator $\widetilde{S} : \mathfrak{J}_d = \{\{\psi_n(x)\} : x \in E\} \to E$ such that $(\{\psi_n\}, \widetilde{S})$ is Banach frame for E.

Now $(\{f_n\}, S)$ is bi-shrinking, so $(\{f_n\}, S)$ is exact. Let $\{x_n\}$ be admissible sequence of vectors to $(\{f_n\}, S)$. Define a sequence of vectors $\{w_n\}$ in E:

$$w_k = \begin{cases} x_k - \alpha_k x_{p_k} & \text{if } k \neq p_n, m_{n-1} \leq k \leq m_n, \\ x_{p_n} & \text{if } k = p_n. \end{cases}, \quad n \in \mathbb{N}.$$

Then, $\psi_m(w_n) = \delta_{n,m}$, for all $n, m \in \mathbb{N}$. That is, $(\{\psi_n\}, S_0)$ is an exact Banach frame for E. Also, by using Lemma 2.7, there exists reconstruction operators T, W such that $(\{w_n\}, T)$ and $(\{\psi_n\}, W)$ are retro Banach frames for E^* and E^{**}, respectively. Hence $(\{\psi_n\}, \widetilde{S})$ is bi-shrinking. $\square$

To conclude the section we observe that if two Banach spaces E and F both have shrinking Banach frames/retro shrinking Banach frames/bi-shrinking Banach frames, then, their product space $E \times F$ also has a shrinking Banach frame (retro shrinking Banach frame/bi-shrinking Banach frame). This is given in the following proposition.

Proposition 6.14. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ and $(\{g_n\}, T)$ $(\{g_n\} \subset F^*, T : F_d \to F)$ be shrinking Banach frames (retro shrinking Banach frames/bi-shrinking Banach frames) for Banach spaces E and F, respectively. Then there exist a sequence $\{h_n\} \subset (E \times F)^*$, and a reconstruction operator $\Theta_0 : (E \times F)_d \to E \times F$ such that $(\{h_n\}, \Theta_0)$ is a normalized tight shrinking Banach frame (*retro shrinking Banach frame/bi-shrinking Banach frame*).*

7. **Dual Weak of Type P^* Banach Frame Systems**

Banach frame systems introduced and studied by Kaushik et al in [20]. Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Let $\{\phi_n\} \subset E^{**}$ be a sequence such that $\phi_n(f_m) = \delta_{n,m}$, for all $n, m \in \mathbb{N}$. If there exists a reconstruction operator $\mathfrak{S}_\circ : (E^*)_d \to E^*$ such that $(\{\phi_n\}, \mathfrak{S}_\circ)$ is a Banach frame for E^* with respect to $(E^*)_d$. Then the pair $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ is called a *Banach frame system* for the Banach space E. The Banach frame $(\{\phi_n\}, \mathfrak{S}_\circ)$ is called an *admissible Banach frame* to the Banach frame $(\{f_n\}, S)$.

Definition 7.1. A Banach frame system $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ is said to be *of type $w'P^*$* (*dual weak of type P^**) if there exists a functional f_0 in E^* such that $\phi_n(f_0) = 1$, for all $n \in \mathbb{N}$.

Regarding existence of Banach frame systems of type $w'P^*$ we have following example.

Example 7.2. Let $E = \ell^1$.

Let $\{f_n\} \subset E^*$, $\{\phi_n\} \subset E^{**}$ be sequences defined by

$$\left.\begin{array}{ll} f_n(x) = \xi_n, & x = \{\xi_n\} \in E \\ \phi_n(f) = \beta_n, & f = \{\beta_n\} \in E^* \end{array}\right\} n = 1, 2, 3, \ldots.$$

Then, there exists reconstruction operators $S : E_d = \{\{f_n(x)\} : x \in E\} \to E$ and $\mathfrak{S}_\circ : (E^*)_d = \{\{\phi_n(f)\} : f \in E^*\} \to E^*$ such that $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ is a Banach frame system for E. Now f_0=(1,1,1,...) in E^* is such that $\phi_n(f_0) = 1$, for all $n \in \mathbb{N}$. Hence $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ is of type $w'P^*$.

Example 7.3. Let $E = c_0$ and let $\{f_n\} \subset E^*$, $\{\phi_n\} \subset E^{**}$ be sequences defined in Example 7.2. Then, $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ is a Banach frame system for c_0 which is not of type $w'P^*$.

Remark 7.4. Let $(\{f_n\}, S)$ be a Banach frame of type P^*. Then, in general, the corresponding Banach frame system $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ is not of type $w'P^*$. Indeed, the Banach frame system given in Example 7.3 is not of type $w'P^*$ but $(\{f_n\}, S)$ is of type P^*.

Remark 7.5. Let $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ be a Banach frame system for a Banach space E which is of type $w'P^*$. Then, in general, $(\{f_n\}, S)$ is not of type P^*. Indeed, let $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ be a Banach frame system for $E = \ell^1$ given in Example 7.2 (which is of type $w'P^*$). But there is no Φ in E^{**} such that $\Phi(f_n) = 1$, for all $n \in \mathbb{N}$. Hence $(\{f_n\}, S)$ is not of type P^*.

Remark 7.6. Let $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_\circ))$ be a Banach frame system for E. Then, in general, admissible Banach frames $(\{\phi_n\}, \mathfrak{S}_\circ)$ is not unique. However, it is unique provided $((\{f_n\}, S)$ is shrinking.

The following proposition gives necessary and sufficient conditions for a Banach frame system to be of type $w'P^*$.

Proposition 7.7. *$((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_o))$ be a Banach frame system for E. Then, the following are equivalent:*

(a) $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_o))$ *of type $w'P^*$.*

(b) *There exists no reconstruction operator $\widetilde{J}$ such that $(\{\phi_n - \phi_{n+1}\}, \widetilde{J})$ is a Banach frame for E^*.*

Proof. This follows from frame inequality of $(\{\phi_n - \phi_{n+1}\}, \widetilde{J})$ and Lemma 2.7. □

8. **Relation between Various Types of Frames in Banach spaces**

In this section we discuss relation between Schauder frames, associated Banach frames, Banach frames of type P and of type P^*. This is given in the form of remarks.

Remark 8.1. Let $(\{f_n\}, S)$ is a Banach frame for E and $(\{x_n\}, W)$ a retro Banach frame for E^*. Then, in general, (x_n, f_n) is not a Schauder frame for E. (Example 9.1).

Remark 8.2. Let (x_n, f_n) be a Schauder frame for E and let $(\{f_n\}, S)$, $(\{x_n\}, T)$ be its associated Banach frame and associated retro Banach frame, respectively. Then:

(I) $(\{f_n\}, S)$ need not be of type P^*. (Example 9.2(a))

(II) $(\{f_n\}, S)$ may be of type P^* but not of type P. (Example 9.2(b))

(III) $(\{f_n\}, S)$ may be shrinking but not of type P^*. (Example 9.2(a))

(IV) $(\{f_n\}, S)$ may be retro shrinking but not of type P^*. (Example 9.3)

(V) $(\{f_n\}, S)$ may be exact and shrinking but not of type P^*. (Example 9.3)

(VI) $(\{x_n\}, T)$ may not be strong. (Example 9.2(c))

(VII) $(\{f_n\}, S)$ may be of type P^* but the corresponding Banach frame system $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_o))$ need not be of type $w'P^*$. (Example 9.2(b))

9. **Counter-examples**

Example 9.1. Let $E = l^2$ and $\{e_n\} \subset E$ be the sequence of unit vectors. Let $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ be sequences defined by

$$\left.\begin{aligned} &x_n = e_n - e_{n+1}, \\ &f_n(x) = \xi_n - \xi_{n+1}, \quad x = \{\xi_n\} \in E \end{aligned}\right\} \; n = 1, 2, 3, \ldots.$$

Then, there exist reconstruction operators S and W such that $(\{f_n\}, S)$ is a Banach frame for E and $(\{x_n\}, W)$ is a retro Banach frame for E^*. But (x_n, f_n) is not a Schauder frame for E.

Example 9.2. Let $E = c_0$ and $\{e_n\} \subset E$ be the sequence of unit vectors.

(a) Let $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ be sequences defined by

$$\left.\begin{array}{llll} x_1 = e_1, & x_2 = e_1, & x_n = e_{n-1}, & \\ f_1(x) = \frac{1}{2}\xi_1, & f_2(x) = \frac{1}{2}\xi_1, & f_n(x) = \xi_{n-1}, & x = \{\xi_n\} \in E \end{array}\right\} \quad n = 3, 4, 5, \ldots.$$

Then, (x_n, f_n), is a Schauder frame for E. Also, there exists a reconstruction operator V such that $(\{f_n\}, V)$ is a shrinking Banach frame (associated) for E which is not of type P^* (hence not of type P).

(b) Let $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ be sequences defined by

$$\left.\begin{array}{l} x_n = e_n, \\ f_n(x) = \xi_n, \quad x = \{\xi_n\} \in E \end{array}\right\} \quad n = 1, 2, 3, \ldots.$$

Then, (x_n, f_n) is a Schauder frame for E. Also, there exists a reconstruction operator S such that $(\{f_n\}, S)$ is an exact Banach frame (associated) for E. Also, Φ=(1,1,1...) in E^{**} is such that $\Phi(f_n) = 1$, for all $n \in \mathbb{N}$. Hence $(\{f_n\}, S)$ is of type P^*. One may observe that $(\{f_n\}, S)$ is not of type of P. Furthermore, Banach frame system $((\{f_n\}, S), (\{\phi_n\}, \mathfrak{S}_o))$ associated with $(\{f_n\}, S)$ is not of type $w'P^*$.

(c) Let $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ be sequences defined by

$$\left.\begin{array}{l} x_n = (\underbrace{2, 2, 2, \ldots, 2}_{n}, 0, 0, 0, \ldots), \\ f_n = (\underbrace{0, 0, 0, 0, \ldots, 0}_{n-1}, \frac{1}{2}, \frac{-1}{2}, 0, 0, 0, 0, \ldots) \end{array}\right\} \quad n \in \mathbb{N}.$$

Then, (x_n, f_n) is a Schauder frame for E. Also, there exists a reconstruction operator $T : (E^*)_d = \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that $(\{x_n\}, T)$ is a retro Banach frame (associated) for E^* with $f_n(x_m) = \delta_{n,m}$, for all $n, m \in \mathbb{N}$. But $\|x_n\| = 2$, for all $n, \in \mathbb{N}$. Hence $(\{x_n\}, T)$ is not strong.

Example 9.3. Let $E = \ell^2$. Let $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ be sequences defined by

$$\left.\begin{array}{l} x_n = e_n, \\ f_n(x) = \xi_n, \quad x = \{\xi_n\} \in E \end{array}\right\} \quad n = 1, 2, 3, \ldots.$$

Then, (x_n, f_n) is a Schauder frame for E. Also, there exists a reconstruction operator S such that $(\{f_n\}, S)$ is shrinking (and retro shrinking) Banach frame (associated) for E with admissible sequence of vectors $\{x_n\}$. But $(\{f_n\}, S)$ is not of type P^*.

10. **Conclusion**

Banach frames of types ωP^* are useful in reconstruction of functions (signals) in Banach spaces. In particular, in a case when there is some error (doping functional) associated with a given Banach frame. Proposition 3.5 shows that how Banach frames of types ωP^* are useful in recovery of a function (signal). In fact, the action of associated functional on a given Banach frame decide the existence of a new pre-frame operator which recover the original function. Shrinking Banach frames and retro Banach frames are defined and discussed towards the development of frames in Banach spaces. Proposition 6.5 gives a necessary condition for reflexivity of Banach spaces (which admits Banach frames). Relation between shrinking and retro-shrinking Banach frames is given in Proposition 6.10. A relation between retro shrinking Banach frames and Banach frames of type ωP^* is given in Theorem 6.12.

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References

[1] P.G. Casazza, The art of frame theory, *Taiwanese J. Math.* **4**(2) (2000), 129–201.

[2] P.G. Casazza, D. Han and D.R. Larson, Frames for Banach spaces, *Contemp. Math.* **247** (1999), 149–182.

[3] P.G. Casazza and O. Christensen, Perturbation of operator and applications to frame theory, *J. Fourier Anal. Appl.* **3** (1997), 543–557.

[4] O. Christensen and C. Heil, Perturbation of Banach frames and atomic decompositions, *Math. Nachr* **185** (1997), 33–47.

[5] O. Christensen, Frames and bases (An introductory course), Birkhäuser, Boston (2008).

[6] R.R. Coifman and G. Weiss, Expensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[7] I. Daubechies, A. Grossmann and Y. Meyer, Painless non-orthogonal exapnsions, *J. Math. Phys.* **27** (1986), 1271–1283.

[8] R.J. Duffin and A.C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952), 341–366.

[9] N. Dunford and J.T. Schwartz, Linear Operator, Vol. I, Interscience Publishers, New York, London, 1958.

[10] H.G. Feichtinger and K. Gröchenig, A unified approach to atomic decompositons via inegrable group representations, *Lecture Notes in Mathematics* **1302** (Springer, Berlin, 1988), 52–73.

[11] S.J. Favier and R.A. Zalik, On the stablity of frames and Riesz bases, *Appl. Comp. Harm. Anal.* **2** (1995), 160–173.

[12] D. Gabor, Theory of communicatons, *Jour. Inst. Elec. Engg.* **93** (1946), 429–457.

[13] K. Gröchenig, Describing functions: Atomic decompositions versus frames, *Monatsh. Math.* **112** (1991), 1–41.

[14] D. Han and D.R. Larson, Frames, bases and group representations, *Mem. Amer. Math. Soc.* **147** (697) (2000), 1–91.

[15] C. Heil, *A Basis Theory Primer*, Birkhäuser (expanded edition)(1998).

[16] H.Heuser, *Functional Analysis*, John Wiley and Sons, New York (1982).
[17] P.K. Jain, S.K. Kaushik and L.K. Vashisht, Banach frames for conjugate Banach spaces, *Zeit. Anal. Anwendungen*, **23** (4) (2004), 713–720.
[18] P.K. Jain, S.K. Kaushik and L.K. Vashisht, On perturbation of Banach frames, *Int. J. Wavelets, Multi. Infor. Processing* (IJWMIP) **4** (3) (2006), 559–565.
[19] P.K. Jain, S.K. Kaushik and L.K. Vashisht, On Banach frames, *Indian J. Pure Appl. Math.* **37** (5) (2006), 265–272.
[20] P.K. Jain, S.K. Kaushik and Nisha Gupta, On frame systems in Banach spaces, *Int. J. Wavelets Multi. Info. Processing* **7** (1) (2009), 1-7.
[21] S.K. Kaushik, A generalization of frames in Banach spaces *J. Contemp. Math. Anal.* **44** (4) (2009), 212–218.
[22] S.K. Kaushik, Some results concerning frames in Banach spaces, *Tamking J. Math.* **38** (3) (2007), 267–276.
[23] S.K. Kaushik, A note on exact Banach frames, *Int. J. Pure Appl. Math.* **31** (2) (2006), 279–286.
[24] S.K. Kaushik, A generalization of frames in Banach frames, *J.Contemp. Math. Anal.* **44** (4) (2009), 212–218.
[25] S.K. Kaushik and L.K. Vashisht, On strong retro Banach frames, *communicated*.
[26] R. Liu and B. Zheng, A characterization of Schauder frames which are near-Schauder bases, *J. Fourier Anal. Appl.* **16** (2010), 791–803.
[27] R. Liu, On shrinking and boundedly complete Schauder frames of Banach spaces, *J. Math. Anal. Appl.* **365** (2010), 385–398.
[28] I. Singer, Basic sequences and reflexivity of Banach spaces, *Studia Math.* **21** (1962), 351–369.
[29] I. Singer, *Bases in Banach Spaces. II* (Springer, New York, 1981).
[30] L.K. Vashisht, On retro Banach frames of type P, *Azerbaijan J. Math.*, **2** (1) (2012), 82-89.
[31] L.K. Vashisht, On $\Phi-$Schauder Frames, *TWMS J. Appl. Eng. Math.*, **2** (1) (2012), 116 - 120.
[32] K. Yoshida, *Functional Analysis*, (Berlin—Heidelber—New York, Springer-Verlag, 1966).
[33] R. Young, *A Introduction to Non-harmonic Fourier Series*, Academic Press, New York, 1980 (revised first edition 2001).

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