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Research Article

Stability Results of the Additive-Quadratic Functional Equations in Random Normed Spaces by Using Direct and Fixed-Point Method

Asha Rani[1,3], Sushma Devi[2] and Manoj Kumar*[3]

[1] *Department of Mathematics, Pandit Neki Ram Sharma Government College (Maharshi Dayanand University), Rohtak 124001, Haryana, India*
[2] *Department of Mathematics, Kanya Mahavidyalaya Kharkhoda (Maharshi Dayanand University), Kharkhoda 131402, Haryana, India*
[3] *Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak 124021, Haryana, India*
***Corresponding author:** manojantil18@gmail.com

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Abstract. In this paper, we prove the Hyers-Ulam stability of different additive-quadratic functional equations in *Random Normed Space* (RN-Space) by direct and fixed-point method.

Keywords. Hyers-Ulam stability, Additive functional equations, Quadratic functional equations, Random normed spaces, Fixed point method, Direct method

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1. Introduction

The concept of stability for various functional equations arises when one replaces a functional equation by an inequality, which acts as a confusion of the equations. A standard problem in the theory of functional equations is the subsequent subject — When is it true that a function which approximately satisfies a functional equation must be close to an approximate solution of the functional equation? We conclude that the functional equation is stable, if the problem satisfies the solution of the functional equation. In 1940, the stability problems of functional equations about group homomorphisms was introduced by Ulam (see [21]). In 1941, Hyers [5] gave an affirmative answer to Ulam's question for additive groups (under the assumption that groups are Banach spaces). Rassias in [13] proved the generalized Hyers theorem for additive mappings. This method is called as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Also, in 1994, Rassias generalization theorem was delivered by Gavruta [3] by replacing a general function $\varepsilon(\|x\|^p + \|y\|^p)$ by $\phi(x,y)$.

In 1982, J. M. Rassias [12] followed the modern approach of the Th. M. Rassias theorem in which he replaced the factor product of norms instead of sum of norms. Ravi *et al.* [16] investigated Gavruta's theorem for the unbounded Cauchy difference in the spirit of Rassias approach.

Many subsequent works employed the fixed-point alternative to get generalized findings in many functional equations in various domains of Hyers-Ulam stability.

The functional equations

$$\phi(s_1+s_2)=\phi(s_1)+\phi(s_2) \tag{1.1}$$

and

$$\phi(s_1+s_2)+\phi(s_1-s_2)=2\phi(s_1)+2\phi(s_2) \tag{1.2}$$

are known as additive functional equation and quadratic functional equation, respectively. Each additive and quadratic solution of a functional equation, in particular, must be an additive mapping and a quadratic mapping.

Some papers representing the stability of quadratic functional equations can be seen in [6,7]. Recently, the stability of many functional equations in various spaces such as Banach spaces, fuzzy normed spaces and random normed spaces have been broadly inspected by a lot of mathematicians ([1,6,8,9,16,20]).

Some notions and conventions of the theory of random normed spaces are taken in our paper as in [9,18].

Throughout the paper, Δ^+ is the distribution functions space, that is, the space of all mappings $V : R \cup \{-\infty,\infty\} \to [0,1]$, such that V is left continuous and increasing on R, $V(0)=0$ and $V(+\infty)=1$. $D^+ \subset \Delta^+$ consisting of all functions $V \in \Delta^+$ for which $l^-V(+\infty)=1$, where $l^-\phi(s)$ denotes $l^-\phi(s)=\lim\limits_{t\to s^-}\phi(s)$. The space Δ^+ is partially ordered by the usual point wise ordering of functions, i.e., $V \le W \iff V(t) \le W(t)\ \forall\ t \in R$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Recently in 2022, Tamilvanan *et al.* [19] proved the general solution and Hyers-Ulam stability of additive quadratic functional equation

$$\begin{aligned} \sum_{1\le i<j\le m} \phi\left(-s_i - s_j + \sum_{k=1, i\neq j\neq k}^{m} s_k\right) &= \left(\frac{m^2-9m+16}{2}\right) \sum_{1\le i<j\le m} \phi(s_i+s_j) \\ &\quad - \left(\frac{m^3-11m^2+26m-16}{2}\right) \sum_{i=1}^{m} \frac{\phi(s_i)+\phi(-s_i)}{2} \\ &\quad - \left(\frac{m^3-11m^2+30m-20}{2}\right) \sum_{i=1}^{m} \frac{\phi(s_i)-\phi(-s_i)}{2}, \end{aligned} \tag{1.3}$$

where $m \ge 4$.

In this paper, we prove the generalized Hyers-Ulam stability of additive-quadratic functional equation (1.3) in random normed space, by using direct method and fixed-point method.

Following definitions and notions will be used in sequel to prove our main results:

Definition 1.1 ([10], t-norm)**.** $\Upsilon : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly t-norm) if Υ satisfies the following conditions:

(i) Υ is commutative and associative;

(ii) Υ is a continuous;

(iii) $\Upsilon(x,1) = x$ for all $x \in [0,1]$;

(iv) $\Upsilon(x,y) \le \Upsilon(z,w)$, whenever $x \le z$ and $y \le w$ for all $x,y,z,w \in [0,1]$.

Examples of continuous t-norm are $\Upsilon(x,y) = xy$, $\Upsilon(x,y) = \max\{x+y-1,0\}$ and $\Upsilon(x,y) = \min(x,y)$.

Recall that, if Υ is a t-norm and $\{s_m\}$ are given numbers in $[0,1]$, then, $\Upsilon_{i=1}^{m} x_i$ is defined by recursively by $\Upsilon_{i=1}^{1} x_1$ and $\Upsilon_{i=1}^{m} x_i = \Upsilon(\Upsilon_{i=1}^{m-1} x_i, x_m)$, for $m \ge 2$.

Definition 1.2 ([2])**.** A *Random Normed space* (briefly RN-space) is a triple (V, Ψ, Υ), where V is a vector space, Υ is a continuous t-norm and $\Psi : V \to D^+$ satisfying the following conditions:

(i) $\Psi_s(t) = \varepsilon_0(t)$, for all $t > 0$ if and only if $s = 0$;

(ii) $\Psi_{\alpha s}(t) = \Psi_s\left(\frac{t}{|\alpha|}\right)$, for all $s \in V$, $t \ge 0$ and $\alpha \in \mathcal{R}$ with $\alpha \neq 0$;

(iii) $\Psi_{s_1+s_2}(t+u) \ge \Upsilon(\Psi_{s_1}(t), \Psi_{s_2}(u))$ for all $s_1, s_2 \in V$ and $t, u \ge 0$.

Definition 1.3 ([2])**.** Let (V,Ψ,Υ) be a RN-space.

(RN1) A sequence $\{s_m\}$ in V is said to be convergent to a point $s\in V$ if $\lim_{m\to\infty}\Psi_{s_m-s}(t)=1$, $t>0$.

(RN2) A sequence $\{s_m\}$ in V is called a Cauchy sequence if $\lim_{m\to\infty}\Psi_{s_m-s_l}(t)=1$, $t>0$.

(RN3) A RN-space (V,Ψ,Υ) is said to be complete if every Cauchy sequence in V is convergent.

2. Solution of Additive and Quadratic Functional Equations

Theorem 2.1 ([19])**.** *If an odd mapping $\phi:V\to W$ satisfies the functional equation* (1.3)*, i.e.,*

$$\sum_{1\le i<j\le m}\phi\left(-s_i-s_j+\sum_{k=1,i\neq j\neq k}^{m}s_k\right)=\left(\frac{m^2-9m+16}{2}\right)\sum_{1\le i<j\le m}\phi(s_i+s_j)-\left(\frac{m^3-11m^2+30m-20}{2}\right)\sum_{i=1}^{m}\phi(s_i),\tag{2.1}$$

for all $s_1,s_2,\dots,s_m\in V$, then the function ϕ is additive.

Theorem 2.2 ([19])**.** *If an even mapping $\phi:V\to W$ satisfies the functional equation* (1.3) *i.e.,*

$$\sum_{1\le i<j\le m}\phi\left(-s_i-s_j+\sum_{k=1,i\neq j\neq k}^{m}s_k\right)=\left(\frac{m^2-9m+16}{2}\right)\sum_{1\le i<j\le m}\phi(s_i+s_j)-\left(\frac{m^3-11m^2+26m-16}{2}\right)\sum_{i=1}^{m}\phi(s_i),\tag{2.2}$$

for all $s_1,s_2,\dots,s_m\in V$, then the function ϕ is quadratic.

3. Main Results

In this section, we investigate the stability of (1.3) in random normed space by using direct and fixed-point approach.

Now, we assume that V is a linear space and (V,Ψ,Υ) is a complete RN-space. We define a mapping $\phi:V\to W$ by

$$\begin{aligned}D\phi(s_1,s_2,\dots,s_m)=&\sum_{1\le i<j\le m}\phi\left(-s_i-s_j+\sum_{k=1,i\neq j\neq k}^{m}s_k\right)\\&-\left(\frac{m^2-9m+16}{2}\right)\sum_{1\le i<j\le m}\phi(s_i+s_j)\\&+\left(\frac{m^3-11m^2+26m-16}{2}\right)\sum_{i=1}^{m}\frac{\phi(s_i)+\phi(-s_i)}{2}\\&+\left(\frac{m^3-11m^2+30m-20}{2}\right)\sum_{i=1}^{m}\frac{\phi(s_i)-\phi(-s_i)}{2},\end{aligned}\tag{3.1}$$

for all $s_1,s_2,\dots,s_m\in V$.

Theorem 3.1. *Let an odd mapping $\phi : V \to W$ for which there exists a mapping $\Phi : V^m \to D^+$ with for some $0 < \alpha < 2$,*

$$\Phi_{2s_1,2s_2,\dots,2s_m}(e) \geq \Phi_{s_1,s_2,\dots,s_n}\left(\frac{e}{\alpha}\right) \tag{3.2}$$

and

$$\lim_{n\to\infty}(\Phi_{2^n s_1,2^n s_2,\dots,2^n s_m}(2^n\varepsilon)) = 1 \tag{3.3}$$

for all $s_1, s_2, \dots, s_m \in V$ and $\varepsilon > 0$ such that the functional inequality with $\phi(0) = 0$ and

$$\Psi_{D\phi(s_1,s_2,\dots,s_m)}(\varepsilon) \geq \Phi_{s_1,s_2,\dots,s_m}(\varepsilon), \tag{3.4}$$

for all $s_1, s_2, \dots, s_m \in V$ and all $\varepsilon > 0$.

Then there exists a unique additive mapping $\chi_2 : V \to W$ satisfying the functional equation (2.1) *with*

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}((2-\alpha)\varepsilon), \tag{3.5}$$

for all $s \in V$ and all $\varepsilon > 0$.

The mapping $\chi_2 : V \to W$ is defined by

$$\Psi_{\chi_2(s)}(\varepsilon) = \lim_{n\to\infty}\frac{\Psi_{\phi(2^n s)}}{2^n}(\varepsilon), \tag{3.6}$$

for all $s \in V$ and all $\varepsilon > 0$.

Proof. Replacing s_1, s_2 by s and putting $s_3 = s_4 = \dots = s_m = 0$ in inequality (3.4), we get

$$\Psi_{\phi(2s)-2\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}(\varepsilon), \tag{3.7}$$

for all $s \in V$ and all $\varepsilon > 0$.

It follows from (3.7) and (RN2) that

$$\Psi_{\frac{\phi(2s)}{2}-\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}(2\varepsilon), \tag{3.8}$$

for all $s \in V$ and all $\varepsilon > 0$.

Replacing s by $2s$ in (3.8), we get

$$\Psi_{\frac{\phi(2^2 s)}{2^2}-\frac{\phi(2s)}{2}}(\varepsilon) \geq \Phi_{2s,2s,0,\dots,0}(2^2\varepsilon), \tag{3.9}$$

for all $s \in V$ and all $\varepsilon > 0$.

Replacing s by $2^n s$, we obtain

$$\begin{aligned}\Psi_{\frac{\phi(2^{n+1}s)}{2^{n+1}}-\frac{\phi(2^n s)}{2^n}}(\varepsilon) &\geq \Phi_{2^n s,2^n s,0,\dots,0}(2^{n+1}\varepsilon)\\ &\geq \Phi_{s,s,0,\dots,0}\left(\frac{2^{n+1}}{\alpha^n}\varepsilon\right),\end{aligned} \tag{3.10}$$

for all $s \in V$ and all $\varepsilon > 0$.

Since

$$\frac{\phi(2^m s)}{2^m} - \phi(s) = \sum_{n=0}^{m-1} \frac{\phi(2^{n+1}s)}{2^{n+1}} - \frac{\phi(2^n s)}{2^n}, \tag{3.11}$$

for all $s \in V$ and all $\varepsilon > 0$.

From inequality (3.10) and (3.11), we get

$$\Psi_{\frac{\phi(2^m s)}{2^m} - \phi(s)} \left(\sum_{n=0}^{m-1} \frac{\alpha^n \varepsilon}{2^{n+1}} \right) \geq \Phi_{s,s,0,\dots,0}(\varepsilon)$$

$$\Psi_{\frac{\phi(2^m s)}{2^m} - \phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=0}^{m-1} \frac{\alpha^n}{2^{n+1}}} \right), \tag{3.12}$$

for all $s \in V$ and all $\varepsilon > 0$.

Replacings by $2^l s$ in inequality (3.12), we obtain

$$\Psi_{\frac{\phi(2^{m+l} s)}{2^{m+l}} - \frac{\phi(2^m s)}{2^m}}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{x=l}^{m+l-1} \frac{\alpha^n}{2^{n+1}}} \right), \tag{3.13}$$

for all $s \in V$ and all $\varepsilon > 0$. As $\Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=l}^{m+l-1} \frac{\alpha^n}{2^{n+1}}} \right) \to 1$ as $l, m \to \infty$, then, $\left\{ \frac{\phi(2^m s)}{2^m} \right\}$ is a Cauchy sequence in (V, Ψ, Υ). Since, (V, Ψ, Υ) is a complete RN-space, this sequence converges to a point $\chi_2(s) \in W$. Fix $s \in V$ and put $l = 0$ in (3.13), we obtain

$$\Psi_{\frac{\phi(2^m s)}{2^m} - \phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=0}^{m-1} \frac{\alpha^n}{2^{n+1}}} \right), \tag{3.14}$$

and so, for every $\zeta > 0$, we get

$$\begin{aligned} \Psi_{\chi_2(s) - \phi(s)}(\varepsilon + \zeta) &\geq \Upsilon\left(\Psi_{\chi_2(s) - \frac{\phi(2^m s)}{2^m}}(\zeta), \Psi_{\frac{\phi(2^m s)}{2^m} - \phi(s)}(\varepsilon) \right) \\ &\geq \Upsilon\left(\Psi_{\chi_2(s) - \frac{\phi(2^m s)}{2^m}}(\zeta), \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=0}^{m-1} \frac{\alpha^n}{2^{n+1}}} \right) \right), \end{aligned} \tag{3.15}$$

for all $s \in V$ and all $\varepsilon, \zeta > 0$.

Taking the limit $m \to \infty$ and using inequality (3.15), we have

$$\Psi_{\chi_2(s) - \phi(s)}(\varepsilon + \zeta) \geq \Phi_{s,s,0,\dots,0}((2 - \alpha)\varepsilon), \tag{3.16}$$

for all $s \in V$ and all $\varepsilon, \zeta > 0$.

Since ζ is arbitrary, by taking $\zeta \to 0$ in (3.16), we get

$$\Psi_{\chi_2(s) - \phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}((2 - \alpha)\varepsilon), \tag{3.17}$$

for all $s \in V$ and all $\varepsilon > 0$.

Replacing $(s_1, s_2, \ldots, s_m)$ by $(2^m s_1, 2^m s_2, \ldots, 2^m s_m)$ in (3.4), we obtain

$$\Psi_{D\phi(2^m s_1, 2^m s_2, \ldots, 2^m s_m)}(\varepsilon) \geq \Phi_{2^m s_1, 2^m s_2, \ldots, 2^m s_m}(2^{m+1}\varepsilon), \tag{3.18}$$

for all $s_1, s_2, \ldots, s_m \in V$ and all $\varepsilon > 0$.

Since

$$\lim_{n\to\infty}(\Phi_{2^n s_1, 2^n s_2, \ldots, 2^n s_m}(2^n\varepsilon)) = 1$$

for all $s \in V$ and all $\varepsilon > 0$.

We conclude that χ_2 satisfies the functional equation (1.3).

Finally, we prove that χ_2 is unique additive mapping. We assume that there exists an additive mapping $\chi_1 : V \to W$ which satisfies the inequality (3.15).

Clearly, $\chi_2(2^m s) = 2^m \chi_2(s)$ and $\chi_1(2^m s) = 2^m \chi_1(s)$, for all $s \in V$.

From (3.17), we have

$$\begin{aligned}
\Psi_{\chi_2(s)-\chi_1(s)}(\varepsilon) &= \lim_{m\to\infty} \Psi_{\frac{\chi_2(2^m s)}{2^m} - \frac{\chi_1(2^m s)}{2^m}}(\varepsilon) \\
\Psi_{\frac{\chi_2(2^m s)}{2^m} - \frac{\chi_1(2^m s)}{2^m}}(\varepsilon) &\geq \Upsilon\left\{\Psi_{\frac{\chi_2(2^m s)}{2^m} - \frac{\phi(2^m s)}{2^m}}\left(\frac{\varepsilon}{2}\right), \Psi_{\frac{\phi(2^m s)}{2^m} - \frac{\chi_1(2^m s)}{2^m}}\left(\frac{\varepsilon}{2}\right)\right\} \\
&\geq \Phi_{2^m s, 2^m s, 0, \ldots, 0}(2^{m+1}(2-\alpha)\varepsilon) \\
&\geq \Phi_{s,s,0,\ldots,0}\left(2^{m+1}\left(\frac{2-\alpha}{\alpha^m}\right)\varepsilon\right),
\end{aligned} \tag{3.19}$$

for all $s \in V$ and all $\varepsilon > 0$.

Since,

$$\lim_{m\to\infty}\left(2^{m+1}\left(\frac{2-\alpha}{\alpha^m}\right)\varepsilon\right) = \infty,$$

we get

$$\lim_{m\to\infty} \Phi_{s,s,0,\ldots,0}\left(2^{m+1}\left(\frac{2-\alpha}{\alpha^m}\right)\varepsilon\right) = 1.$$

Therefore, it follows that $\Psi_{\chi_2(s)-\chi_1(s)}(\varepsilon) = 1$, for all $s \in V$and all $\varepsilon > 0$. and so $\chi_2(s) = \chi_1(s)$. Hence, the proof is complete. □

Theorem 3.2. *Let an even mapping $\phi : V \to W$ for which there exists a mapping $\Phi : V^m \to D^+$ with some $0 < \alpha < 4$,*

$$\Phi_{2s_1, 2s_2, \ldots, 2s_m}(e) \geq \Phi_{s_1, s_2, \ldots, s_n}\left(\frac{e}{\alpha}\right) \tag{3.20}$$

and

$$\lim_{n\to\infty}(\Phi_{2^n s_1, 2^n s_2, \ldots, 2^n s_m}(2^{2n}\varepsilon)) = 1 \tag{3.21}$$

for all $s_1, s_2, \ldots, s_m \in V$ *and* $\varepsilon > 0$ *such that the functional inequality with* $\phi(0) = 0$ *and*

$$\Psi_{D\phi(s_1,s_2,\ldots,s_m)}(\varepsilon) \geq \Phi_{s_1,s_2,\ldots,s_m}(\varepsilon) \tag{3.22}$$

for all $s_1, s_2, \ldots, s_m \in V$ *and all* $\varepsilon > 0$.

Then there exists a unique quadratic mapping $\chi_2 : V \to W$ *satisfying the functional equation* (2.2) *with*

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\ldots,0}(2(2^2 - \alpha)\varepsilon), \tag{3.23}$$

for all $s \in V$ *and all* $\varepsilon > 0$.

The mapping $\chi_2 : V \to W$ *is defined by*

$$\Psi_{\chi_2(s)}(\varepsilon) = \lim_{n\to\infty} \frac{\Psi_{\phi(2^n s)}}{2^{2n}}(\varepsilon), \tag{3.24}$$

for all $s \in V$ *and all* $\varepsilon > 0$.

Proof. Replacing s_1, s_2 by s and putting $s_3 = s_4 = \cdots = s_m = 0$ in inequality (3.22), we get

$$\Psi_{2\phi(2s)-8\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\ldots,0}(\varepsilon), \tag{3.25}$$

for all $s \in V$ and all $\varepsilon > 0$.

It follows from (3.25) and (RN2), we obtain

$$\Psi_{\frac{\phi(2s)}{2^2}-\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\ldots,0}(8\varepsilon), \tag{3.26}$$

for all $s \in V$ and all $\varepsilon > 0$.

Replacing s by $2s$ in (3.23), we get

$$\Psi_{\frac{\phi(2^2 s)}{2^4}-\frac{\phi(2s)}{2^2}}(\varepsilon) \geq \Phi_{2s,2s,0,\ldots,0}(2\,.\,2^4\varepsilon), \tag{3.27}$$

for all $s \in V$ and all $\varepsilon > 0$.

Replacing s by $2^n s$ in inequality (3.27), we obtain

$$\begin{aligned}\Psi_{\frac{\phi(2^{n+1}s)}{2^{2(n+1)}}-\frac{\phi(2^n s)}{2^{2n}}}(\varepsilon) &\geq \Phi_{2^n s,2^n s,0,\ldots,0}(2^{2(n+1)}2\varepsilon)\\ &\geq \Phi_{s,s,0,\ldots,0}\left(\frac{2^{2(n+1)}}{\alpha^n}2\varepsilon\right),\end{aligned} \tag{3.28}$$

for all $s \in V$ and all $\varepsilon > 0$.

Since

$$\frac{\phi(2^m s)}{2^{2m}} - \phi(s) = \sum_{n=0}^{m-1} \frac{\phi(2^{n+1}s)}{2^{2(n+1)}} - \frac{\phi(2^n s)}{2^{2n}}, \tag{3.29}$$

for all $s \in V$ and all $\varepsilon > 0$.

From inequality (3.25) and (3.26), we get

$$\Psi_{\frac{\phi(2^m s)}{2^{2m}}-\phi(s)}\left(\sum_{n=0}^{m-1}\frac{\alpha^n\varepsilon}{2^{2(n+1)}2}\right)\geq \Phi_{s,s,0,\dots,0}(\varepsilon)$$

$$\Psi_{\frac{\phi(2^m s)}{2^{2m}}-\phi(s)}\geq \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=0}^{m-1}\frac{\alpha^n}{2^{n+1}}}\right), \tag{3.30}$$

for all $s\in V$ and all $\varepsilon>0$.

Replacings by $2^l s$ in inequality (3.30), we obtain

$$\Psi_{\frac{\phi(2^{m+l}s)}{2^{2(m+l)}}-\frac{\phi(2^m s)}{2^{2m}}}(\varepsilon)\geq \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=l}^{m+l-1}\frac{\alpha^n}{2^{2(n+1)}2}}\right), \tag{3.31}$$

for all $s\in V$ and all $\varepsilon>0$.

As $\Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=l}^{m+l-1}\frac{\alpha^n}{2^{2(n+1)}2}}\right)\to 1$ as $l,m\to\infty$, then $\left\{\frac{\phi(2^m s)}{2^{2m}}\right\}$ is a Cauchy sequence in (V,Ψ,Υ). Since (V,Ψ,Υ) is a complete RN-space, this sequence converges to some point $\chi_2(s)\in W$.

Fix $s\in V$ and put $l=0$ in (3.31), we obtain

$$\Psi_{\frac{\phi(2^m s)}{2^{2m}}-\phi(s)}(\varepsilon)\geq \Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=0}^{m-1}\frac{\alpha^n}{2^{2(n+1)}2}}\right), \tag{3.32}$$

and so, for every $\zeta>0$, we get

$$\begin{aligned}\Psi_{\chi_2(s)-\phi(s)}(\varepsilon+\zeta)&\geq \Upsilon\left(\Psi_{\chi_2(s)-\frac{\phi(2^m s)}{2^{2m}}}(\zeta),\Psi_{\frac{\phi(2^m s)}{2^{2m}}-\phi(s)}(\varepsilon)\right)\\&\geq \Upsilon\left(\Psi_{\chi_2(s)-\frac{\phi(2^m s)}{2^{2m}}}(\zeta),\Phi_{s,s,0,\dots,0}\left(\frac{\varepsilon}{\sum\limits_{n=0}^{m-1}\frac{\alpha^n}{2^{2(n+1)}2}}\right)\right),\end{aligned} \tag{3.33}$$

for all $s\in V$ and all $\varepsilon,\zeta>0$.

Taking the limit $m\to\infty$ and using inequality (3.33), we have

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon+\zeta)\geq \Phi_{s,s,0,\dots,0}(2(2^2-\alpha)\varepsilon), \tag{3.34}$$

for all $s\in V$ and all $\varepsilon,\zeta>0$.

Since ζ is arbitrary, by taking $\zeta\to 0$ in (3.34), we get

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon)\geq \Phi_{s,s,0,\dots,0}(2(2^2-\alpha)\varepsilon), \tag{3.35}$$

for all $s\in V$ and all $\varepsilon>0$.

Replacing $(s_1,s_2,\dots,s_m)$ by $(2^m s_1,2^m s_2,\dots,2^m s_m)$ in (3.22), we obtain

$$\Psi_{D\phi(2^m s_1,2^m s_2,\dots,2^m s_m)}(\varepsilon)\geq \Phi_{2^m s_1,2^m s_2,\dots,2^m s_m}(2^{2m}\varepsilon), \tag{3.36}$$

for all $s_1,s_2,\dots,s_m\in V$ and all $\varepsilon>0$.

Since

$$\lim_{n\to\infty}(\Phi_{2^n s_1,2^n s_2,\dots,2^n s_m}(2^{2n}\varepsilon)) = 1$$

for all $s \in V$ and all $\varepsilon > 0$.

We conclude that χ_2 satisfies the functional equation (1.3). Finally, we prove that χ_2 is unique quadratic mapping. We assume that there exists a quadratic mapping $\chi_1 : V \to W$ which satisfies the inequality (3.35).

Clearly, $\chi_2(2^m s) = 2^{2m}\chi_2(s)$ and $\chi_1(2^m s) = 2^{2m}\chi_1(s)$, for all $s \in V$.

From (3.35), we have

$$\Psi_{\chi_2(s)-\chi_1(s)}(\varepsilon) = \lim_{m\to\infty}\Psi_{\frac{\chi_2(2^m s)}{2^{2m}}-\frac{\chi_1(2^m s)}{2^{2m}}}(\varepsilon)$$

$$\begin{aligned}\Psi_{\frac{\chi_2(2^m s)}{2^{2m}}-\frac{\chi_1(2^m s)}{2^{2m}}}(\varepsilon) &\geq \Upsilon\left\{\Psi_{\frac{\chi_2(2^m s)}{2^{2m}}-\frac{\phi(2^m s)}{2^{2m}}}\left(\frac{\varepsilon}{2}\right), \Psi_{\frac{\phi(2^m s)}{2^{2m}}-\frac{\chi_1(2^m s)}{2^{2m}}}\left(\frac{\varepsilon}{2}\right)\right\}\\ &\geq \Phi_{2^m s,2^m s,0,\dots,0}(2^{2m}2(2^2-\alpha)\varepsilon)\\ &\geq \Phi_{2^m s,2^m s,0,\dots,0}\left(\left(\frac{2^{2m}2(2^2-\alpha)}{\alpha^m}\right)\varepsilon\right),\end{aligned} \tag{3.37}$$

for all $s \in V$ and all $\varepsilon > 0$.

Since, $\lim_{m\to\infty}\left(\left(\frac{2^{2m}2(2^2-\alpha)}{\alpha^m}\right)\varepsilon\right) = \infty$, we get

$$\lim_{m\to\infty}\Phi_{s,s,0,\dots,0}\left(\left(\frac{2^{2m}2(2^2-\alpha)}{\alpha^m}\right)\varepsilon\right) = 1.$$

Therefore, it follows that $\Psi_{\chi_2(s)-\chi_1(s)}(\varepsilon) = 1$, for all $s \in V$and all $\varepsilon > 0$.

Therefore, $\chi_2(s)= \chi_1(s)$. □

Theorem 3.3. *Let a mapping $\phi : V \to W$ for which there exists a mapping $\Phi : V^m \to D^+$ with*

$$\lim_{n\to\infty}\left(\Phi_{\frac{s_1}{2^n},\frac{s_2}{2^n},\frac{s_3}{2^n},\dots,\frac{s_m}{2^n}}\left(\frac{\varepsilon}{2^{2n}}\right)\right) = 1 \tag{3.38}$$

for all $s_1, s_2, \dots, s_m \in V$ and $\varepsilon > 0$ such that the functional inequality with $\phi(0) = 0$ and

$$\Psi_{D\phi(s_1,s_2,\dots,s_m)}(\varepsilon) \geq \Phi_{s_1,s_2,\dots,s_m}(\varepsilon), \tag{3.39}$$

for all $s_1, s_2, \dots, s_m \in V$ and all $\varepsilon > 0$.

Then there exists a unique quadratic mapping $\chi_2 : V \to W$ satisfying the functional eq. (1.3) *with*

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon) \geq \Phi_{s,s,0,\dots,0}(2(2^2-\alpha)\varepsilon), \tag{3.40}$$

for all $s \in V$ and all $\varepsilon > 0$.

The mapping $\chi_2 : V \to W$ is defined by

$$\Psi_{\chi_2(s)}(\varepsilon) = \lim_{n\to\infty}\Psi_{2^{2n}\phi(\frac{s}{2^n})}(\varepsilon), \tag{3.41}$$

for all $s \in V$ and all $\varepsilon > 0$.

Corollary 3.1. *Let a ζ be a positive real number. If $\phi : V \to W$ be a quadratic mapping which satisfies*

$$\Psi_{D\phi(s_1,s_2,\dots,s_m)} \geq \Phi_{\zeta}(\varepsilon),$$

for all $s_1, s_2, \dots, s_m \in V$ and all $\varepsilon > 0$. Then, there exists a unique quadratic mapping $\chi_2 : V \to W$ satisfying the functional equation (1.3) *with*

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon) \geq \Phi_{\frac{\varepsilon}{2|2^2-1|}}(\varepsilon),$$

for all $s \in V$ and all $\varepsilon > 0$.

Proof. If $s_1, s_2, \dots, s_m = \zeta$, then, the proof is complete from Theorem 3.1 and Theorem 3.2 by taking $\alpha = 2^0$. □

4. Stability Results by Fixed Point Method

In this section, we consider the functional equation (1.3) in random normed space using by fixed-point approach. For notational convenience, we define ζ_a as follows:

$$\zeta_a = \begin{cases} 2, & a = 0, \\ \frac{1}{2}, & a = 1. \end{cases}$$

Now, we come to the main result of this section.

Theorem 4.1. *Let $\phi : V \to W$ be an odd mapping for which there exists a mapping $\Phi : V^m \to D^+$ satisfying*

$$\lim_{n\to\infty} \Phi_{\zeta_a{}^n s_1, \zeta_a{}^n s_2, \dots, \zeta_a{}^n s_m}(\zeta_a{}^{2n}) = 1, \tag{4.1}$$

for all $s_1, s_2, \dots, s_m \in V$ and all $\varepsilon > 0$, satisfying the inequality

$$\Psi_{D\phi(s_1,s_2,\dots,s_m)}(\varepsilon) \geq \Phi_{s_1,s_2,\dots,s_m}(\varepsilon), \tag{4.2}$$

for all $s_1, s_2, \dots, s_m \in V$ and all $\varepsilon > 0$.

If there exists $L = L(a)$ such that the function $s \to \tau(s,\varepsilon) = \Phi_{\frac{s}{2},\frac{s}{2},0\dots,0}(2\varepsilon)$ has the property, that

$$\tau(s,\varepsilon) \leq L \frac{1}{\zeta_a{}^2} \tau(\zeta_a s, \varepsilon), \tag{4.3}$$

for all $s \in V$ and all $\varepsilon > 0$.

Then, there exists a unique additive mapping $\chi_2 : V \to W$ satisfies the functional equation (2.1) *and satisfies*

$$\Psi_{\chi_2(s)-\phi(s)}\left(\frac{L^{1-a}}{1-L}\varepsilon\right) \geq \tau(s,\varepsilon), \tag{4.4}$$

for all $s \in V$ and all $\varepsilon > 0$.

Proof. Consider a general metric d on X such that

$$d(n_1,n_2)=\inf\{v\in(0,\infty) \text{ s.t. } \Psi_{n_1(s)-n_2(s)}(v\varepsilon)\geq\tau(s,\varepsilon),\ s\in V,\ \varepsilon>0\}.$$

It is easy to view that (X,d) is complete metric space. Let us consider a mapping $\Upsilon:X\to X$ by $\Upsilon n_1(s)=\frac{1}{{\zeta_a}^2}n_1(\zeta_a s)$, for all $s\in V$. Now for $n_1,n_2\in X$, we have $d(n_1,n_2)\leq v$.

$$\begin{aligned}
\implies\quad & \Psi_{n_1(s)-n_2(s)}(v\varepsilon)\geq\tau(s,\varepsilon)\\
& \Psi_{\Upsilon n_1(s)-\Upsilon n_2(s)}\left(\frac{v\varepsilon}{{\zeta_a}^2}\right)\geq\tau(s,\varepsilon)\\
& d(\Upsilon n_1(s),\Upsilon n_2(s))\leq vL\\
\implies\quad & d(\Upsilon n_1,\Upsilon n_2)\leq Ld(n_1,n_2),
\end{aligned}\tag{4.5}$$

for all $n_1,n_2\in X$.

Therefore, v is strictly contractive mapping on X with Lipschitz constant L. It follows from (4.5) that

$$\Psi_{\phi(2s)-2\phi(s)}(\varepsilon)\geq\Phi_{s,s,0,\dots,0}(\varepsilon),\tag{4.6}$$

for all $s\in V$ and all $\varepsilon>0$.

It follows from (4.6) that

$$\Psi_{\frac{\phi(2s)}{2}-\phi(s)}(\varepsilon)\geq\Phi_{s,s,0,\dots,0}(2\varepsilon),\tag{4.7}$$

for all $s\in V$ and all $\varepsilon>0$.

Using (4.3) for $a=0$, it reduces to

$$\Psi_{\frac{\phi(2s)}{2}-\phi(s)}(\varepsilon)\geq L\tau(s,\varepsilon),$$

for all $s\in V$ and all $\varepsilon>0$. Hence, we obtain

$$d(\Psi_{\Upsilon\phi(s)-\phi(s)})\geq L=L^{1-a}<\infty\tag{4.8}$$

for all $s\in V$ and all $\varepsilon>0$.

Replacing s by $\frac{s}{2}$ in (4.7), we obtain

$$\Psi_{\frac{\phi(s)}{2}-\phi(\frac{s}{2})}(\varepsilon)\geq\Phi_{\frac{s}{2},\frac{s}{2},0,\dots,0}(2\varepsilon),\tag{4.9}$$

for all $s\in V$ and all $\varepsilon>0$.

Using (4.3) for $a=1$, it reduces to

$$\Psi_{\frac{\phi(s)}{2}-\phi(\frac{s}{2})}(\varepsilon)\geq\tau(s,\varepsilon),$$

for all $s\in V$ and all $\varepsilon>0$.

Hence, we arrive

$$d(\Psi_{\Upsilon\phi(s)-\phi(s)})\geq L=L^{1-a}<\infty\tag{4.10}$$

for all $s \in V$.

From (4.8) and (4.10), we can conclude

$$d(\Psi_{\Upsilon\phi(s)-\phi(s)}) \leq \infty \tag{4.11}$$

for all $s \in V$.

In order to prove $\chi_2 : V \to W$ satisfies the functional equation (1.3), the remaining proof is similar as in Theorem 3.1.

As the function χ_2 is unique fixed point of Υ in $\Omega = \{\phi \in X \text{ s.t. } d(\phi, \chi_2) < \infty\}$. Finally, χ_2 is an unique function such that

$$\Psi_{\chi_2(s)-\phi(s)}\left(\frac{L^{1-a}}{1-L}\varepsilon\right) \geq \tau(s,\varepsilon),$$

for all $s \in V$ and all $\varepsilon > 0$. Thus, the proof is complete. □

5. General Solution of Mixed Type Functional Equation

Next, in this section we take a new generalized m-variable mixed type functional equation of the form

$$\begin{aligned}\sum_{i=1,j=i+1}^{m-1} (\phi(ks_i + s_j) + \phi(ks_m + s_1)) - k\left[\sum_{i=1,j=i+1}^{m-1} (\phi(s_i + s_j) + \phi(s_m + s_1))\right]\\ = \frac{(1-k)^2}{2}\sum_{i=1}^{m}(\phi(s_i) + \phi(-s_i)) - \frac{1-k}{k^2-k}\sum_{i=1}^{m}(k^2\phi(s_i) - \phi(ks_i)),\end{aligned} \tag{5.1}$$

for positive integer $m, k \geq 2$ and investigate Hyers-Ulam stability of equation (5.1) in random normed spaces using the direct method. Let V and W be real vector spaces.

Theorem 5.1 ([11]). *Let an odd mapping $\phi : V \to W$ satisfy functional equation*

$$\begin{aligned}\sum_{i=1,j=i+1}^{m-1} (\phi(ks_i + s_j) + \phi(ks_m + s_1)) - k\left[\sum_{i=1,j=i+1}^{m-1} (\phi(s_i + s_j) + \phi(s_m + s_1))\right]\\ = \frac{1-k}{k^2-k}\sum_{i=1}^{m}(k^2\phi(s_i) - \phi(ks_i)),\end{aligned} \tag{5.2}$$

for positive integer $m, k \geq 2$, then, ϕ is additive.

Theorem 5.2 ([11]). *Let an even mapping $\phi : V \to W$ satisfy functional equation*

$$\begin{aligned}\sum_{i=1,j=i+1}^{m-1} (\phi(ks_i + s_j) + \phi(ks_m + s_1)) - k\left[\sum_{i=1,j=i+1}^{m-1} (\phi(s_i + s_j) + \phi(s_m + s_1))\right]\\ = (1-k)^2\sum_{i=1}^{m}(\phi(s_i)) + \frac{1-k}{k^2-k}\sum_{i=1}^{m}(k^2\phi(s_i) - \phi(ks_i)),\end{aligned} \tag{5.3}$$

for positive integer $m, k \geq 2$, then, ϕ is quadratic.

Now, we assume that V is a linear space and (V,Ψ,Υ) is a complete RN-space. Now, we prove the result by taking $k=2$ and define a mapping $\phi: V\to W$ by

$$D\phi(s_1,s_2,\dots,s_n)=\sum_{i=1,j=i+1}^{m-1}(\phi(2s_i+s_j)+\phi(2s_m+s_1))-2\left[\sum_{i=1,j=i+1}^{m-1}(\phi(s_i+s_j)+\phi(s_m+s_1))\right] -\frac{1}{2}\sum_{i=1}^{m}(\phi(s_i)+\phi(-s_i))+\frac{1}{2}\sum_{i=1}^{m}(4\phi(s_i)-\phi(2s_i)), \tag{5.4}$$

for all $s_1,s_2,\dots,s_m\in V$, for positive integer $m,k=2$.

Theorem 5.3. *Let a mapping $\phi: V\to W$ for which there exists a mapping $\Phi: V^m\to D^+$ with*

$$\lim_{n\to\infty}(\Phi_{2^n s_1,2^n s_2,\dots,2^n s_m}(2^n\varepsilon))=1, \tag{5.5}$$

for all $s_1,s_2,\dots,s_m\in V$ and $\varepsilon>0$ such that the functional inequality with $\phi(0)=0$ and

$$\Psi_{D\phi(s_1,s_2,\dots,s_m)}(\varepsilon)\ge\Phi_{s_1,s_2,\dots,s_m}(\varepsilon), \tag{5.6}$$

for all $s_1,s_2,\dots,s_m\in V$ and all $\varepsilon>0$.

Then there exists a unique additive mapping $\chi_2: V\to W$ satisfying the functional equation (5.2) *with*

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon)\ge\Phi_{s,s,0,\dots,0}((2-\alpha)\varepsilon), \tag{5.7}$$

for all $s\in V$ and all $\varepsilon>0$.

The mapping $\chi_2: V\to W$ is defined by

$$\Psi_{\chi_2(s)}(\varepsilon)=\lim_{n\to\infty}\frac{\Psi_{\phi(2^n s)}}{2^n}(\varepsilon), \tag{5.8}$$

for all $s\in V$ and all $\varepsilon>0$.

Proof. The proof is same as Theorem 3.1. □

Theorem 5.4. *Let a mapping $\phi: V\to W$ for which there exists a mapping $\Phi: V^m\to D^+$ with*

$$\lim_{n\to\infty}(\Phi_{2^n s_1,2^n s_2,\dots,2^n s_m}(2^{2n}\varepsilon))=1 \tag{5.9}$$

for all $s_1,s_2,\dots,s_m\in V$ and $\varepsilon>0$ such that the functional inequality with $\phi(0)=0$ such that

$$\Psi_{D\phi(s_1,s_2,\dots,s_m)}(\varepsilon)\ge\Phi_{s_1,s_2,\dots,s_m}(\varepsilon), \tag{5.10}$$

for all $s_1,s_2,\dots,s_m\in V$ and all $\varepsilon>0$.

Then, there exists a unique quadratic mapping $\chi_2: V\to W$ satisfying the functional equation (5.3) *with*

$$\Psi_{\chi_2(s)-\phi(s)}(\varepsilon)\ge\Phi_{s,s,0,\dots,0}(2(2^2-\alpha)\varepsilon), \tag{5.11}$$

for all $s\in V$ and all $\varepsilon>0$.

The mapping $\chi_2 : V \to W$ *is defined by*

$$\Psi_{\chi_2(s)}(\varepsilon) = \lim_{n\to\infty} \frac{\Psi_{\phi(2^n s)}}{2^{2n}}(\varepsilon), \tag{5.12}$$

for all $s \in V$ *and all* $\varepsilon > 0$.

Proof. The proof is same here as Theorem 3.2. □

6. Conclusion

In this paper, we use some notation, results and some theorems and demonstrate the direct and fixed-point methods Hyers-Ulam stability for various additive-quadratic functional equations in random normed Spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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