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$\mathscr{A}\mathscr{D}$-Frames satisfying Property $\mathscr{B}$

S.K. Kaushik and Varinder Kumar

Abstract. $\mathscr{A}\mathscr{D}$-frames in Banach spaces have been introduced and studied. Some necessary conditions for existence of $\mathscr{A}\mathscr{D}$-frames have been given. Property $\mathscr{B}$ for $\mathscr{A}\mathscr{D}$-frames is defined and a characterization of $\mathscr{A}\mathscr{D}$-frames satisfying property $\mathscr{B}$ has been obtained. Also, we gave a sufficient condition for an $\mathscr{A}\mathscr{D}$-frame to satisfy property $\mathscr{B}$ and a necessary condition for a particular type of $\mathscr{A}\mathscr{D}$-frame satisfying property $\mathscr{B}$. Finally, a result regarding quasi-reflexivity of Banach spaces having $\mathscr{A}\mathscr{D}$-frames satisfying property $\mathscr{B}$ is proved.

1. Introduction

Frames were introduced in 1952 by Duffin and Schaeffer [8]. It took more than 30 years to realize the importance of frames. In 1986, Daubechies, Grossmann and Meyer [7], reintroduced frames and thereafter the theory of frames began to be more widely studied. One of the intrinsic properties of a frame is that, given a frame, we can get properties of a function and reconstruct it only from the frame co-efficients, a sequence of complex numbers. We refer to [1, 2, 5] for an introduction to frame theory and its applications.

Besides traditional applications, a number of new applications have emerged which cannot be modeled naturally by one single frame system. In such cases, data assigned to one single frame system becomes too large to be handled numerically. So, it would be beneficial to split large frame system into a set of smaller systems and then to process the data locally within each subsystem effectively. Thus, a distributed frame theory for a set of local frame systems is required. In this direction, a theory based on fusion frames was developed in [3, 4] which provides a framework to deal with these applications.

In the present paper, we introduce and study $\mathscr{A}\mathscr{D}$-frames for Banach spaces and observe that a Banach space E having an $\mathscr{A}\mathscr{D}$-frame also possesses a fusion Banach frame but the converse is not true. Some necessary conditions for existence of $\mathscr{A}\mathscr{D}$-frames have been given. We define property $\mathscr{B}$ for $\mathscr{A}\mathscr{D}$-frames and obtain

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a characterization of $\mathscr{A}\mathscr{D}$-frames satisfying property $\mathscr{B}$. Also, we give a sufficient condition for an $\mathscr{A}\mathscr{D}$-frame to satisfy property $\mathscr{B}$ and a necessary condition for a particular type of $\mathscr{A}\mathscr{D}$-frame satisfying property $\mathscr{B}$. Finally, we prove a result regarding quasi-reflexivity of Banach spaces having $\mathscr{A}\mathscr{D}$-frames satisfying property $\mathscr{B}$.

2. **Preliminaries**

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the dual space of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$.

A sequence $\{x_n\}$ in E is said to be *complete* if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be *total* over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. A sequence of projections $\{v_n\}$ on E is *total* on E if $\{x \in E : v_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$.

Feichtinger and Gröchenig introduced the concept of atomic decompositions in Banach spaces. Gröchenig [10] introduced the concept of Banach frames which were further studied in [11, 12, 13].

A generalization of this concept namely, fusion Banach frames was introduced and studied in [14].

Definition 2.1 ([14]). Let E be a Banach space. Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. Let $\mathscr{A}$ be a Banach space associated with E and $S : \mathscr{A} \to E$ be an operator. Then $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (or, *fusion Banach frame*) for E with respect to $\mathscr{A}$, if

(i) $\{v_n(x)\} \in \mathscr{A}$, for all $x \in E$,

(ii) there exist positive constants A, B $(0 < A \le B < \infty)$ such that

$$A\|x\|_E \le \|\{v_n(x)\}\|_{\mathscr{A}} \le B\|x\|_E, \quad x \in E, \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{v_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the frame of subspaces $(\{G_n, v_n\}, S)$. But they are not unique. In fact, any pair of positive constants A, B for which the inequality (2.1), called frame inequality for fusion Banach frame $(\{G_n, v_n\}, S)$, hold good are called frame bounds. The supremum of all lower frame bounds and infimum of all upper frame bounds (of $(\{G_n, v_n\}, S)$) are called optimal bounds or best bounds for the fusion Banach frame $(\{G_n, v_n\}, S)$. The operator $S : \mathscr{A} \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*).

Note. The Banach space $\mathscr{A}$ associated with a given Banach space E is a space of vector-valued sequences, indexed by $\mathbb{N}$. This associated Banach space may not be unique. Further, if $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathscr{A}$, then $\{v_n(x)\} \in \mathscr{A}$, $x \in E$, i.e., $\{\{v_n(x)\} : x \in E\} \subset \mathscr{A}$. Moreover, if $\{v_n\}$ is total over E, then the associated Banach space $\{\{v_n(x)\} : x \in E\}$ serves the purpose of $\mathscr{A}$ and is linearly isometric to E.

We give below a result, in the form of a lemma, which is referred in this paper.

Lemma 2.2 ([14]). *Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over E, then $\mathscr{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|x\|_E = \|\{v_n(x)\}\|_{\mathscr{A}}$, $x \in E$.*

Note that the associated Banach space $\mathscr{A}$ is linearly isometric to E.

The characteristic of a linear subspace V of dual Banach space E^* is the greatest number $r = \gamma(V)$ such that the unit ball $\{f \in V : \|f\| \leq 1\}$ of V is $\sigma(E^*, E)$-dense in the r-ball $\{f \in E^* : \|f\| \leq r\}$ of E^*.

The space bv_0 is the linear space of all sequences $\{\alpha_n\}$ of scalars with $\lim\limits_{n\to\infty} \alpha_n = 0$ and for which the norm $\|\{\alpha_n\}\| = \sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$ is finite.

A Banach space E is said to be *quasi-reflexive* of order n, if $\operatorname{codim}_{E^{**}} \pi(E) = n$ and is called *reflexive* if it is quasi-reflexive of order zero.

An infinite real matrix $(\alpha_{ij})_{i,j=1}^{\infty}$ is said to be a *regular summability method* ($\mathscr{R}\mathscr{S}\mathscr{M}$) if for every convergent sequence (b_n) of real numbers, we have

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij} b_j = \lim_{n\to\infty} b_n \,.$$

3. $\mathscr{A}\mathscr{D}$-Frames

We begin this section with the following definition of $\mathscr{A}\mathscr{D}$-frames

Definition 3.1. Let E be a Banach space. Let $\{G_n\}$ be a sequence of closed subspaces of E and $\{v_n\}$ be a sequence of non-zero bounded linear projections on E such that $v_n(E) = G_n$, $n \in \mathbb{N}$. Let $\mathscr{A}$ be a Banach space associated with E. Then $(\{G_n\}, \{v_n\})$ is called an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$ if

(i) $\{v_n(x)\} \in \mathscr{A}$, $x \in E$,

(ii) there exist constants A, B $(0 < A \leq B < \infty)$ such that

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathscr{A}} \leq B\|x\|_E, \quad x \in E,$$

(iii) $x = \sum\limits_{n=1}^{\infty} v_n(x)$, $x \in E$.

The constants A and B are called bounds for the $\mathscr{A}\mathscr{D}$-frame $(\{G_n\}, \{v_n\})$.

In view of above definition, we have the following observation

Observations. If a Banach space E has an $\mathscr{A}\mathscr{D}$-frame, then it also has a fusion Banach frame. Indeed, let $(\{G_n\}, \{v_n\})$ be an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$.

Define $S : \mathscr{A} \to E$ by $S(\{v_n(x)\}) = x$, $x \in E$. Then S is a bounded linear operator on $\mathscr{A}$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathscr{A}$.

However, the converse of this Observation is not true as shown in the following example

Example 3.2. Let $E = \ell^\infty(\chi) = \{\{x_n\} : x_n \in \chi; \sup \|x_n\|_\chi < \infty\}$ be a Banach space with norm given by $\|\{x_n\}\|_E = \sup\limits_{1\le n<\infty} \|x_n\|_\chi$, $\{x_n\} \in E$, where $(\chi, \|\cdot\|_\chi)$ is a Banach space.

For each $n \in \mathbb{N}$, define $G_n = \{\delta_n^x : x \in \chi\}$ and for each $x = \{x_n\} \in E$ define $v_n(x) = \delta_n^{x_n}$, where $\delta_n^x = (0, 0, \ldots, \underset{\substack{\downarrow \\ n\text{th place}}}{x}, 0, \ldots)$, for all $n \in \mathbb{N}$ and $x \in \chi$.

Then there exists an associated Banach space $\mathscr{A} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{v_n(x)\}\|_{\mathscr{A}} = \|x\|_E$, $x \in E$. Define $S : \mathscr{A} \to E$ by $S(\{v_n(x)\}) = x$, $x \in E$. Then $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathscr{A}$. But $(\{G_n\}, \{v_n\})$ is not an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$, since $\{G_n\}$ is not complete in E.

The following result gives necessary conditions for an $\mathscr{A}\mathscr{D}$-frame.

Theorem 3.3. *Let $(\{G_n\}, \{v_n\})$ be an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$ such that $K = \sup\limits_{1\le n<\infty} \left\| \sum\limits_{i=1}^{n} v_i \right\| < \infty$. Then*

(a) $\gamma\left(\left[\bigcup\limits_{n=1}^{\infty} v_n^*(E^*)\right]\right) \ge \dfrac{1}{K} > 0$

(b) $\|f\|_{E^*} \le \sup\limits_{1\le n<\infty} \left\| \sum\limits_{i=1}^{n} v_i^*(f) \right\| \le K\|f\|_{E^*}$, $f \in E^*$.

Proof. (a) Since $x = \sum\limits_{n=1}^{\infty} v_n(x)$, $x \in E$, we have

$$\sum_{i=1}^{n} v_i^*(f) = \left(\sum_{i=1}^{n} v_i\right)^*(f) \xrightarrow{w^*} f, \quad f \in E^*. \tag{3.1}$$

But $\sum\limits_{i=1}^{n} v_i^*(f) \in \left[\bigcup_{m=1}^{\infty} v_m^*(E^*)\right]$, $f \in E^*$, $n = 1, 2, \ldots$.

So $\left\| \sum\limits_{i=1}^{n} v_i^*(f) \right\| \le 1$, for all $f \in E^*$ with $\|f\| \le \dfrac{1}{K}$, $n \in \mathbb{N}$.

Hence, the unit ball of $\left[\bigcup\limits_{n=1}^{\infty} v_n^*(E^*)\right]$ is $\sigma(E^*, E)$-dense in the $\dfrac{1}{K}$-ball of E^*.

(b) By (3.1), we have $\|f\| \le \sup_{1\le n<\infty} \left\| \sum\limits_{i=1}^{n} v_i^*(f) \right\|$, $f \in E^*$.

The other inequality is straightforward. □

Next result also gives a necessary condition for an $\mathscr{A}\mathscr{D}$-frame under stronger hypotheses.

Theorem 3.4. *Let $(\{G_n\},\{v_n\})$ be an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$ with $v_i v_j = 0$, for all $i \neq j$. Then for every $x \in E$, there exists a sequence $\{\alpha_i\} \in bv_0$ and $y \in E$ such that $v_j(x) = \alpha_j v_j(y)$, $j = 1, 2, \ldots$.*

Proof. For each k, write $u_k = \sum\limits_{i=1}^{k} v_i$. Then, for any arbitrary $x \in E$, we have $\lim\limits_{k\to\infty} u_k(x) = x$. Therefore, there exists a sequence $\{m_n\}$ of positive integers such that $\|x - u_k(x)\| \leq \frac{1}{4^{n+1}}$, $k \geq m_n$, $n = 1, 2, \ldots$.

Take $y_n = \sum\limits_{i=m_{n-1}+1}^{m_n} v_i(x)$, $n = 1, 2, \ldots$.

Then $\|y_n\| \leq \frac{2}{4^n}$, $n = 1, 2, \ldots$.

So $\sum\limits_{n=1}^{\infty} 2^{n-1}\|y_n\| \leq \sum\limits_{n=1}^{\infty} 2^{-n}$.

Thus, the series $\sum\limits_{n=1}^{\infty} 2^{n-1}\|y_n\|$ converges.

Put $y = \sum\limits_{n=1}^{\infty} 2^{n-1} y_n$ and $\alpha_j = 2^{1-n}$, $m_{n-1} + 1 \leq j \leq m_n$, $n = 1, 2, \ldots$.

Then $\{\alpha_j\} \in bv_0$ and $v_j(y) = 2^{n-1} v_j(x)$, $m_{n-1} + 1 \leq j \leq m_n$, $n = 1, 2, \ldots$. Hence $v_j(x) = \alpha_j v_j(y)$, $j = 1, 2, \ldots$. $\square$

Regarding the converse of Theorem 3.4 and converse of Observation, we have

Theorem 3.5. *Let E be a Banach space. Let $\{G_n\}$ be a sequence of closed subspaces of E and $\{v_n\}$ be a sequence of non-zero bounded linear projections on E with $v_n(E) = G_n$, $n \in \mathbb{N}$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathscr{A}$ with $v_i v_j = 0$ for all $i \neq j$. Then $(\{G_n\},\{v_n\})$ is an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$ if for every $x \in E$, there exist $\{\alpha_i\} \in bv_0$ and $y \in E$ such that $v_j(x) = \alpha_j v_j(y)$, $j = 1, 2, \ldots$ and $\sup\limits_{1\leq n<\infty} \left\| \sum\limits_{j=1}^{n} v_j(y) \right\| < \infty$.*

Proof. Let $p, q \in \mathbb{N}$ be such that $p < q$. Then

$$\left\| \sum_{i=p}^{q} v_i(x) \right\| = \left\| \sum_{i=p}^{q} \alpha_i \left(\sum_{j=1}^{i} v_j(y) - \sum_{j=1}^{i-1} v_j(y) \right) \right\|$$

$$\leq \left(|\alpha_p| + \sum_{i=p}^{q-1} |\alpha_i - \alpha_{i+1}| + |\alpha_q| \right) \sup_{1\leq n<\infty} \left\| \sum_{j=1}^{n} v_j(y) \right\|$$

Since $\{\alpha_i\} \in bv_0$, we conclude that $\left\{ \sum\limits_{i=1}^{n} v_i(x) \right\}$ is a Cauchy sequence in E and so it converges in E. Also since $\{v_n\}$ is total on E and

$$v_j \left(x - \lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) \right) = 0, \quad j = 1, 2, \ldots,$$

it follows that $(\{G_n\},\{v_n\})$ is an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$. □

4. $\mathscr{A}\mathscr{D}$-Frames satisfying property $\mathscr{B}$

In this section, W will denote $\left[\bigcup_{i=1}^{\infty} v_i^*(E^*)\right]$. We start this section by defining property $\mathscr{B}$ for an $\mathscr{A}\mathscr{D}$-frame

Definition 4.1. An $\mathscr{A}\mathscr{D}$-frame $(\{G_n\},\{v_n\})$ is said to satisfy property $\mathscr{B}$ if every bounded $\sigma(E,W)$-Cauchy sequence in E is $\sigma(E,W)$-convergent in E, where $W = \left[\bigcup_{i=1}^{\infty} v_i^*(E^*)\right]$.

The following theorem gives a characterization of $\mathscr{A}\mathscr{D}$-frames satisfying property $\mathscr{B}$.

Theorem 4.2. *An $\mathscr{A}\mathscr{D}$-frame $(\{G_n\},\{v_n\})$ with $v_iv_j = 0$, for all $i \neq j$ satisfies property $\mathscr{B}$ if and only if each G_n is weakly sequentially complete and for every bounded sequence $\{y_j\}$ in E satisfying $\lim_{j\to\infty} f(v_n(y_j)) = f(z_n), f \in E^*$, there exists an $x \in E$ such that $v_n(x) = z_n, n \in \mathbb{N}$.*

Proof. Suppose first that the $\mathscr{A}\mathscr{D}$-frame $(\{G_n\},\{v_n\})$ satisfies property $\mathscr{B}$. For a fixed $n \in \mathbb{N}$, let $\{y_j\}$ be a bounded sequence in G_n such that for each $f \in G_n^*$, $\lim_{j\to\infty} f(y_j)$ exists. Then $\lim_{j\to\infty} f(y_j)$ exists for each $f \in W$. Therefore, by hypothesis, there exists an $x \in E$ such that $\lim_{j\to\infty} f(y_j) = f(x), f \in W$. Since $(\{G_n\},\{v_n\})$ is an $\mathscr{A}\mathscr{D}$-frame, $E = G_n \oplus \left[\bigcup_{j\neq n} G_j\right]$. So there exists $y \in G_n$ and $z \in \left[\bigcup_{j\neq n} G_j\right]$ such that $x = y + z$.

Now $f(v_k(z)) = f(v_k(x)) = \lim_{j\to\infty}(v_k^* f(y_j)) = 0$, for all $k \neq n$, $f \in E^*$.

This gives $z = 0$. So $x = y \in G_n$.

Also, if $\{y_j\}$ is a bounded sequence in E such that for all $f \in E^*$, $\lim_{j\to\infty} f(v_n(y_j)) = f(z_n), n \in \mathbb{N}$. Then $\lim_{j\to\infty} f(y_j)$ exists for all $f \in W$. So there exists an $x \in E$ such that $\{y_j\}$ is $\sigma(E,W)$-convergent to x. Then

$$f(z_n) = \lim_{j\to\infty}(v_n^* f)(y_j) = f(v_n(x)), \quad f \in E^*, n \in \mathbb{N}$$

Therefore, $v_n(x) = z_n$, for all $n \in \mathbb{N}$.

Conversely, let $\{y_j\}$ be a bounded $\sigma(E,W)$-Cauchy sequence in E. Then, for each $n \in \mathbb{N}$, $\{v_n(y_j)\}$ is a bounded $\sigma(G_n, G_n^*)$-Cauchy sequence in G_n. Therefore, by hypotheses, there exists a sequence $\{z_n\}$ in E with $z_n \in G_n$, $n \in \mathbb{N}$ such that

$$\lim_{j\to\infty} f(v_n(y_j)) = f(z_n), \quad f \in G_n^*, n \in \mathbb{N}.$$

Hence, there exists an $x \in E$ such that $v_n(x) = z_n$, $n \in \mathbb{N}$. □

Next result gives a sufficient condition for an $\mathscr{A}\mathscr{D}$-frame to satisfy property $\mathscr{B}$.

Theorem 4.3. *An $\mathscr{A}\mathscr{D}$-frame $(\{G_n\},\{v_n\})$ satisfies property $\mathscr{B}$ if each G_n is weakly sequentially complete and E is canonically isomorphic to W^*.*

Proof. Let $\{y_j\}$ be a bounded sequence in E such that there exists $z_n \in G_n$, $n \in \mathbb{N}$ and $\lim_{j\to\infty} f(v_n(y_j)) = f(z_n)$, $f \in E^*$, $n \in \mathbb{N}$.

Define $h : W \to \mathbb{R}$ by

$$h(f) = \lim_{j\to\infty} f(y_j), \quad f \in W$$

Then, h is a well defined bounded linear functional on W. Since E is canonically isomorphic to W^*, there exists an $x \in E$ such that $h = u(x)$, where u is the canonical embedding of E into W^*. Therefore,

$$\begin{aligned} f(z_n) &= \lim_{j\to\infty} f(v_n(y_j)) \\ &= h(v_n^* f) \\ &= (u(x))(v_n^* f) \\ &= f(v_n(x)), \quad f \in W, n \in \mathbb{N}. \end{aligned}$$

This gives $v_n(x) = z_n$, for all $n \in \mathbb{N}$. Hence, by Theorem 3.7, $(\{G_n\},\{v_n\})$ satisfies property $\mathscr{B}$. □

In the following theorem, we obtain a necessary condition for a particular type of $\mathscr{A}\mathscr{D}$-frame satisfying property $\mathscr{B}$.

Theorem 4.4. *Let $(\{G_n\},\{v_n\})$ be an $\mathscr{A}\mathscr{D}$-frame satisfying property $\mathscr{B}$ and each G_n is weakly sequentially complete. Then E is canonically isomorphic to W^* if W is separable and $\gamma(W) > 0$.*

Proof. Let u be the canonical embedding of E into W^*. Then, by hypotheses, u is an isomorphism from E into W^*. Let $h \in W^*$ be such that $\|h\| = 1$. Since closed unit ball of W^* is metrizable in the $\sigma(W^*,W)$-topology, there exists a sequence $\{y_j\}$ in E with $\|y_j\| \leq 1$, $j \in \mathbb{N}$ such that $\{u(y_j)\}$ is $\sigma(W^*,W)$-convergent to h. Then, for all $f \in W$, $h(f) = \lim_{j\to\infty} f(y_j)$. So $\lim_{j\to\infty} f(v_k(y_j))$ exists for all $f \in E^*$ and $k \in \mathbb{N}$. In particular, $\lim_{j\to\infty} g(v_k(y_j))$ exists for all $g \in G_n^*$, $k \in \mathbb{N}$. Thus, for each $k \in \mathbb{N}$, $\{v_k(y_j)\}$ is a bounded $\sigma(G_k, G_k^*)$-Cauchy sequence in G_k. Therefore, by hypotheses, there exists a sequence $\{z_k\}$ in E with $z_k \in G_k$, $k \in \mathbb{N}$ such that

$$\lim_{j\to\infty} g(v_k(y_j)) = g(z_k), \quad g \in G_k^*, k \in \mathbb{N}.$$

This gives

$$\lim_{j\to\infty} f(v_k(y_j)) = f(z_k), \quad f \in E^*, k \in \mathbb{N}.$$

Since $(\{G_n\},\{v_n\})$ satisfies property $\mathscr{B}$, there exists an $x \in E$ such that $v_k(x) = z_k$, $k \in \mathbb{N}$. So

$$h(f) = \lim_{j\to\infty} f(y_j) = u(x)(f), \quad \text{for all } f \in E^*. \qquad \square$$

Next using regular summability method, we shall define property $\mathscr{RSM}$ for an $\mathscr{AD}$-frame

Definition 4.5. An $\mathscr{AD}$-frame $(\{G_n\},\{v_n\})$ for E is said to satisfy property $\mathscr{RSM}$ if for every sequence $\{y_n\}$ in the unit ball B_E of E, there exists a regular summability method $\{\alpha_{ij}\}_{i,j\in\mathbb{N}}$, a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ and a $y \in E$ such that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} |\alpha_{ij}| = 1 \quad \text{and} \quad \sigma(E,W) - \lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij} y_{n_j} = y.$$

In the following theorem, we give sufficient conditions, in terms of property $\mathscr{RSM}$ for an $\mathscr{AD}$-frame, in a weakly compactly generated Banach space, to satisfy property $\mathscr{B}$

Theorem 4.6. *Let $(\{G_n\},\{v_n\})$ be an $\mathscr{AD}$-frame for a weakly compactly generated Banach space E such that*

(a) *the norm given by $|||x||| = \sup\{|f(x)| : f \in W, \|f\| = 1\}$ on E is equivalent to the initial norm on E.*
(b) *$(\{G_n\},\{v_n\})$ satisfies property $\mathscr{RSM}$*
(c) *each G_n is weakly sequentially complete.*

Then $(\{G_n\},\{v_n\})$ satisfies property $\mathscr{B}$.

Proof. Let $\{y_j\}$ be a sequence in B_E and $\{z_n\}$ be a sequence with $z_n \in G_n$, $n \in \mathbb{N}$ satisfying

$$\lim_{j\to\infty} f(v_n(y_j)) = f(z_n), \quad (f \in E^*, n \in \mathbb{N}).$$

Thus $\lim_{j\to\infty} f(y_j)$ exists for all $f \in W$.

Define $\phi \in W^*$ by $\phi(f) = \lim_{j\to\infty} f(y_j)$, $f \in W$.

Let $u : E \to W^*$ be a canonical embedding of E into W^*. If $\phi \notin u(B_E)$, then by hypothesis, there exists an $f \in W$ such that it does not attain its supremum on B_E. Let $\{x_n\}$ be a sequence in B_E such that $\lim_{n\to\infty} f(x_n) = \|f\|$. Since $(\{G_n\},\{v_n\})$ satisfies property $\mathscr{RSM}$, there exists a regular summability method $\{\alpha_{ij}\}_{i,j\in\mathbb{N}}$, a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and an $x \in E$ such that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} |\alpha_{ij}| = 1 \quad \text{and} \quad \lim_{i\to\infty} f\left(\sum_{j=1}^{\infty} \alpha_{ij} x_{n_j}\right) = f(x), \quad f \in W$$

Therefore, by convexity of B_E, x is in $\sigma(E,W)$-closure of B_E and $f(x) = \lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij} f(x_{nj}) = \|f\|$.

This is a contradiction. Hence, there exists an element x_0 in B_E such that $u(x_0) = \phi$. So $z_n = v_n(x_0)$, $n \in \mathbb{N}$. whence, by Theorem 3.7, $(\{G_n\}, \{v_n\})$ satisfies property $\mathscr{B}$. □

Remark. There exist $\mathscr{A}\mathscr{D}$-frames satisfying property $\mathscr{B}$ which does not satisfy property $\mathscr{R}\mathscr{S}\mathscr{M}$. Let $E = L^1[0,1]$ and let G be the complemented subspace (isomorphic to ℓ^1). Let G_1 be the complement of G in E. Since E is weakly sequentially complete, G_1 is weakly sequentially complete. Let v_1 and P be the continuous linear projections, respectively, of E onto G_1 and G. Let $\{x_n\}$ be a boundedly complete Schauder basis of G with a.s.c.f $\{f_n\} \subset G^*$. Write $G_n = [x_{n-1}]$, $n \geq 2$ and $v_n(x) = f_{n-1}(Px)x_{n-1}$, $n \geq 2$, $x \in E$. Then $\mathscr{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathscr{A}} = \|x\|_E$, $x \in E$, such that $(\{G_n\}, \{v_n\})$ is an $\mathscr{A}\mathscr{D}$-frame for E with respect to $\mathscr{A}$ which satisfies property $\mathscr{B}$. But $(\{G_n\}, \{v_n\})$ does not satisfy property $\mathscr{R}\mathscr{S}\mathscr{M}$, since otherwise E is canonically isomorphic to W^*.

Finally, we prove that following result regarding $\mathscr{A}\mathscr{D}$-frames and quasi-reflexivity of Banach spaces.

Theorem 4.7. *Let E be a Banach space such that E^* is separable. Let $(\{G_n\}, \{v_n\})$ be an $\mathscr{A}\mathscr{D}$-frame for E with $v_i v_j = 0$, for all $i \neq j$ satisfying property $\mathscr{B}$ such that $\gamma(W) > 0$. Let k be a positive integer such that $\operatorname{codim}_{E^*} W = k$. Then E is quasi-reflexive of order k.*

Proof. By hypotheses and Theorem 4.2, each G_n is weakly sequentially complete. Since E^* is separable, W is separable. Therefore, by Theorem 4.4, W^* is canonically isomorphic to E. Also, since $\operatorname{codim}_{E^*}(W) = k$, it follows that E is quasi-reflexive of order k. □

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S.K. Kaushik, *Department of Mathematics, Kirori Mal College, University of Delhi, Delhi 110007, India*.
E-mail: shikk2003@yahoo.co.in

Varinder Kumar, *Department of Mathematics, Shaheed Bhagat Singh College, University of Delhi, Delhi 110017, India*.
E-mail: varinder1729@gmail.com