Communications in Mathematics and Applications
Vol. 14, No. 1, pp. 67–79, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i1.2113

http://www.rgnpublications.com

Research Article

Applications of an Efficient Iterative Scheme for Finding Zeros of Nonlinear Equations and Its Basins of Attraction

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Received: November 4, 2022 **Accepted:** March 21, 2023

Abstract. The recent research focuses on building several iterative methods over existing or classical numerical methods, such as *Newton's Method* (NM), to solve nonlinear equations to attain higher-order convergence with an improving efficiency index over the produced models. To solve nonlinear equations, a three-step iterative strategy is suggested in this study. Additionally, we used our method in real-time applications for the azeotropic point of a binary solution, beam designing models, chemical engineering, fractional conversion, parachutist's problem, Planck's constant, classical projectile problem, and vertical stress. The numerical results demonstrate our method's superior efficiency to some other existing methods of the same order. To illustrate the dynamic behaviour of basins of attraction in the complex plane, we also studied them.

Keywords. Nonlinear equations, Iterative method, Functional evaluations, Efficiency index, Convergence order, Basins of attraction

Mathematics Subject Classification (2020). 41A25, 65H04, 65H05, 65H20, 65K05

1. Introduction

One of the most difficult problems in numerical analysis is solving nonlinear equations, which occur in practically all areas of science and engineering and are difficult to solve using analytical approaches. As a result, iterative techniques are employed. Various techniques, including Taylor's series, the quadrature formula, the decomposition method, variational iteration method, and the predictor-corrector method, are used to develop these methods. The *Newton-Raphson Method* (NR) [9], is the most well-known iterative technique for locating the root of the nonlinear equation is described by,

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)} \tag{1}$$

which is a single-step iterative method. It has two functional evaluations and quadratic convergence for a simple zero of a nonlinear equation $h(x) = 0$ and its efficiency index $E.I$ is 1.4142. This paper suggests a new sixth-order convergent three-step iterative method free from second derivatives. We derive this method from Taylor's series expansion and finite difference use. To evaluate the applicability, validity, and accuracy of the suggested iteration technique, it has been used to solve algebraic and transcendental problems as well as real-world applications. Here are a few recently created, established techniques for solving (1):

A sixth-order iterative approach (PM) proposed by Chand *et al*. [2] is provided by

$$\left.\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ z_n &= x_n - \left(1 + 2\left(\frac{h(y_n)}{h(x_n)}\right)^2\right)\left(\frac{h(x_n)+h(y_n)}{h'(x_n)}\right), \\ x_{n+1} &= z_n - \left(1 + 2\left(\frac{h(y_n)}{h(x_n)}\right)\right)\frac{h(z_n)}{h'(x_n)}. \end{aligned}\right\} \tag{2}$$

A sixth-order iterative method (FM) given by Chand *et al*. [1] is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{h(x_n)}{h'(x_n)}, \\ z_n &= x_n - \left(-\frac{1}{4} + \frac{3}{4}\frac{h'(x_n)}{h'(y_n)} + \frac{1}{2}\frac{h'(y_n)}{h'(x_n)}\right)\cdot\frac{2h(x_n)}{h'(x_n)+h'(y_n)}, \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(y_n)}\cdot\left(\frac{1}{2} + \frac{1}{2}\frac{h'(x_n)}{h'(y_n)}\right). \end{aligned}\right\} \tag{3}$$

A sixth-order iterative method (IM) developed by Quasi *et al*. [5] and given by

$$\left.\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ x_{n+1} &= y_n - \frac{h(y_n)}{h'(y_n)} - \frac{2h(y_n)^2 h'(y_n) Q(x_n,y_n)}{4(h'(y_n))^4 - 4h(y_n)(h'(y_n))^2 Q(x_n,y_n) + (h(y_n))^2 (Q(x_n,y_n))^2}, \end{aligned}\right\} \tag{4}$$

where

$$h''(y_n) = \frac{2}{x_n - y_n}\left[3\frac{h(x_n)-h(y_n)}{x_n-y_n} - 2h'(y_n) - h'(x_n)\right] = Q(x_n,y_n).$$

Sharma and Panday's [6] sixth-order iterative method (EM) is provided by

$$\left.\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ x_{n+1} &= y_n - \frac{h(y_n)}{h'(y_n)} - \left(\frac{h(y_n)}{h'(y_n)}\left(\frac{-2M_1 + M_2 + M_3 * M_4}{2M_1 - 2M_2 - M_3 * M_4}\right)\right)^2 \frac{Q(x_n,y_n)}{2h'(y_n)}, \end{aligned}\right\} \tag{5}$$

where

$$M_1 = h(x_n)^3 h'(y_n)^2 Q(x_n, y_n), \quad M_2 = h(x_n)^3 h(y_n) Q(x_n, y_n)^2, \quad M_3 = h'(x_n) h(y_n) h'(y_n),$$

$$M_4 = h(x_n)^2 Q(x_n, y_n) - 2h(y_n) h'(x_n)^2, \text{ and}$$

$$h''(y_n) = \frac{2}{x_n - y_n}\left[3\frac{h(x_n) - h(y_n)}{x_n - y_n} - 2h'(y_n) - h'(x_n)\right] = Q(x_n, y_n).$$

The sections of this research paper are as follows: After providing the fundamental concepts and doing a literature review in Section 1, we created the novel three-step iterative technique in Section 2. We demonstrated that this approach has sixth-order convergence in Section 3. We built our method for real-time applications in Section 4, and numerical comparisons with other available methods of the same order are provided. The matching fractal pictures for each iteration plan for the test issues are supplied in Section 5 to show the consistency of the suggested methods. In Section 6, we draw the conclusion that, in terms of iterations and error, our suggested solution outperforms other methods of the same order currently in use.

2. Sixth Order Convergent Method

Consider x^* is an exact root of the nonlinear equation $h(x) = 0$ where $h(x)$ is continuous and has well-defined first-order derivatives. Let x_n be the root of nth approximation and is

$$x^* = x_n + \varepsilon_n, \tag{6}$$

where ε_n is the error. Thus, we get

$$h(x^*) = 0. \tag{7}$$

By Taylor's series, we have

$$\begin{aligned} h(x^*) &= h(x_n) + (x^* - x_n)h'(x_n) + \frac{(x^* - x_n)^2}{2!}h''(t_n) + \dots \\ &= h(x_n) + \varepsilon_n h'(x_n) + \frac{\varepsilon_n^2}{2!}h''(x_n) + \dots. \end{aligned} \tag{8}$$

Neglecting from ε_n^3 on wards, using (7) and (8), we have

$$\varepsilon_n^2 h''(x_n) + 2\varepsilon_n h'(x_n) + 2h(x_n) = 0,$$

$$\varepsilon_n = \left[-2h'(x_n) \pm \sqrt{4h'(x_n) - 8h(x_n)h''(x_n)}\right] \div 2h''(x_n). \tag{9}$$

On substituting x^* by x_{n+1} in (6) and from (9), we get

$$z_n = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)} \cdot \frac{1}{1 + \sqrt{1 - 2\rho_n}}\right], \tag{10}$$

where $\rho_n = \dfrac{h'(x_n) - h'(y_n)}{h'(x_n)}$, $h'(y_n) = 2h(y_n, x_n) - h'(x_n)$, $y_n = x_n - \dfrac{h(x_n)}{h'(x_n)}$, $H(\tau) = 1 - \tau$ and $\tau = \dfrac{h(y_n)}{h(x_n)}$ is the weight function.

By using (1) as the first step, (10) as the second step, and Newton's variation technique as the third step, we create the algorithm.

Algorihtm. The iterative scheme is computed as x_{n+1},

$$\text{(i)}\quad y_n = x_n - \frac{h(x_n)}{h'(x_n)},$$

$$\text{(ii)}\quad z_n = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)}\cdot\frac{1}{1+\sqrt{1-2\rho_n}}\right], \tag{11}$$

where

$$\rho_n = \frac{h'(x_n)-h'(y_n)}{h'(x_n)},$$

$$h'(y_n) = 2h[y_n,x_n]-h'(x_n),\quad H(\tau)=1-\tau \text{ and}$$

$$\tau = \frac{h(y_n)}{h(x_n)},$$

$$\text{(iii)}\quad x_{n+1} = z_n - \frac{h(z_n)}{h(z_n)-h(y_n)}[z_n-y_n].$$

Three functional assessments are conducted using this method, and the order of convergence is enhanced up to six. The method (11) has Efficiency Index $(E.I) = 6^{\frac{1}{4}} = 1.5650$ and it is denoted with KNM.

3. Convergence Criteria

Theorem ([4])**.** *Let $x_0 \in D$ be a single zero of a sufficiently differentiable function h for an open interval D. If x_0 is in the neighborhood of x^*. Then, the algorithm* (11) *has sixth-order convergence.*

Proof. Let the single zero of $h(x)=0$ be x^* and $x^* = x_n+\varepsilon_n$. Using Taylor's series, we get

$$h(x_n) = h'(x^*)(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + \dots), \tag{12}$$

$$h'(x_n) = h'(x^*)(1+2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + \dots). \tag{13}$$

Now, we get

$$y_n = x^* + t_n,$$

where

$$t_n = c_2\varepsilon_n^2 + (2c_3-2c_2^2)\varepsilon_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)\varepsilon_n^4 + \dots. \tag{14}$$

We have

$$h(y_n) = c_2\varepsilon_n^2 + (2c_3-2c_2^2)\varepsilon_n^3 + (3c_4-7c_2c_3+5c_2^3)\varepsilon_n^4 + \dots$$

and

$$\frac{h(y_n)}{h'(y_n)} = c_2\varepsilon_n^2 + (2c_3-2c_2^2)\varepsilon_n^3 + (3c_2^3-6c_2c_3+3c_4)\varepsilon_n^4 + \dots, \tag{15}$$

$$\rho_n = 2c_2\varepsilon_n + (4c_3-6c_2^2)\varepsilon_n^2 + (6c_4+16c_2^3-20c_2c_3)\varepsilon_n^3 + \dots. \tag{16}$$

From (16), on simplification

$$(1+\sqrt{1-2\rho_n})^{-1} = \frac{1}{2}(1+c_2\varepsilon_n + (2c_3-c_2^2)\varepsilon_n^2 + (3c_4-2c_2c_3)\varepsilon_n^3 + (37c_2^4 + 8c_3^2 + 12c_2c_4 - 52c_2^2c_3)\varepsilon_n^4). \tag{17}$$

Using (14) and (16), we get

$$\frac{2h(y_n)}{h'(y_n)}\left(\frac{1}{1+\sqrt{1-2\rho_n}}\right)=c_2\varepsilon_n^2+(2c_3-c_2^2)\varepsilon_n^3+(3c_4-2c_2c_3)\varepsilon_n^4\dots \tag{18}$$

and

$$H(\tau)=1-\frac{h(y_n)}{h(x_n)}=(1-c_2\varepsilon_n-(2c_3-3c_2^2)\varepsilon_n^2-(3c_4-10c_2c_3+8c_2^3)\varepsilon_2^3+\dots). \tag{19}$$

From the second step of (11), we get

$$z_n=x^*+K_1\varepsilon_n^4+K_2\varepsilon_n^5+K_3\varepsilon_n^6+\dots, \tag{20}$$

where $K_1=-c_2c_3$, $K_2=c_2c_4-c_3^2+c_2^4$, $K_3=c_2c_5+6c_2^2c_4+4c_2c_3^2+5c_2^3c_3-c_2^5-c_3c_4-13c_2c_3c_4$,

$$h(z_n)=h'(x^*)(K_1\varepsilon_n^4+K_2\varepsilon_n^5+K_3\varepsilon_n^6+\dots). \tag{21}$$

Using (20), (21) in the third step of (11), we get

$$\varepsilon_{n+1}=-c_2^3c_3\varepsilon_n^6+o(\varepsilon_n^7).$$

Thus, the order of convergence of this method is six, and its efficiency index ($E.I$) is $6^{\frac{1}{4}}=1.5650$.

4. Numerical Examples

We will evaluate the effectiveness and convergence of our suggested algorithm in this part. In order to do this, we take a look at certain real-time engineering applications that are stated in the eight applications. We will now compare the number of iterations, errors, and functional evaluations of our suggested technique (KNM) with those of other sixth-order methods already in use from PM, FM, IM, EM and NR. mpmath-PYTHON is used for all numerical calculations, and we use the halting criterion $|f(x_n)|<\varepsilon$ with the required accuracy set to 290 decimal places and the tolerance set to $\varepsilon=10^{-100}$.

Applications

In this section, we discuss various real-time engineering applications, compare our findings to some established methodologies in Table 2, and display the efficiency index in Table 1:

Application 1 (Azeotropic point of a binary solution, [8]). To determine the azeotropic point of the nonlinear equation:

$$h_1(x)=\frac{PQ[Q(1-x)^2-Px^2]}{[x(P-Q)+Q]^2}+0.14845.$$

The zero of the above equation is a 0.69147373574714144.

Application 2 (Beam designing model, [7]). The following nonlinear equation determines a sheet-pile wall's embedment depth:

$$h_2(x)=\frac{1}{4.62}(x^3+2.87x^2-10.28)-x.$$

The approximated root is 2.0021187789538272.

Application 3 (A chemical engineering, [2])**.** The equation provides the chemical concentration in a mixed reactor can be expressed as

$$h_3(t) = 1 - 0.75e^{-0.05t}.$$

The approximate root is −5.753641449035618.

Application 4 (Fractional conversion, [3])**.** The equation for the fractional conversion is as follows:

$$h_4(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674.$$

The real root is 0.2777595428417206.

Application 5 (Parachutist's problem, [1])**.** The total force for parachutists is calculated as

$$h_5(x) = \frac{gm}{x}(1 - e^{-\frac{x}{m}t}) - v.$$

Here 12.533522848184467 is the root.

Application 6 (Plank's constant, [7])**.** The solution to Planck's radiation law problem generates an isothermal blackbody's energy density is

$$h_6(x) = e^{-x} - 1 + \frac{x}{5}.$$

This equation's approximative root is 4.96511423174427630369.

Application 7 (Projectile problem, [7])**.** The equation describes the moment of an electron in the space between two parallel plates.

$$h_7(x) = x - 0.5\cos x + \frac{\pi}{4}.$$

This function has a simple root at $x^* \approx -0.309466139208214$.

Application 8 (The vertical stress, [7])**.** Vertical stress is one of the basic stresses experienced by finite underground structures, is

$$h_8(x) = \frac{x + \cos x \sin x}{\pi} - \frac{1}{4}.$$

The nonlinear equation has a root of 0.4160444988100767043.

Table 1. Analogy of efficiency index

Methods	Order of convergence	Functional evaluations	Efficiency index
NR	2	2	1.4142
PM	6	5	1.4309
FM	6	4	1.5650
SM	6	4	1.5650
EM	6	4	1.5650
KNM	6	4	1.5650

Table 2. Comparison of different methods

Method	x_0	n	$\lvert x_{n+1}-x_n\rvert$	$\lvert h(x_{n+1})\rvert$	x_0	n	$\lvert x_{n+1}-x_n\rvert$	$\lvert h(x_{n+1})\rvert$
$h_1(x)$	0.9				0.2			
NR		8	4.55e-119	5.01e-119		9	6.72e-138	7.39e-138
PM		4	3.74e-160	4.12e-160		5	3.92e-290	2.60e-291
FM		4	6.67e-128	7.35e-128		5	4.81e-291	2.61e-291
IM		4	3.80e-155	4.18e-155		5	5.61e-291	9.22e-291
EM		5	5.61e-291	2.61e-291		5	5.91e-181	6.50e-181
KNM		4	9.53e-201	1.05e-200		4	1.43e-119	1.57e-119
$h_2(x)$	1.4				3.5			
NR		9	3.36e-115	1.37e-114		10	2.94e-178	1.20e-177
PM		5	1.42e-111	5.84e-111		5	3.11e-283	1.27e-282
FM		5	9.67e-226	3.96e-225		5	5.79e-202	2.37e-201
IM		4	1.12e-107	4.61e-137		5	1.76e-118	7.21e-118
EM		5	1.40e-189	5.74e-189		6	2.81e-282	1.15e-281
KNM		4	1.55e-108	6.35e-108		4	1.33e-100	5.46e-100
$h_3(x)$	-10				6			
NR		8	7.83e-126	3.92e-127		9	7.02e-126	3.51e-127
PM		4	4.54e-172	2.27e-171		4	1.61e-103	8.05e-105
FM		4	4.29e-131	2.14e-132		5	8.89e-267	4.44e-267
IM		4	4.42e-161	2.21e-162		5	2.64e-141	1.32e-142
EM		5	2.31e-147	1.15e-148		6	5.09e-198	2.54e-199
KNM		4	7.91e-220	3.95e-221		4	1.72e-111	8.64e-113
$h_4(x)$	0.5				0.1			
NR		8	6.93e-106	6.23e-105		9	1.58e-158	1.42e-157
PM		4	1.61e-127	1.44e-126		5	2.44e-290	3.52e-290
FM		4	2.26e-102	2.03e-101		5	3.02e-263	2.71e-262
IM			Divergent				Divergent	
EM		5	2.02e-245	1.81e-244		5	2.29e-183	2.06e-182
KNM		4	6.98e-172	6.27e-171		4	1.74e-225	1.56e-224
$h_5(x)$	3				15			
NR		9	1.74e-133	3.20e-133		8	3.92e-136	7.22e-136
PM		5	6.15e-289	8.72e-289		5	2.79e-182	5.15e-182
FM		5	1.45e-281	2.66e-281		4	2.89e-141	5.32e-141
IM		5	1.38e-288	8.72e-289		4	8.05e-210	1.47e-209
EM		6	4.35e-221	8.01e-221		5	3.70e-178	6.81e-178
KNM		4	7.83e-109	1.44e-108		4	6.11e-235	1.12e-234
$h_6(x)$	3				9			
NR		8	5.99e-142	5.78e-142		7	7.77e-103	7.49e-103
PM		4	2.06e-113	1.99e-113		4	2.45e-220	2.36e-220
FM		5	2.57e-290	5.77e-290		4	1.31e-122	1.26e-122
IM		5	1.04e-269	1.00e-269		4	4.06e-221	3.92e-221
EM		5	3.24e-146	3.13e-146		5	6.41e-290	3.21e-290
KNM		4	8.88e-194	8.57e-194		4	6.66e-286	6.43e-286

Table 2 *Contd.*

Method	x_0	n	$\lvert x_{n+1}-x_n\rvert$	$\lvert h(x_{n+1})\rvert$	x_0	n	$\lvert x_{n+1}-x_n\rvert$	$\lvert h(x_{n+1})\rvert$
$h_7(x)$	-0.7				0.5			
NR		8	1.34e-120	1.14e-120		9	4.15e-187	3.52e-187
PM		4	3.59e-151	3.04e-151		4	1.06e-114	8.94e-115
FM		4	2.24e-126	1.89e-126		5	4.01e-291	1.60e-291
IM		4	6.72e-143	5.69e-143		4	7.08e-110	6.01e-110
EM		5	3.00e-256	2.54e-256		5	5.45e-179	4.62e-179
KNM		4	6.04e-213	5.11e-213		4	7.93e-176	6.72e-176
$h_8(x)$	0				0.9			
NR		8	2.21e-128	1.17e-128		9	1.61e-134	8.58e-135
PM		4	1.75e-143	9.34e-144		9	6.81e-291	2.61e-291
FM		4	1.46e-108	7.79e-109		5	1.34e-189	7.17e-190
IM		4	6.66e-143	3.55e-143		7	1.96e-206	1.04e-206
EM		5	8.73e-227	4.65e-227		5	1.17e-104	6.22e-105
KNM		4	1.94e-187	1.03e-183		4	3.47e-123	1.85e-123

5. Basins of Attraction

We provide an in-depth graphical comparison of the newly developed sixth-order iterative method with the existing sixth-order methods by looking at several polynomials using computer technology. By examining the rational function's attractive regions, the iterative method provides information regarding convergence and stability. A method for analyzing how a calculation acts as a function of the several starting places is known as the basin of attraction. It is an alternate approach to talking about iterative processes. Consider a grid point region with $R\times R=[-2,2]\times[-2,2]$ and 250×250 values. We study iterative methods in the z^0 grid points of the square region. With the requirement that $|f(z_k)|<10^{-16}$ and a maximum of 100 iterations. We get to the conclusion that z^0 is in this zero's basin of attraction. We choose a yellow color for diverging grid points if $N>100$. For the suggested method and some existing same-order iterative methods. We outline the areas of interest for finding complex $f_1(z)=1-z^2$, $f_2(z)=1-z^3$, and $f_3(z)=1-z^4$ roots. Figures 1-3 depict the basins of attraction for our cutting-edge iterative method as well as other traditional techniques. For each point $z^0\in C$, we assign a color based on the root at which the corresponding iterative procedure beginning at z^0 converges. The roots of the employed functions are stated, and computations are done with PYTHON are presented. In the vertical bar graph, the purple color area represents strong convergence, moderate convergence was represented by green color area, and weak convergence was represented by yellow color.

Example 1. $f_1(z) = 1 - z^2$

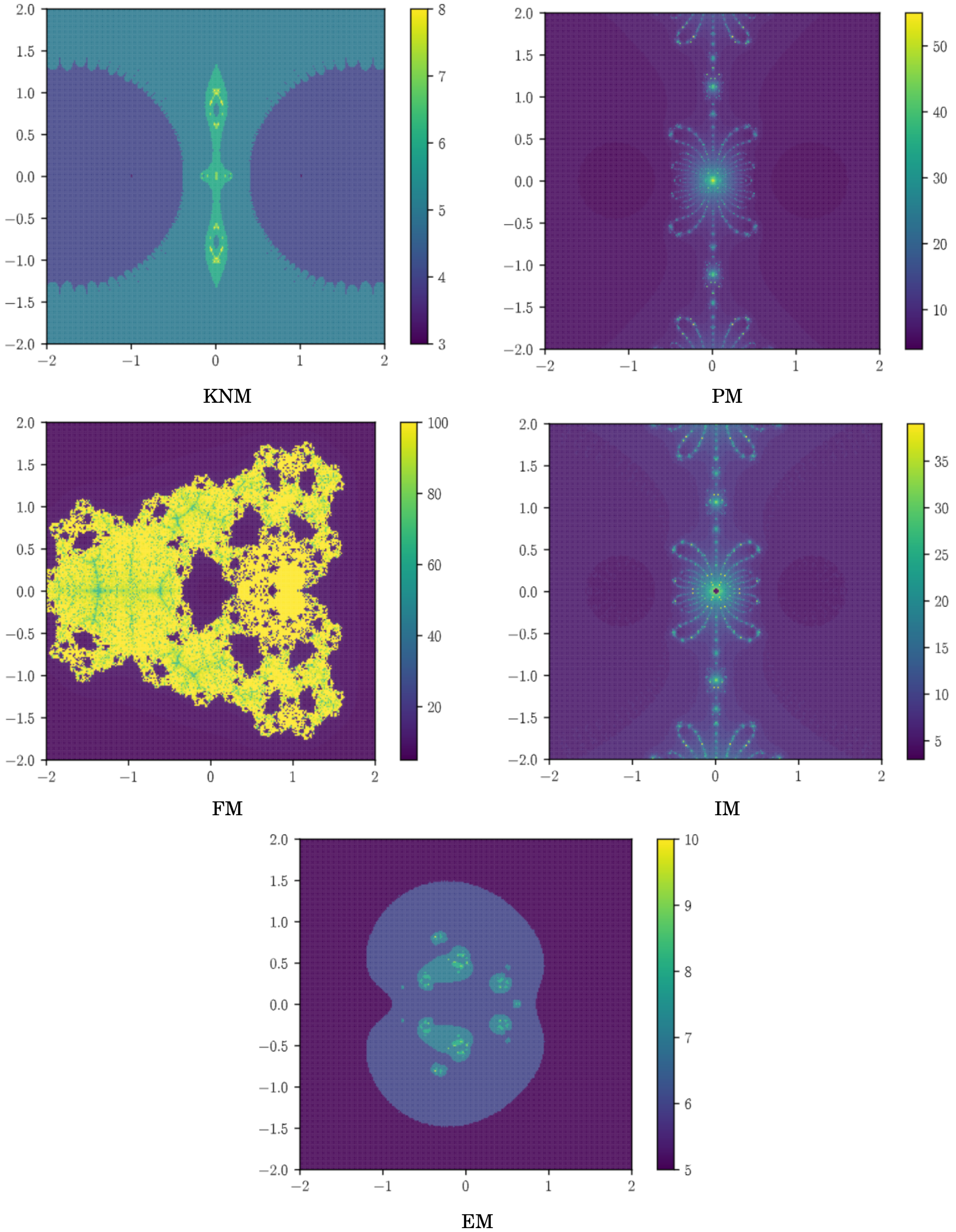

Figure 1. The polynomiographs were obtained with the provided methods KNM, PM, FM, IM and EM for $f_1(z)$

Example 2. $f_2(z) = 1 - z^3$

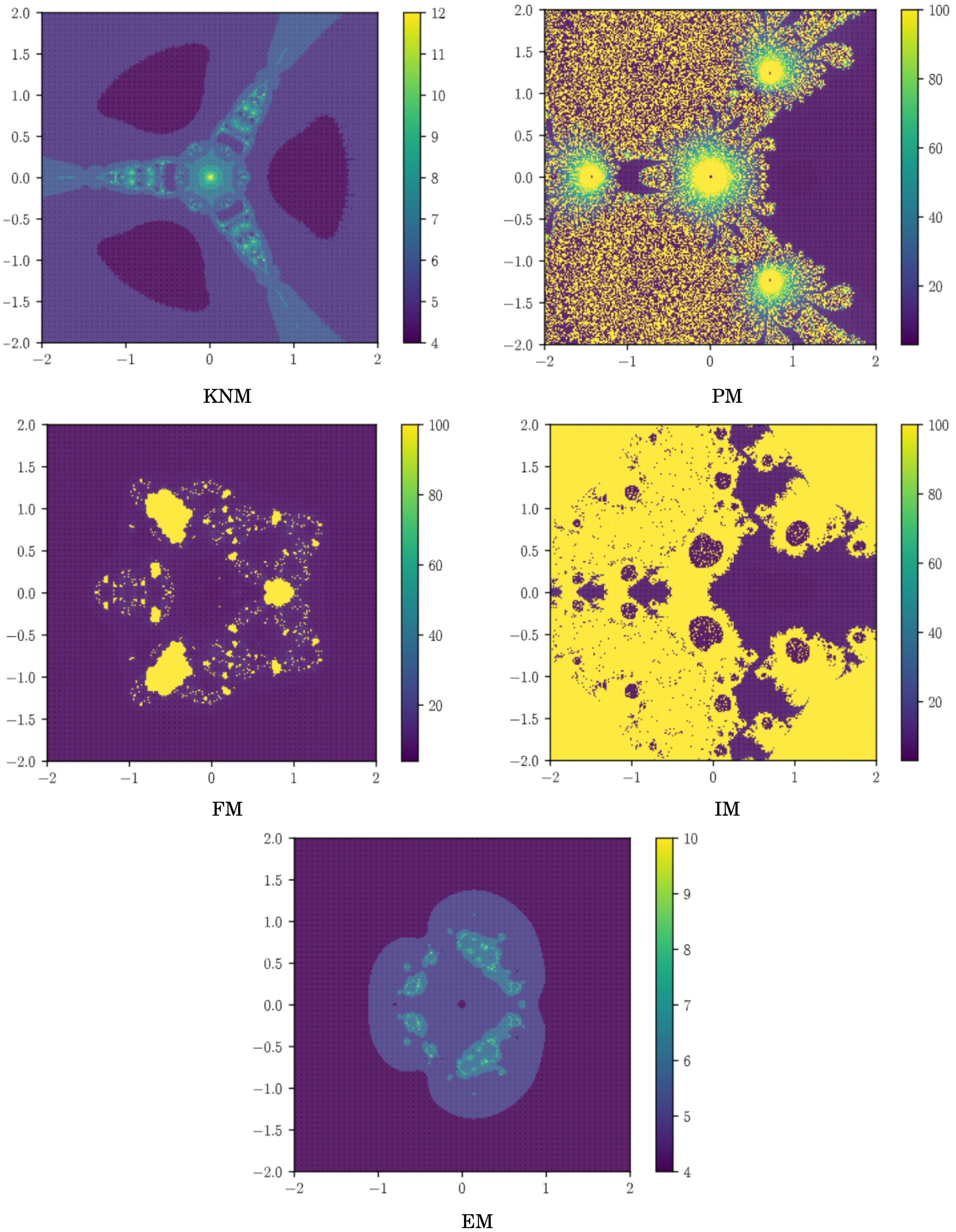

Figure 2. The polynomiographs were obtained with the provided methods KNM, PM, FM, IM and EM for $f_2(z)$

Example 3. $f_3(z) = 1 - z^4$

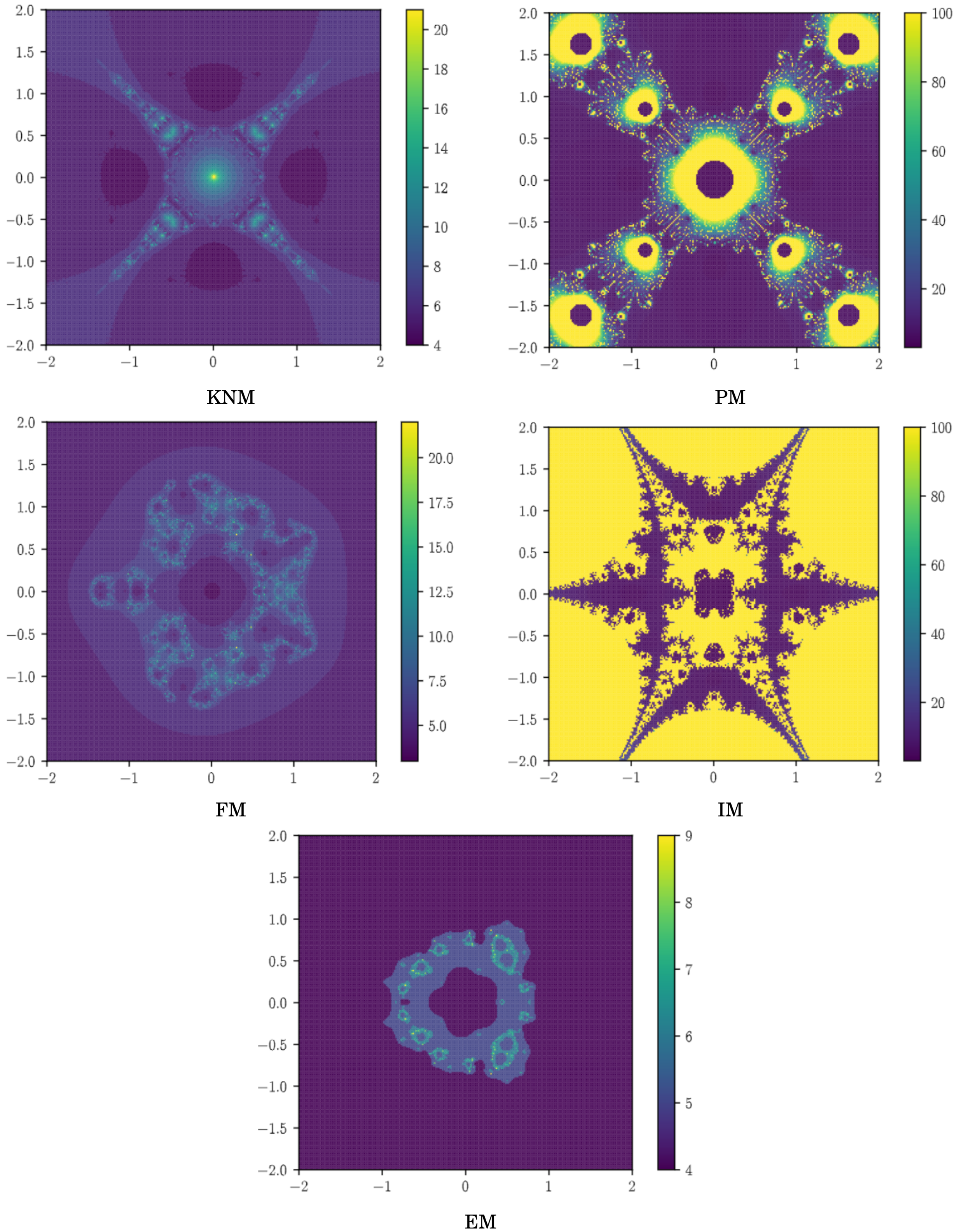

Figure 3. The polynomiographs were obtained with the provided methods KNM, PM, FM, IM and EM for $f_3(z)$

In all three examples, it is clear that the developed method, KNM, needs fewer iterations than the current methods. Figure 1 to Figure 3 demonstrates how the suggested method (KNM) outperforms other methods currently in use for the same order.

6. Conclusion

The zeros of a nonlinear equation can be found utilizing a novel three-step sixth-order iterative method that we suggested in this study. The proposed method's convergence demonstrates the sixth order. In some application issues in science and engineering, the new approach outperforms some other compared methods of the same order in terms of results. Regarding the number of iterations and error, our suggested solution outperforms existing methods for all application problems. The proposed method's dynamical behaviour has also been looked at to assess its stability. The findings demonstrate that the method has substantial basins of attraction, ensuring its stability in the presence of a variety of nonlinear problems. Finally, we draw the conclusion that our suggested strategy is more effective than the other ones now in use.

Acknowledgments

The authors would like to be thankful to GITAM (Deemed to be University) for their support and providing the resources.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] P. B. Chand, F. I. Chicharro and P. Jain, On the design and analysis of high-order Weerakoon-Fernando methods based on weight functions, *Computational and Mathematical Methods* **2**(5) (2020), e1114, DOI: 10.1002/cmm4.1114.

[2] P. B. Chand, F. I. Chicharro, N. Garrido and P. Jain, Design and complex dynamics of Potra–Pták-type optimal methods for solving nonlinear equations and its applications, *Mathematics* **7**(10) (2019), 942, DOI: 10.3390/math7100942.

[3] M. S. K. Mylapalli, P. R. Kumar and S. Ramadevi, A ninth order iterative method for solving nonlinear equations with high efficiency-index, *Advances in Mathematics: Scientific Journal* **9**(7) (2020), 5283 – 5290, DOI: 10.37418/amsj.9.7.97.

[4] M. S. K. Mylapalli, R. K. Palli, P. Chaganti and R. Sri, An optimal fourth order iterative method for solving non-linear equations, *IAENG International Journal of Applied Mathematics* **52**(3) (2022), 732 – 741, URL: https://www.iaeng.org/IJAM/issues_v52/issue_3/IJAM_52_3_25.pdf.

[5] R. Qudsi, M. Imran and Syamsudhuha, Another sixth-order iterative method free from derivative for solving multiple roots of a nonlinear equation, *Applied Mathematical Sciences* **11**(43) (2017), 2121 – 2129, DOI: 10.12988/ams.2017.76208.

[6] E. Sharma and S. Panday, Efficient sixth order iterative method free from higher derivatives for nonlinear equations, *Journal of Mathematical and Computational Science* **12** (2022), Article ID 46, 1 – 13, DOI: 10.28919/jmcs/6950.

[7] P. Sivakumar and J. Jayaraman, Some new higher order weighted newton methods for solving nonlinear equation with applications, *Mathematical and Computational Applications* **24**(2) (2019), 59, DOI: 10.3390/mca24020059.

[8] O. S. Solaiman and I. Hashim, Optimal eighth-order solver for nonlinear equations with applications in chemical engineering, *Intelligent Automation & Soft Computing* **27**(2) (2021), 379 – 390, DOI: 10.32604/iasc.2021.015285.

[9] J. F. Traub, *Iterative Methods for the Solution of Equations*, AMS Chelsea Publishing Vol. **312**, 310 pp., (1964), URL: https://bookstore.ams.org/chel-312.