Communications in Mathematics and Applications
Vol. 14, No. 1, pp. 49–58, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i1.2056

http://www.rgnpublications.com

Research Article

A Study on $\mathcal{I}$-localized Sequences in S-metric Spaces

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Received: September 17, 2022 **Accepted:** February 12, 2023

Abstract. In this paper, we study the notion of $\mathcal{I}$-localized and $\mathcal{I}^*$-localized sequences in S-metric spaces. Also, we investigate some properties related to $\mathcal{I}$-localized and $\mathcal{I}$-Cauchy sequences and give the idea of $\mathcal{I}$-barrier of a sequence in the same space. Finally, we use this idea for an $\mathcal{I}$-localized sequence to be $\mathcal{I}$-Cauchy when the ideal $\mathcal{I}$ satisfies the condition (AP).

Keywords. Ideal, S-metric space, $\mathcal{I}$-locator, $\mathcal{I}$-localized sequence, $\mathcal{I}^*$-localized sequence, $\mathcal{I}$-barrier

Mathematics Subject Classification (2020). 40A35, 40A30, 54A20, 40A05

1. Introduction

After long fifty years of introduction of the notion of statistical convergence [5, 12, 14] the idea of $\mathcal{I}$-convergence was given by Kostyrko *et al.* [10] in 2000 where $\mathcal{I}$ is an ideal of subsets of the set of natural numbers. Then this idea of ideal convergence was studied by several authors in many directions [1–4].

The notion of localized sequences was introduced by Krivonosov [9] in metric spaces in 1974 as a generalization of a Cauchy sequence. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X,d) is said to be localized in some subset $M \subset X$ if the number sequence $\alpha_n = d(x_n, x)$ converges for $x \in M$. The maximal subset of X on which the sequence $\{x_n\}_{n\in\mathbb{N}}$ is localized is called the locator of $\{x_n\}_{n\in\mathbb{N}}$ and it is denoted by $\operatorname{loc}(x_n)$. If $\{x_n\}_{n\in\mathbb{N}}$ is localized on X then it is called localized everywhere in X. If the locator of a sequence $\{x_n\}_{n\in\mathbb{N}}$ contains all elements of this sequence, except for a finite number of elements of it then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is called localized in itself.

After long years, in 2020, Nabiev *et al.* [11] introduced the idea of $\mathcal{I}$-localized and $\mathcal{I}^*$-localized sequences in metric spaces and investigated some basic properties of the $\mathcal{I}$-localized sequences related with $\mathcal{I}$-Cauchy sequences. At the same time, Gürdal *et al.* [8] studied A-statistically localized sequences in n-normed spaces, Yamanci *et al.* [15] have extended this idea of localized sequences to statistically localized sequences in 2-normed spaces and interestingly this notion has been generalized in ideal context in 2-normed spaces by Yamanci *et al.* [16]. In 2021, Granados and Bermudez [7] studied on $\mathcal{I}_2$-localized double sequences and Granados [6] nurtured this notion with the help of triple sequences using ideals in metric spaces.

In 2012, Sedghi *et al.* [13] introduced the interesting notion of S-metric spaces and proved some basic properties in this space. For an admissible ideal $\mathcal{I}$, $\mathcal{I}^*$-convergence and $\mathcal{I}^*$-Cauchy criteria in X imply $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy criteria in X respectively. Moreover, for admissible ideal with the property (AP), $\mathcal{I}$ and $\mathcal{I}^*$-convergence ($\mathcal{I}$ and $\mathcal{I}^*$-Cauchy criteria) in X are equivalent [1]. In this paper we have studied the notion of $\mathcal{I}$ and $\mathcal{I}^*$-localized sequences and have investigated some results related to $\mathcal{I}$-Cauchy sequences in S-metric spaces.

2. Preliminaries

Now we recall some basic definitions and notations from [10]. If X is a non-empty set then a collection $\mathcal{I}$ of subsets of X is said to be an ideal of X if (i) $A,B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$, $B \subset A \Rightarrow B \in \mathcal{I}$. Clearly, $\{\phi\}$ and 2^X, the power set of X, are the trivial ideals of X. A non-trivial ideal $\mathcal{I}$ is said to be an admissible ideal if $\{x\} \in \mathcal{I}$ for each $x \in X$. If $\mathcal{I}$ is a non-trivial ideal of X then the family of sets $\mathcal{F}(\mathcal{I}) = \{A \subset X : X \setminus A \in \mathcal{I}\}$ is clearly a filter on X. This filter is called the filter associated with the ideal $\mathcal{I}$. An admissible ideal $\mathcal{I}$ of $\mathbb{N}$, the set of natural numbers, is said to satisfy the condition (AP) if for every countable family $\{A_1,A_2,A_3,\dots\}$ of sets belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1,B_2,B_3,\dots\}$ such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i\in\mathbb{N}} B_i \in \mathcal{I}$. Note that $B_i \in \mathcal{I}$ for all $i \in \mathbb{N}$.

Now, we recall some basic definitions and some properties from [13].

Definition 2.1. Let X be a non-empty set. The S-metric on X is a function $S : X \times X \times X \to [0,\infty)$, such that for each $x,y,z,a \in X$,

(i) $S(x,y,z) \geq 0$;

(ii) $S(x,y,z) = 0$ if and only if $x = y = z$;

(iii) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called a S-metric space. Several examples may be seen from [13]. In a S-metric space, we have $S(x,x,y) = S(y,y,x)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,S) is said to converge to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x) < \varepsilon$ for all $n \geq n_0$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,S) is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq n_0$.

We recall the following definitions in an S-metric space from [1] which will be useful in the sequel.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $x\in X$ if for each $\varepsilon>0$, the set $A(\varepsilon)=\{n\in\mathbb{N}:S(x_n,x_n,x)\geq\varepsilon\}\in\mathcal{I}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}^*$-convergent to $x\in X$ if and only if there exists a set $M\in\mathcal{F}(\mathcal{I})$, $M=\{m_1<m_2<\cdots<m_k<\cdots\}\subset\mathbb{N}$ such that $\lim\limits_{k\to\infty}S(x_{m_k},x_{m_k},x)=0$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is called an $\mathcal{I}$-Cauchy sequence if for every $\varepsilon>0$, there exists a positive integer $n_0=n_0(\varepsilon)$ such that the set $A(\varepsilon)=\{n\in\mathbb{N}:S(x_n,x_n,x_{n_0})\geq\varepsilon\}\in\mathcal{I}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is called an $\mathcal{I}^*$-Cauchy sequence if there exists a set $M=\{m_1<m_2<\ldots<m_k\ldots\}\subset\mathbb{N}$, $M\in\mathcal{F}(\mathcal{I})$, such that the subsequence $\{x_{m_k}\}$ is an ordinary Cauchy sequence in X i.e., for each preassigned $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ such that $S(x_{m_k},x_{m_k},x_{m_r})<\varepsilon$ for all $k,r\geq k_0$.

3. Main Results

Throughout the discussion, $\mathbb{N}$ stands for the set of natural numbers, $\mathcal{I}$ for an admissible ideal of $\mathbb{N}$ and X stands for a S-metric space unless otherwise stated. Now we introduce some definitions and properties regarding localized sequences with respect to the ideal $\mathcal{I}$ in S-metric spaces.

Definition 3.1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be localized in the subset $M\subset X$ if for each $x\in M$, the non-negative real sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ converges in $\mathbb{R}$.

Definition 3.2. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-localized in the subset $M\subset X$ if for each $x\in M$, $\mathcal{I}$-$\lim\limits_{n\to\infty}S(x_n,x_n,x)$ exists i.e., if the non-negative real sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent.

The maximal subset of X on which a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is $\mathcal{I}$-localized is called the $\mathcal{I}$-locator of $\{x_n\}_{n\in\mathbb{N}}$ and it is denoted by $\mathrm{loc}_{\mathcal{I}}(x_n)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-localized everywhere if the $\mathcal{I}$-locator of $\{x_n\}_{n\in\mathbb{N}}$ is the whole set X. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-localized in itself if the set $\{n\in\mathbb{N}:x_n\in\mathrm{loc}_{\mathcal{I}}(x_n)\}\in\mathcal{F}(\mathcal{I})$.

Now we introduce an important result in S-metric spaces which will be useful in the sequel.

Lemma 3.1. *The inequality* $|S(x,x,\xi)-S(\xi,\xi,y)|\leq 2S(x,x,y)$ *holds good for any* $x,y,\xi\in X$.

Proof. Now for $x,y,\xi\in X$, we have

$$\begin{aligned}S(x,x,\xi)&\leq S(x,x,y)+S(x,x,y)+S(\xi,\xi,y)\\&=2S(x,x,y)+S(\xi,\xi,y).\end{aligned}$$

Therefore

$$S(x,x,\xi)-S(\xi,\xi,y)\leq 2S(x,x,y). \tag{3.1}$$

Again, we have

$$\begin{aligned}S(\xi,\xi,y)-S(x,x,\xi)&=S(y,y,\xi)-S(x,x,\xi)\\&\leq S(y,y,x)+S(y,y,x)+S(\xi,\xi,x)-S(x,x,\xi)\\&=S(x,x,y)+S(x,x,y)+S(x,x,\xi)-S(x,x,\xi)\\&=2S(x,x,y).\end{aligned}$$

Therefore

$$S(\xi,\xi,y)-S(x,x,\xi)\le 2S(x,x,y). \tag{3.2}$$

From eqs. (3.1) and (3.2) we have $|S(x,x,\xi)-S(\xi,\xi,y)|\le 2S(x,x,y)$. This completes the proof. □

Lemma 3.2. *If $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathfrak{I}$-Cauchy sequence in X then it is $\mathfrak{I}$-localized everywhere.*

Proof. By the condition, for every $\varepsilon>0$, there exists a positive integer $n_0=n_0(\varepsilon)$ such that the set $A(\varepsilon)=\{n\in\mathbb{N}:S(x_n,x_n,x_{n_0})\ge\frac{\varepsilon}{2}\}\in\mathfrak{I}$. Let $\xi\in X$. Using Lemma 3.1, we have $|S(x_n,x_n,\xi)-S(\xi,\xi,x_{n_0})|\le 2S(x_n,x_n,x_{n_0})$. Therefore $\{n\in\mathbb{N}:|S(x_n,x_n,\xi)-S(\xi,\xi,x_{n_0})|\ge\varepsilon\}\subset\{n\in\mathbb{N}:S(x_n,x_n,x_{n_0})\ge\frac{\varepsilon}{2}\}\in\mathfrak{I}$. This shows that the number sequence $\{S(x_n,x_n,\xi)\}_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent for each $\xi\in X$. Hence the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathfrak{I}$-localized everywhere. □

Corollary 3.1. *By Lemma* 3.2*, it follows that every $\mathfrak{I}$-convergent sequence in X is $\mathfrak{I}$-localized everywhere.*

Also, if $\mathfrak{I}$ is an admissible ideal then every localized sequence in X is $\mathfrak{I}$-localized sequence in X.

Definition 3.3. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathfrak{I}^*$-localized in X if the real sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathfrak{I}^*$-convergent for each $x\in X$.

Theorem 3.1. *Let $\mathfrak{I}$ be an admissible ideal. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is $\mathfrak{I}^*$-localized on the subset $M\subset X$ then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathfrak{I}$-localized on the set M and* $\mathrm{loc}_{\mathfrak{I}^*}(x_n)\subset\mathrm{loc}_{\mathfrak{I}}(x_n)$.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be $\mathfrak{I}^*$-localized on the subset $M\subset X$. Then, by Definition 3.3, the real sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathfrak{I}^*$-convergent for each $x\in M$. Now since $\mathfrak{I}$ is an admissible ideal, the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent for each $x\in M$ which implies that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathfrak{I}$-localized on the set M. □

But the converse of Theorem 3.1 does not hold in general. It can be shown by the following example.

Example 3.1. First, we define the S-metric on $\mathbb{R}$ by $S(x,y,z)=d(x,z)+d(y,z)$, $\forall\ x,y,z\in\mathbb{R}$ where d is the usual metric on $\mathbb{R}$. Let $\mathbb{N}=\bigcup_{j=1}^{\infty}\Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i\cap\Delta_j=\phi$ for $i\ne j$. Let $\mathfrak{I}$ be the class of all those subsets of $\mathbb{N}$ which intersects only a finite number of Δ_j's. Then $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $(\mathbb{R},S)$ defined by $x_n=\frac{1}{j}$, for $n\in\Delta_j$. Let $\varepsilon>0$ be given. Now since the sequence $\left\{\frac{1}{j}\right\}_{j\in\mathbb{N}}$ in $(\mathbb{R},d)$ converges to zero, so there exists $p\in\mathbb{N}$ such that $d\left(\frac{1}{j},0\right)<\frac{\varepsilon}{4}$ for all $j\ge p$. Now

$$S(x_n,x_n,0)=d(x_n,0)+d(x_n,0)=d\left(\frac{1}{j},0\right)+d\left(\frac{1}{j},0\right)<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2},\quad \text{for all } j\ge p. \tag{3.3}$$

Let $x\in\mathbb{R}$. Now using Lemma 3.1 and eq. (3.3), we have

$$|S(x_n,x_n,x)-S(x,x,0)|\le 2S(x_n,x_n,0)<\varepsilon,\quad \text{for all } j\ge p.$$

Hence $\{n \in \mathbb{N} : |S(x_n,x_n,x) - S(x,x,0)| \geq \varepsilon\} \subseteq \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_p \in \mathcal{I}$. Therefore, $\{n \in \mathbb{N} : |S(x_n,x_n,x) - S(x,x,0)| \geq \varepsilon\} \in \mathcal{I}$. Hence for each $x \in \mathbb{R}$, the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent. Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-localized in $(\mathbb{R},S)$

Now we show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}^*$-localized in $(\mathbb{R},S)$. If possible, let the sequence $\{x_n\}_{n\in\mathbb{N}}$ be $\mathcal{I}^*$-localized in $(\mathbb{R},S)$. So the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent for each $x \in \mathbb{R}$. So there exists $A \in \mathcal{I}$ such that, for $M = \mathbb{N} \setminus A = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$, the subsequence $\{S(x_n,x_n,x)\}_{n\in M}$ is convergent. Now, by the definition of $\mathcal{I}$, there is a positive integer t such that $A \subseteq \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_t$. But then $\Delta_i \subset \mathbb{N} \setminus A = M$ for all $i \geq t+1$. In particular Δ_{t+1}, $\Delta_{t+2} \subset M$. Since $\Delta_j' s$ are infinite, there are infinitely many $k's$ for which $x_{m_k} = \frac{1}{t+1}$ when $m_k \in \Delta_{t+1}$ and $x_{m_k} = \frac{1}{t+2}$ when $m_k \in \Delta_{t+2}$. So

$$S(x_{m_k},x_{m_k},0) = \begin{cases} d\left(\dfrac{1}{t+1},0\right) + d\left(\dfrac{1}{t+1},0\right) = \dfrac{2}{t+1}, & \text{when } m_k \in \Delta_{t+1}, \\ d\left(\dfrac{1}{t+2},0\right) + d\left(\dfrac{1}{t+2},0\right) = \dfrac{2}{t+2}, & \text{when } m_k \in \Delta_{t+2}. \end{cases}$$

So for $0 \in \mathbb{R}$ there are infinitely many terms of the form $\frac{2}{t+1}$ and $\frac{2}{t+2}$. So $\{S(x_{m_k},x_{m_k},0)\}_{k\in\mathbb{N}}$ can not be convergent which leads to a contradiction. Hence the sequence $\{x_n\}_{n\in\mathbb{N}}$ can not be $\mathcal{I}^*$-localized.

Remark 3.1. If X has no limit point then $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence coincide. Therefore, by Definitions 3.2 and 3.3 and by Theorem 3.1, we have $\mathrm{loc}_{\mathcal{I}}(x_n) = \mathrm{loc}_{\mathcal{I}^*}(x_n)$. Also, if X has a limit point ξ then there is an admissible ideal $\mathcal{I}$ for which there exists an $\mathcal{I}$-localized sequence $\{y_n\}_{n\in\mathbb{N}}$ in X but $\{y_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}^*$-localized.

Now we shall formulate the necessary and sufficient condition for the ideal $\mathcal{I}$ under which $\mathcal{I}$ and $\mathcal{I}^*$-localized sequences are equivalent.

Theorem 3.2. (i) *If $\mathcal{I}$ satisfies the condition (AP) and $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-localized on the set $M \subset X$ then it is $\mathcal{I}^*$-localized on M.*

(ii) *If X has a limit point and every $\mathcal{I}$-localized sequence implies $\mathcal{I}^*$-localized then $\mathcal{I}$ will have the property (AP).*

Proof. (i): Suppose that $\mathcal{I}$ satisfies the condition (AP) and $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-localized on the set $L \subset X$. Then, by the definition, the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent for $x \in L$. Let $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ be $\mathcal{I}$-convergent to $\beta = \beta(x) \in \mathbb{R}$. Then for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |S(x_n,x_n,x) - \beta| \geq \varepsilon\} \in \mathcal{I}$. Now suppose $A_1 = \left\{n \in \mathbb{N} : |S(x_n,x_n,x) - \beta| \geq 1\right\}$ and $A_k = \left\{n \in \mathbb{N} : \frac{1}{k} \leq |S(x_n,x_n,x) - \beta| < \frac{1}{k-1}\right\}$ for $k \geq 2, k \in \mathbb{N}$. Obviously, $A_1, A_k \in \mathcal{I}$ for $k \geq 2$, $k \in \mathbb{N}$ and $A_i \cap A_j = \phi$, for $i \neq j$. Since $\mathcal{I}$ satisfies the condition (AP), there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is finite for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. Now we shall show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-localized. By the definition, it is enough to prove that the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent for every $x \in L$. We show, for $\mathbb{N} \setminus B = M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$, $\displaystyle\lim_{n\to\infty, n\in M} S(x_n,x_n,x) = \beta$. Let $\theta > 0$ and $k \in \mathbb{N}$

be such that $\frac{1}{k+1} < \theta$. Then $\{n \in \mathbb{N} : |S(x_n, x_n, x) - \beta| \geq \theta\} \subset \bigcup_{j=1}^{k+1} A_j$. Since $A_j \Delta B_j$, $j = 1, 2, \cdots k+1$, is finite, we have an $n_0 \in \mathbb{N}$ such that $\left(\bigcup_{j=1}^{k+1} B_j\right) \cap \{n \in \mathbb{N} : n > n_0\} = \left(\bigcup_{j=1}^{k+1} A_j\right) \cap \{n \in \mathbb{N} : n > n_0\}$. If $n > n_0$ and $n \notin B$, then $n \notin \bigcup_{j=1}^{k+1} B_j$ and so $n \notin \bigcup_{j=1}^{k+1} A_j$. But then $|S(x_n, x_n, x) - \beta| < \frac{1}{k+1} < \theta$. Thus the number sequence $\{S(x_n, x_n, x)\}_{n \in \mathbb{N}}$, $x \in L$, is $\mathcal{I}^*$-convergent. Therefore the sequence $\{x_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}^*$-localized.

(ii): The proof is parallel to [10, Theorem 3.2]. Therefore, it is omitted. □

Definition 3.4. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X. Then $\{x_n\}_{n \in \mathbb{N}}$ is said to be $\mathcal{I}$-bounded if there exists $x \in X$ and $G > 0$ such that the set $\{n \in \mathbb{N} : S(x_n, x_n, x) > G\} \in \mathcal{I}$.

Proposition 3.1. *Every $\mathcal{I}$-localized sequence is $\mathcal{I}$-bounded.*

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be $\mathcal{I}$-localized on a subset $M \subset X$. Then the number sequence $\{S(x_n, x_n, \xi)\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-convergent for every $\xi \in M$. Let $\{S(x_n, x_n, \xi)\}_{n \in \mathbb{N}}$ converge to $\alpha = \alpha(\xi) \in \mathbb{R}$. Let $G > 0$ be given. Then $\{n \in \mathbb{N} : |S(x_n, x_n, \xi) - \alpha| > G\} \in \mathcal{I}$. This implies that $\{n \in \mathbb{N} : S(x_n, x_n, \xi) - \alpha > G\} \cup \{n \in \mathbb{N} : S(x_n, x_n, \xi) - \alpha < -G\} \in \mathcal{I}$. Therefore, $\{n \in \mathbb{N} : S(x_n, x_n, \xi) > \alpha + G\} \in \mathcal{I}$, which shows that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-bounded. □

Theorem 3.3. *Let $\mathcal{I}$ be an admissible ideal with the condition (AP) and $L = loc_{\mathcal{I}}(x_n)$ and let $z \in X$ be a point such that for any $\varepsilon > 0$ there exists $x \in L$ satisfying*

$$\{n \in \mathbb{N} : |S(x_n, x_n, x) - S(x_n, x_n, z)| \geq \varepsilon\} \in \mathcal{I}. \tag{3.4}$$

Then $z \in L$.

Proof. Let $\varepsilon > 0$ be given and $x \in L = loc_{\mathcal{I}}(x_n)$ be a point satisfying the condition (3.4). Let $A = \{n \in \mathbb{N} : |S(x_n, x_n, x) - S(x_n, x_n, z)| \geq \varepsilon\} \in \mathcal{I}$. Then $M = \mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$. Therefore, for $n \in M$, we have $|S(x_n, x_n, x) - S(x_n, x_n, z)| < \varepsilon$. Now since $x \in L = \mathrm{loc}_{\mathcal{I}}(x_n)$, the number sequence $\{S(x_n, x_n, x)\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-convergent. So the number sequence $\{S(x_n, x_n, x)\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-Cauchy. Again since $\mathcal{I}$ satisfies the condition (AP), the number sequence $\{S(x_n, x_n, x)\}_{n \in \mathbb{N}}$ is $\mathcal{I}^*$-Cauchy. Then there exists $B \subset \mathbb{N}$, $B \in \mathcal{F}(\mathcal{I})$ such that the subsequence $\{S(x_n, x_n, x)\}_{n \in B}$ is an ordinary Cauchy sequence i.e., there exists $n_0 \in \mathbb{N}$ such that $|S(x_n, x_n, x) - S(x_m, x_m, x)| < \varepsilon$ for all $n, m > n_0$ and $n, m \in B$. Let $K = M \cap B$. Then $K \in \mathcal{F}(\mathcal{I})$. Now, for $p, q \in K$ and $p, q > n_0$, we have

$$\begin{aligned}
|S(x_p, x_p, z) - S(x_q, x_q, z)| \leq{}& |S(x_p, x_p, z) - S(x_p, x_p, x)| + |S(x_p, x_p, x) - S(x_q, x_q, x)| \\
& + |S(x_q, x_q, x) - S(x_q, x_q, z)| \\
<{}& \varepsilon + \varepsilon + \varepsilon \\
={}& 3\varepsilon.
\end{aligned}$$

Therefore, we have the subsequence $\{S(x_n, x_n, z)\}_{n \in K}$ is a Cauchy Sequence. So the number sequence $\{S(x_n, x_n, z)\}_{n \in K}$ is convergent. Therefore, the number sequence $\{S(x_n, x_n, z)\}_{n \in \mathbb{N}}$ is $\mathcal{I}^*$-convergent. Since $\mathcal{I}$ is an admissible ideal, the number sequence $\{S(x_n, x_n, z)\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-convergent. Therefore, the sequence $\{x_n\}_{n \in \mathbb{N}}$ is $\mathcal{I}$-localized and $z \in L$. This proves the theorem. □

Definition 3.5 (cf. [11])**.** Let (X,S) be a S-metric space and $\xi \in X$. Then ξ is said to be an $\mathcal{I}$-limit point of the sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ if there is a set $M = \{m_1 < m_2 < \cdots\}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, \xi) = 0$, and the point ξ is said to be an $\mathcal{I}$-cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ if and only if for each $\varepsilon > 0$ we have $\{n \in \mathbb{N} : S(x_n, x_n, \xi) < \varepsilon\} \notin \mathcal{I}$.

Definition 3.6 (cf. [11])**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$. If $M \in \mathcal{I}$, then the subsequence $\{x_n\}_{n\in M}$ of $\{x_n\}_{n\in\mathbb{N}}$ is called $\mathcal{I}$-thin subsequence of $\{x_n\}_{n\in\mathbb{N}}$. On the other hand, if $M \notin \mathcal{I}$, then the subsequence $\{x_n\}_{n\in M}$ of $\{x_n\}_{n\in\mathbb{N}}$ is called $\mathcal{I}$-nonthin subsequence of $\{x_n\}_{n\in\mathbb{N}}$.

Proposition 3.2. *If $z \in X$ is an $\mathcal{I}$-limit point (respectively $\mathcal{I}$-cluster point) of a sequence $\{x_n\}_{n\in\mathbb{N}} \in X$, then for each $y \in X$ the number $S(z,z,y)$ is an $\mathcal{I}$-limit point (respectively $\mathcal{I}$-cluster point) of the number sequence $\{S(x_n,x_n,y)\}_{n\in\mathbb{N}}$.*

Proof. Let $z \in X$ be an $\mathcal{I}$-limit point of $\{x_n\}_{n\in\mathbb{N}} \in X$. Then there is a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \notin \mathcal{I}$ such that $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, z) = 0$. Then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_{m_k}, x_{m_k}, z) < \frac{\varepsilon}{2}$ for all $k > n_0$. Let $y \in X$. Now, by Lemma 3.1, we have $|S(x_{m_k}, x_{m_k}, y) - S(y,y,z)| \le 2S(x_{m_k}, x_{m_k}, z) < \varepsilon$, $\forall\ k > n_0$. Therefore, $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, y) = S(y,y,z) = S(z,z,y)$. Hence, according to the definition of $\mathcal{I}$-limit point of a real sequence, $S(z,z,y)$ is an $\mathcal{I}$-limit point of the number sequence $\{S(x_n,x_n,y)\}_{n\in\mathbb{N}}$.

Next, let $z \in X$ be an $\mathcal{I}$-cluster point of $\{x_n\}_{n\in\mathbb{N}} \in X$. Then for each $\varepsilon > 0$ we have $\{n \in \mathbb{N} : S(x_n,x_n,z) < \frac{\varepsilon}{2}\} \notin \mathcal{I}$. Let $y \in X$. Now using Lemma 3.1, we get $|S(x_n,x_n,y) - S(y,y,z)| \le 2S(x_n,x_n,z)$. Therefore, $\left\{n \in \mathbb{N} : S(x_n,x_n,z) < \frac{\varepsilon}{2}\right\} \subset \{n \in \mathbb{N} : |S(x_n,x_n,y) - S(y,y,z)| < \varepsilon\}$. Hence $\{n \in \mathbb{N} : |S(x_n,x_n,y) - S(y,y,z)| < \varepsilon\} \notin \mathcal{I}$. Therefore, the number $S(y,y,z) = S(z,z,y)$ is an $\mathcal{I}$-cluster point of the number sequence $\{S(x_n,x_n,y)\}_{n\in\mathbb{N}}$. □

Now we prove the following theorem in S-metric spaces which will be needed to prove some results.

Theorem 3.4. *Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence in a S-metric space (X,S) such that $\mathcal{I}$-$\lim x_n = \xi$. If $\Lambda_x(\mathcal{I})_S$ and $\Gamma_x(\mathcal{I})_S$ are the sets of all $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points of x respectively, then we have $\Lambda_x(\mathcal{I})_S = \Gamma_x(\mathcal{I})_S = \{\xi\}$.*

Proof. If possible, let $\alpha \in \Lambda_x(\mathcal{I})_S$ where $\xi \neq \alpha$. Then there exist two sets $K_1 = \{s_1 < s_2 < \cdots < s_i < \cdots\} \subset \mathbb{N}$ and $K_2 = \{t_1 < t_2 < \cdots < t_j < \cdots\} \subset \mathbb{N}$ such that $K_1 \notin \mathcal{I}$ and $\lim_{i\to\infty} S(x_{s_i}, x_{s_i}, \xi) = 0$, $K_2 \notin \mathcal{I}$ and $\lim_{j\to\infty} S(x_{t_j}, x_{t_j}, \alpha) = 0$. Let $\varepsilon > 0$ be given. Then, there exists $j_0 \in \mathbb{N}$ such that $S(x_{t_j}, x_{t_j}, \alpha) < \varepsilon$ for all $j > j_0$. Therefore, the set $A = \{t_j \in K_2 : S(x_{t_j}, x_{t_j}, \alpha) \ge \varepsilon\} \subset \{t_1, t_2, \ldots, t_{j_0}\}$. Since $\mathcal{I}$ is an admissible ideal, $A \in \mathcal{I}$. Choose $B = \{t_j \in K_2 : S(x_{t_j}, x_{t_j}, \alpha) < \varepsilon\}$. Clearly, $B \notin \mathcal{I}$. For, if $B \in \mathcal{I}$ then $K_2 = A \cup B \in \mathcal{I}$ which is a contradiction. Now since $\mathcal{I}$-$\lim x_n = \xi$, we have $M = \{n \in \mathbb{N} : S(x_n,x_n,\xi) \ge \varepsilon\} \in \mathcal{I}$. Consequently, $M^c = \{n \in \mathbb{N} : S(x_n,x_n,\xi) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. Since $\xi \neq \alpha$, we have $B \cap M^c = \phi$. So $B \subset M$. Since $M \in \mathcal{I}$ therefore $B \in \mathcal{I}$. But this contradicts the fact $B \notin \mathcal{I}$. Therefore $\Lambda_x(\mathcal{I})_S = \{\xi\}$.

Next, we assume that $\eta \in \Gamma_x(\mathcal{I})_S$ where $\xi \neq \eta$. Let $\varepsilon > 0$ be given. Then $E_1 = \{n \in \mathbb{N} : S(x_n,x_n,\xi) < \frac{\varepsilon}{4}\} \notin \mathcal{I}$ and $E_2 = \{n \in \mathbb{N} : S(x_n,x_n,\eta) < \frac{\varepsilon}{2}\} \notin \mathcal{I}$. Since $\xi \neq \eta$, we have $E_1 \cap E_2 = \phi$. If not, let $m \in E_1 \cap E_2$. Then $S(\xi,\xi,\eta) \le S(\xi,\xi,x_m)+S(\xi,\xi,x_m)+S(\eta,\eta,x_m) = 2S(x_m,x_m,\xi)+S(x_m,x_m,\eta) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. Since $\varepsilon > 0$ be arbitrary therefore $S(\xi,\xi,\eta) = 0$. This gives $\xi = \eta$. But it is a contradiction. So we have $E_2 \subset E_1^c$. Since $\mathcal{I}$-$\lim x_n = \xi$, the set $E_1^c = \left\{n \in \mathbb{N} : S(x_n,x_n,\xi) \ge \frac{\varepsilon}{4}\right\} \in \mathcal{I}$. Hence $E_2 \in \mathcal{I}$, which contradicts the fact that $E_2 \notin \mathcal{I}$. Therefore, $\Gamma_x(\mathcal{I})_S = \{\xi\}$. This completes the proof of the theorem. □

Lemma 3.3. *If $\alpha,\beta \in X$ are $\mathcal{I}$-limit points (respectively $\mathcal{I}$-cluster points) of an $\mathcal{I}$-localized sequence $\{x_n\}_{n\in\mathbb{N}}$ then $S(\alpha,\alpha,x) = S(\beta,\beta,x)$ for each $x \in \operatorname{loc}_{\mathcal{I}}(x_n)$.*

Proof. Let $x \in \operatorname{loc}_{\mathcal{I}}(x_n)$ and $y = \{y_n\} = \{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$. Let α,β be any two $\mathcal{I}$-limit points (respectively $\mathcal{I}$-cluster points) of $\{x_n\}_{n\in\mathbb{N}}$. Then by Proposition 3.2, $S(\alpha,\alpha,x)$, $S(\beta,\beta,x)$ are the $\mathcal{I}$-limit points (respectively $\mathcal{I}$-cluster points) of the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ i.e., $S(\alpha,\alpha,x)$, $S(\beta,\beta,x) \in \Lambda_y(\mathcal{I})$ (respectively $\Gamma_y(\mathcal{I})$). Since $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-localized sequence and $x \in \operatorname{loc}_{\mathcal{I}}(x_n)$, the number sequence $\{S(x_n,x_n,x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent. Let $y_n \xrightarrow{\mathcal{I}} \xi$. Then by Theorem 3.4, $\Lambda_y(\mathcal{I}) = \Gamma_y(\mathcal{I}) = \{\xi\}$. Therefore, $S(\alpha,\alpha,x) = S(\beta,\beta,x)$ for each $x \in \operatorname{loc}_{\mathcal{I}}(x_n)$. This completes the proof. □

Lemma 3.4. $\operatorname{loc}_{\mathcal{I}}(x_n)$ *does not contain more than one $\mathcal{I}$-limit point (respectively $\mathcal{I}$-cluster point) of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.*

Proof. If possible, let $z_1,z_2 \in \operatorname{loc}_{\mathcal{I}}(x_n)$ be two distinct $\mathcal{I}$-limit points (respectively $\mathcal{I}$-cluster points) of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Then, by Lemma 3.3, we have $S(z_1,z_1,z_1) = S(z_2,z_2,z_1)$. But $S(z_1,z_1,z_1) = 0$. Consequently, $S(z_2,z_2,z_1) = 0$. This gives $z_1 = z_2$ which leads to a contradiction. This proves the lemma. □

Remark 3.2. We know from Theorem 3.4 that if $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x then $\mathcal{I}$-limit point is unique. But converse result holds if the $\mathcal{I}$-limit point belongs to $\mathcal{I}$-locator of $\{x_n\}_{n\in\mathbb{N}}$ which is shown in the following proposition.

Proposition 3.3. *If the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an $\mathcal{I}$-limit point $y \in \operatorname{loc}_{\mathcal{I}}(x_n)$, then $\mathcal{I}$-$\lim_{n\to\infty} x_n = y$.*

Proof. Since $y \in \operatorname{loc}_{\mathcal{I}}(x_n)$ is an $\mathcal{I}$-limit point of $\{x_n\}_{n\in\mathbb{N}}$, then by Proposition 3.2, $S(y,y,y)$ is an $\mathcal{I}$-limit point of the number sequence $\{S(x_n,x_n,y)\}_{n\in\mathbb{N}}$. By the condition $y \in \operatorname{loc}_{\mathcal{I}}(x_n)$, so the number sequence $t = \{t_n\}_{n\in\mathbb{N}} = \{S(x_n,x_n,y)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent. Let $\mathcal{I}$-$\lim_{n\to\infty} S(x_n,x_n,y) = \xi$. Now since $S(y,y,y) \in \Lambda_t(\mathcal{I})$ and, by Theorem 3.4, we have $\Lambda_t(\mathcal{I}) = \{\xi\}$, therefore $S(y,y,y) = \xi$. So $\mathcal{I}$-$\lim_{n\to\infty} S(x_n,x_n,y) = \xi = S(y,y,y) = 0$ i.e., $\mathcal{I}$-$\lim_{n\to\infty} S(x_n,x_n,y) = 0$. So for each $\varepsilon > 0$ the set $\{n \in \mathbb{N} : S(x_n,x_n,y) \ge \varepsilon\} \in \mathcal{I}$ which gives $\mathcal{I}$-$\lim_{n\to\infty} x_n = y$. This completes the proof. □

Definition 3.7 (cf. [11])**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be an $\mathcal{I}$-localized sequence with the $\mathcal{I}$-locator $L = \operatorname{loc}_{\mathcal{I}}(x_n)$. Then the number $\sigma = \inf_{x\in L}\left(\mathcal{I}\text{-}\lim_{n\to\infty} S(x_n,x_n,x)\right)$ is called the $\mathcal{I}$-barrier of $\{x_n\}_{n\in\mathbb{N}}$.

Theorem 3.5. *Let $\mathcal{I}$ satisfies the condition (AP). Then, an $\mathcal{I}$-localized sequence is an $\mathcal{I}$-Cauchy sequence if and only if $\sigma = 0$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an $\mathcal{I}$-Cauchy sequence in X. So it is $\mathcal{I}^*$-Cauchy sequence, since $\mathcal{I}$ satisfies the condition (AP). Therefore, there exists a set $K = (k_n)$ such that $K \in \mathcal{F}(\mathcal{I})$ and $\lim_{n,m\to\infty} S(x_{k_n}, x_{k_n}, x_{k_m}) = 0$. So for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_{k_n}, x_{k_n}, x_{k_{n_0}}) < \varepsilon$ for all $n \geq n_0$. Since $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-localized sequence, $\mathcal{I}\text{-}\lim_{n\to\infty} S(x_n, x_n, x_{k_{n_0}})$ exists. Therefore, we have $\mathcal{I}\text{-}\lim_{n\to\infty} S(x_{k_n}, x_{k_n}, x_{k_{n_0}}) \leq \varepsilon$. Hence $\sigma \leq \varepsilon$. As, $\varepsilon > 0$, $\sigma = 0$.

Conversely assume that $\sigma = 0$. Then by definition of σ, for each $\varepsilon > 0$ there is an $x \in \mathrm{loc}_{\mathcal{I}}(x_n)$ such that $\beta(x) = \mathcal{I}\text{-}\lim_{n\to\infty} S(x_n, x_n, x) < \varepsilon$. So $\{n \in \mathbb{N} : |S(x_n, x_n, x) - \beta(x)| \geq \varepsilon - \beta(x)\} \in \mathcal{I}$, as $\varepsilon - \beta(x) > 0$. Now, since $S(x_n, x_n, x) = |S(x_n, x_n, x) - \beta(x) + \beta(x)| \leq |S(x_n, x_n, x) - \beta(x)| + \beta(x)$, therefore $\{n \in \mathbb{N} : S(x_n, x_n, x) \geq \varepsilon\} \in \mathcal{I}$ i.e. the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-Cauchy sequence. This proves the theorem. □

Remark 3.3. From the proof of the above theorem we can conclude that converse part holds without the condition (AP).

Theorem 3.6. *If the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-localized in itself and $\{x_n\}_{n\in\mathbb{N}}$ contains an $\mathcal{I}$-nonthin Cauchy subsequence, then $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-Cauchy sequence.*

Proof. Let $\{y_n\}_{n\in\mathbb{N}}$ be an $\mathcal{I}$-nonthin Cauchy subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Without any loss of generality we suppose that all the members of $\{y_n\}_{n\in\mathbb{N}}$ are in $\mathrm{loc}_{\mathcal{I}}(x_n)$. Since $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then, by Theorem 3.5, we have $\inf_{y_n \in \mathrm{loc}_{\mathcal{I}}(x_n)} \mathcal{I}\text{-}\lim_{m\to\infty} S(y_m, y_m, y_n) = 0$. Now since $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-localized in itself, then the number sequence $\{S(x_m, x_m, y_n)\}_{m\in\mathbb{N}}$, $y_n \in \mathrm{loc}_{\mathcal{I}}(x_n)$, is $\mathcal{I}$-convergent. Therefore, we have $\mathcal{I}\text{-}\lim_{m\to\infty} S(x_m, x_m, y_n) = \mathcal{I}\text{-}\lim_{m\to\infty} S(y_m, y_m, y_n) = 0$. This shows that $\sigma = 0$. Therefore, by Theorem 3.5, we have $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-Cauchy sequence. This completes the proof. □

Acknowledgement

We express great sense of gratitude and deep respect to the referees for their valuable comments which improved the quality of the paper. Also, the second author is grateful to The Council of Scientific and Industrial Research (CSIR), HRDG, India, for the grant of Junior Research Fellowship during the preparation of this paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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