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Shift Invariant Spaces and Shift Generated Dual Frames for Local Fields

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Abstract. Let G be a locally compact Abelian group with a compact open subgroup H and X be a shift invariant subspace of $L^2(G)$ which forms a frame for a closed subspace of $L^2(G)$, then the dual frame of X which is a shift invariant space, is called shift generated dual frame.

In the present paper, we first define shift generated dual frame of type I and type II for a locally compact Abelian group with a compact open subgroup. Next, we present a characterization of shift generated dual frame in terms of fibers $H^{\perp}$.

1. Introduction

In this paper we define shift generated dual frame for shift invariant spaces which form a frame for $L^2(G)$, where G is a locally compact Abelian group with a compact open subgroup.

The theory of shift invariant spaces play an important role in image processing, signal processing and sampling theory. The shift invariant subspaces of $L^2(\mathbb{R})$ introduced by Helson in [11]. Bownik in [6] presented a characterization of shift invariant subspaces of $L^2(\mathbb{R}^n)$. Kamyabi Gol and Raisi Tousi defined shift invariant subspaces of $L^2(G)$ where G is a locally compact Abelian group with a uniform lattice (a discrete and cocompact subgroup of G) [12]. Finally, in [1] the authors introduced shift invariant subspaces for local fields.

Dual frames are useful tools for expansion of a signal into its frame components. Frames introduced by Duffin and Schaefer [8]. Askari Hemmat and Gabardo in [3] presented a characterization of shift generated dual frames for shift invariant spaces of $L^2(\mathbb{R}^n)$ which form a frame for $L^2(\mathbb{R}^n)$.

In this paper we characterize shift generated dual frames for $L^2(G)$ where G is a locally compact Abelian group with a compact open subgroup.

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The present paper has been organized as follows. In section 2, we recall some definition and theorems which we need them. In section 3, we recall the definition of shift invariant spaces for local fields. Finally, in section 4, we define shift generated dual frames of type I and type II and present our main result theorem.

2. Preliminaries

In this section we fix the terminology which we require from frame theory, locally compact Abelian groups.

For definitions and theorems of locally compact Abelian groups we refer to [9].

Throughout this paper we assume that G be a *locally compact Abelian* (LCA) group with a compact open subgroup H with Haar measure μ which is normalized on H, i.e., $\mu(H) = 1$. We denote the dual group G by $\widehat{G}$ and annihilator of H by $H^{\perp}$, that is $H^{\perp} = \{\gamma \in \widehat{G} : (x,\gamma) = 1 \text{ for all } x \in H\}$, where (x,γ) denotes the action of the duality between G and $\widehat{G}$, with Haar measure ν such that $\nu(H^{\perp}) = 1$. Since H is a compact open subgroup of G then $H^{\perp}$ is a compact open subgroup $\widehat{G}$, also G/H and $\widehat{G}/H^{\perp}$ are discrete.

The ***Fourier transform*** $\wedge : L^1(G) \to \mathscr{C}_0(\widehat{G})$, is defined by

$$\wedge(f)(\gamma) = \widehat{f}(\gamma) = \int_G f(x)\overline{(x,\gamma)}d\mu(x).$$

The ***Weil's Formula*** is the identity

$$\int_G f(x)d\mu(x) = \int_{G/H}\int_H f(xy)d\mu(y)d\mu_{G/H}(xH), \quad \text{for } f \in L^1(G).$$

A ***local field*** is an algebraic field and a topological properties of locally compact non discrete, complete and totally disconnected. Now, we present two examples of local fields:

Example 2.1 ([4]). A good example for local fields is Q_p, where p is a prime number. That is the completion of $\mathbb{Q}$ with respect to a certain natural metric topology. One of the most important application of these groups is in quantum physics ([10]).

Another example for local fields is called cantor dyadic group which is the field $\mathbb{F}_2((t))$ of formal Laurent series consists of all infinite formal sums $c_{n_0}t^{n_0} + c_{n_0+1}t^{n_0+1} + c_{n_0+2}t^{n_0+2} + \ldots$, where $n_0 \in \mathbb{Z}$, each $c_n \in \mathbb{F}_2$ where $\mathbb{F}_2$ is the field of elements $\{0,1\}$.

Here, we recall definition of frame and Riesz family for a Hilbert space $\mathscr{H}$. For more details the authors refer to [7].

A family Ω is called a ***frame*** for a separable Hilbert space $\mathscr{H}$ if there exists constants $0 < A \leq B < \infty$ which satisfies in the following inequality

$$A\|f\|^2 \leq \sum_{\eta\in\Omega} |\langle f,\eta\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathscr{H}. \tag{2.1}$$

The family Ω is called a ***Bessel family*** if the right hand inequality 2.1 holds.

A family $\mathscr{X}$ is called a ***Riesz family*** for a Hilbert space $\mathscr{H}$ if there exist two constants $0 < A \leq B < \infty$ such that the following inequality holds.

$$A\sum_{\eta\in\mathscr{X}}|h_\eta|^2 \leq \sum_{\eta\in\mathscr{X}}\|h_\eta\eta\|^2 \leq B\sum_{\eta\in\mathscr{X}}|h_\eta|^2,$$

for all finitely supported $\{h_\eta\}_{\eta\in\mathscr{X}} \subseteq \mathbb{C}$.

For Bessel collection $F = \{f_n\}_{n\in\mathbb{N}}$, we recall the definitions of ***analysis operator*** $T_F : \mathscr{H} \to l^2(\mathbb{N})$ by $T_F(f) = \{\langle f, f_n\rangle\}_{n\in\mathbb{N}}$, $f \in \mathscr{H}$. The ***synthesis operator*** $T_F^* : l^2(\mathbb{N}) \to \mathscr{H}$ is defined

$$T_X^*(\{c_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} c_n f_n, \qquad \{c_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N}).$$

The operator $S = T_F^*T_F : \mathscr{H} \to \mathscr{H}$ is called ***frame operator***. If F is a frame then S is a bounded and invertible operator from $\mathscr{H}$ onto $\mathscr{H}$. The collection $\{S^{-1}f_n\}_{n\in\mathbb{N}}$ is called the ***standard dual*** frame of the frame F.

3. **Review on shift invariant spaces**

The authors in [1] defined shift invariant spaces for an LCA group G with a compact open subgroup.

For the definition of shift invariant spaces we need to recall the definitions of the translation operator on $L^2(G)$. It has been defined in [4] and [5]. We recall it in the following

Definition 3.1. The maps $\theta = \theta_D : \widehat{G} \to \mathscr{D}$, $\eta = \eta_D : \widehat{G} \to H^\perp$ are defined by $\theta(\gamma) =$ the unique $\sigma_\gamma \in \mathscr{D}$ and $\eta_{\mathscr{D}}(\gamma) = \gamma - \theta(\gamma)$, where $\mathscr{D} \subseteq \widehat{G}$ is the set of coset representatives in $\widehat{G}$ for the quotient $\widehat{G}/H^\perp$. Also for any fixed $[s] \in G/H$ and for any $f \in L^2(G)$, the translation operator $\tau_{[s]}f$ is defined by $\tau_{[s]}f = f * \check{\omega}_{[s]}$, where $\omega_{[s]}(\gamma) = \overline{(s, \eta_{\mathscr{D}}(\gamma))}$ and $\check{\omega}_{[s]}$ is the inverse Fourier transform of $\omega_{[s]}$.

Now we are ready to recall the definition of shift invariant spaces.

Definition 3.2. A closed subspace $V \subseteq L^2(G)$ is called ***shift invariant space*** with respect to G/H, if $f \in V$ implies $\tau_{[s]}f \in V$ for all $[s] \in G/H$, where $\tau_{[s]}$ is the translation operator defined in Definition 3.1. For countable subset $\Phi \subset L^2(G)$ we define ***principal shift invariant space*** by $V_\Phi = \overline{span}\{\tau_{[s]}\phi : [s] \in G/H, \phi \in \Phi\}$.

We consider $l_\perp^2 := l^2(\widehat{G}/H^\perp)$ and define the Hilbert space

$$L_*^2 := L^2(H^\perp, l_\perp^2) = \left\{\Phi : H^\perp \to l_\perp^2; \int_{H^\perp} \|\Phi(\gamma)\|_{l_\perp^2}^2\, d\nu(\gamma) < \infty\right\}.$$

With inner product $\langle f, g\rangle = \int_{H^\perp}\langle f(\xi), g(\xi)\rangle_{l_\perp^2}\, d\nu(\xi)$, and norm $\|f\|_{L_*^2} = \int_{H^\perp}\|f(\xi)\|_{l_\perp^2}^2\, d\nu(\xi)$. Also, the mapping $\digamma : L^2(G) \to L_*^2$ defined by $(\digamma g)(\gamma) = \{\widehat{g}(\gamma + \eta)\}_{[\eta]\in\widehat{G}/H^\perp}$ is an isometric isomorphism between $L^2(G)$ and L_*^2 (for more details see [2, 13]).

Definition 3.3. A ***range function*** is a mapping

$$J : H^{\perp} \to \{\text{closed subspaces of } l^2_{\perp}\}$$

J is called measurable if the orthogonal projections $P(\xi) : l^2_{\perp} \to J(\xi)$ for a.e. $\xi \in H^{\perp}$ are measurable i.e. $\xi \to \langle P(\xi)a, b\rangle$ is measurable for all $a, b \in l^2_{\perp}$.

In [2] we used the ideas proof of Theorems 3.1 of [13] and 2.3 of [6] to prove the following theorem.

Theorem 3.4. *Assume that G is an LCA group with compact open subgroup H and J is a measurable range function. A closed subspace $V \subseteq L^2(G)$ is shift invariant, with respect to the lattice induced by G/H, if and only if*

$$V = \{f \in L^2(G) \;\; \mathcal{F} f(\xi) \in J(\xi) \text{ for a.e. } \xi \in H^{\perp}\}, \tag{3.1}$$

Also, if V_Φ is a shift invariant subspace of $L^2(G)$ generated by countable set $\Phi \subset L^2(G)$, then $J(\xi) = \overline{span}\{\mathcal{F}\phi(\xi); \phi \in \Phi\}$.

4. Shift generated dual frames

Our main goal in this section is to generalize a characterization of shift generated dual frame for shift invariant subspaces of $L^2(G)$ in terms of fibers of $H^{\perp}$. For this, we need to define shift generated dual frames of type I and type II. If the shift invariant system X is a frame for closed subspace $\mathcal{M}$ of $L^2(G)$ but is not a Riesz family, then there exists a dual frame except standard dual frame for X, which is shift invariant. We recall definition of three types of dual frame for a Hilbert space $\mathcal{H}$.

Definition 4.1. If $F = \{f_n\}_{n\in\mathbb{N}}$ is a frame for the closed subspace $\mathcal{M}$ of the Hilbert space $\mathcal{H}$ with inner product denoted by $\langle\cdot,\cdot\rangle$.

- (i) A dual frame for the frame F is a Bessel collection $K = \{k_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ satisfying $\sum_{n\in\mathbb{N}} \langle f, k_n\rangle f_n = f$, $f \in \mathcal{M}$.
- (ii) A dual frame of type I for frame F is a frame $\{k_n\}_{n\in\mathbb{N}}$ such that $k_n \in \mathcal{M}$, for each $n \in \mathbb{N}$.
- (iii) A dual frame of type II for the frame F is a frame $K = \{k_n\}_{n\in\mathbb{N}}$ with the property $Range(T_K) \subset Range(T_F)$, where T_F and T_K denotes the analysis operators associated with F and K, respectively.

Now, we are ready to define three types of shift generated dual frames, for shift invariant subspace X of $L^2(G)$ which form frame for $\overline{span}X$.

Definition 4.2. Let $X = \{\tau_{[s]}\phi;\; [s] \in G/H,\; \phi \in \Phi\}$ be a frame for closed subspace $V_\Phi \subset L^2(G)$. Let $R : \Phi \to L^2(G)$ be a mapping and $Y = \{\tau_{[s]}R\phi; [s] \in G/H, \phi \in \Phi\}$.

- (i) We say that Y is ***shift generated (SG)-dual frame*** for X if it is a dual frame for X as in Definition 4.1(i).

(ii) We say that Y is ***shift generated (SG)-dual frame*** of type I (resp. type II) for X, if it is a dual frame for X as in Definition 4.1(ii)(resp. (iii)).

To prove Theorem 4.5 we must to establish two following lemmas. The function $h \in L^2(G)$ is called $\mathscr{D}$-periodic function whenever $f(\xi+\eta) = f(\xi)$, $\eta \in \mathscr{D}$ and for a.e. $\xi \in H^\perp$.

Lemma 4.3. *If m_ϕ is a measurable $\mathscr{D}$-periodic function for $\phi \in \Phi$, such that $\int_{H^\perp} \sum_{\phi\in\Phi} |m_\phi(\xi)|^2 d\nu(\xi) < \infty$. Then for $X = \{\tau_{[s]}\phi; \phi \in \Phi, [s] \in G/H\}$ with the Bessel property, the following are equivalent*

(a) $\sum_{\phi\in\Phi} \int_{H^\perp} \langle \digamma f(\xi), \digamma\phi(\xi)\rangle \overline{m_\phi(\xi)} d\nu(\xi) = 0$*, for all* $f \in V_\Phi$.

(b) *For a.e.* $\xi \in H^\perp$*, we have* $\sum_{\phi\in\Phi} \langle \digamma f(\xi), \digamma\phi(\xi)\rangle \overline{m_\phi(\xi)} d\nu(\xi) = 0$*, for all* $f \in V_\Phi$.

Proof. Let (a) holds. If $\{e_{[\eta]}\}_{[\eta]\in\widehat{G}/H^\perp}$ is the standard orthonormal basis for $l^2_\perp$ then $\digamma f(\xi) = \sum_{[\eta]\in\widehat{G}/H^\perp} P(\xi)e_{[\eta]}$, where $P(\xi) : l^2_\perp \to J(\xi)$ is a orthogonal projection projection and $J(\xi)$ is range function. Thus

$$\sum_{\phi\in\Phi} \left\langle \sum_{[\eta]\in\widehat{G}/H^\perp} P(\xi)e_{[\eta]}, \digamma\phi(\xi) \right\rangle = \sum_{\phi\in\Phi} \langle \digamma f(\xi), \digamma\phi(\xi)\rangle = 0.$$

Now we assume that (b) fails. Then there exists a measurable subset E of $H^\perp$ such that $\nu(E) > 0$ and $h(\xi) = \sum_{[\eta]\in\widehat{G}/H^\perp} \langle P(\xi)e_{[\eta]}, \digamma\phi(\xi)\rangle \overline{m_\phi(\xi)} \neq 0$. Therefore we have subsets

$$E_1 = \{\xi \in E; Re\, h(\xi) > 0\}, \qquad E_2 = \{\xi \in E; Im\, h(\xi) > 0\},$$

$$E_3 = \{\xi \in E; Re\, h(\xi) < 0\}, \qquad E_4 = \{\xi \in E; Im\, h(\xi) < 0\}.$$

Set $\digamma f_1 = \chi_{E_1} e_{[\eta]}$ then $f_1 \in \mathscr{M}$ and $\sum_{\phi\in\Phi} \int_{H^\perp} \langle \digamma f_1(\xi), \digamma\phi(\xi)\rangle \overline{m_{\phi(\xi)}} d\nu(\xi) \neq 0$. That is a contradiction and proof is completed. It is clear that (b) implies that (a). □

Lemma 4.4. *Let $X = \{\tau_{[s]}\phi; \phi \in \Phi, [s] \in G/H\}$ be a frame for $\overline{span} X$ and let $R : \Phi \to L^2(G)$ be a mapping with property that $Y = \{\tau_{[s]}R\phi; \phi \in \Phi, [s] \in G/H\}$ is Bessel. Then, the following are equivalent*

(a) *Range* $T_Y \subset$ *Range* T_X

(b) *The range of the analysis operator associated with collection* $\{\digamma R\phi(\xi)\}_{\phi\in\Phi}$ *in* $l^2_\perp$ *is contained in the range of analysis operator associated with collection* $\{\digamma\phi(\xi)\}_{\phi\in\Phi}$.

Proof. Define $\widetilde{X} = \{\digamma\phi(\xi)\}_{\phi\in\Phi}$ and $\widetilde{Y} = \{\digamma R\phi(\xi)\}_{\phi\in\Phi}$. Set $\mathscr{I} = \Phi \times G/H$. We must show that *range*$(T_X)^\perp \subset$ *range*$(T_Y)^\perp$ (note that *range* T_X and *range* T_Y are closed subspace of $l^2(\mathscr{I})$.) Let $\{c_{\phi,[s]}\} \in l^2(\mathscr{I})$ satisfies $\sum_{(\phi,[s])\in\mathscr{I}} \langle f, \tau_{[s]}\phi\rangle \overline{c_{\phi,[s]}} = 0$.

We claim that $\sum_{(\phi,[s])\in\mathscr{I}} \langle f, \tau_{[s]}R\phi\rangle \overline{c_{\phi,[s]}} = 0$.

Let (b) holds, then $(range\, T_{\widetilde{X}})^{\perp} \subset (range\, T_{\widetilde{Y}})^{\perp}$. Thus for $\{d_\phi\} \in l^2(\Phi)$ and $b \in J(\xi)$, if

$$\sum_{\phi\in\Phi} \langle b, \mathcal{F}\phi(\xi)\rangle \overline{d_\phi} = 0$$

then,

$$\sum_{\phi\in\Phi} \langle b, \mathcal{F}(R\phi)(\xi)\rangle \overline{d_\phi} = 0,$$

for a.e. $\xi \in H^{\perp}$.

By Plancherel Theorem and Weil's Formula we have

$$\begin{aligned}
&\sum_{(\phi,[s])\in\mathcal{I}} \langle f, \tau_{[s]}\phi\rangle \\
&= \sum_{(\phi,[s])\in\mathcal{I}} \langle \widehat{f}, \omega_{[s]}\widehat{\phi}\rangle \\
&= \sum_{(\phi,[s])\in\mathcal{I}} \int_{\widehat{G}} \widehat{f}(\gamma)\widehat{\phi}(\gamma)\overline{\omega_{[s]}(\gamma)}d\nu(\gamma) \\
&= \sum_{(\phi,[s])\in\mathcal{I}} \int_{H^{\perp}} \left(\sum_{[\eta]\in\widehat{G}/H^{\perp}} \widehat{f}(\gamma+\eta)\overline{\widehat{\phi(\gamma+\eta)}} \right) \overline{\omega_{[s]}(\gamma)}d\nu(\gamma)\overline{c_{\phi,[s]}}. \qquad (4.1)
\end{aligned}$$

Set $m_\phi = \sum_{[s]\in G/H} c_{\phi,[s]}\overline{\omega_{[s]}}$. The function $\omega_{[s]}$ is $\mathcal{D}$-periodic so is m_ϕ, and since $\omega_{[s]}$ is a unimodular function then

$$\int_{H^{\perp}} \sum_{\phi\in\Phi} |m_\phi(\xi)|^2 d\nu(\xi) = \int_{H^{\perp}} \sum_{(\phi,[s])\in\mathcal{I}} |c_{\phi,[s]}|^2 d\nu(\xi) < \infty.$$

Thus $\{m_\phi(\xi)\}_{\phi\in\Phi} \in l^2(\Phi)$ for a.e. $\xi \in H^{\perp}$.

For $\{c_{([s],\phi)}\}_{([s],\phi)\in\mathcal{I}}$ and for all $f \in V_\Phi$ we have

$$\sum_{(\phi,[s])\in\mathcal{I}} \langle f, \tau_{[s]}\phi\rangle \overline{c_{([s],\phi)}} = \int_{H^{\perp}} \sum_{\phi\in\Phi} \langle \mathcal{F}f(\xi), \mathcal{F}\phi(\xi)\rangle \overline{m_\phi(\xi)} d\nu(\xi). \qquad (4.2)$$

$$\sum_{(\phi,[s])\in\mathcal{I}} \langle f, \tau_{[s]}R\phi\rangle \overline{c_{([s],\phi)}} = \int_{H^{\perp}} \sum_{\phi\in\Phi} \langle \mathcal{F}f(\xi), \mathcal{F}R\phi(\xi)\rangle \overline{m_\phi(\xi)} d\nu(\xi). \qquad (4.3)$$

By Lemma 4.3 and (4.2) we have $\sum_{\phi\in\Phi} \langle \mathcal{F}f(\xi), \mathcal{F}\phi(\xi)\rangle \overline{m_\phi}(\xi) = 0$, that is $\{m_\phi(\xi)\}_{\phi\in\Phi} \in (Range\, T_{\widetilde{X}})^{\perp}$ and since (b) holds, then

$$\sum_{\phi\in\Phi} \langle \mathcal{F}f(\xi), \mathcal{F}R\phi(\xi)\rangle \overline{m_\phi(\xi)} = 0, \quad \text{for all } f \in V_\Phi.$$

Thus the left-hand equality (4.3) is zero, which shows that (a) is true.

Conversely, let (a) holds. If $\sum_{\phi\in\Phi}\langle \digamma f(\xi), \digamma\phi(\xi)\rangle\overline{a_\phi} = 0$, for $\{a_\phi\}_{\phi\in\Phi} \in (range\, T_{\widetilde{X}})^\perp$. Set $m_\phi = a_\phi$, then by (4.2), (4.3) and Lemma 4.3, we have $\sum_{\phi\in\Phi}\langle \digamma f(\xi), \digamma R\phi(\xi)\rangle\overline{a_\phi(\xi)} = 0$, for a.e. $\xi \in H^\perp$. Therefore (b) is true. □

The following theorem and corollaries present a characterization of SG-dual frames, SG-dual of type I and SG-dual of type II for frame $X = \{\tau_{[s]}\phi; [s] \in G/H, \phi \in \Phi\}$.

Theorem 4.5. *We assume that $X = \{\tau_{[s]}\phi; [s] \in G/H, \phi \in \Phi\}$ is a frame for closed subspace $M \subseteq L^2(G)$ and $R : \Phi \to L^2(G)$ is a mapping that the collection $Y = \{\tau_{[s]}R\phi; [s] \in G/H, \phi \in \Phi\}$ is Bessel, then Y is a SG-dual for X if and only if the collection $\widetilde{Y} = \{\digamma R\phi(\xi)\}_{\phi\in\Phi}$ is a SG-dual frame for $\widetilde{X} = \{\digamma\phi(\xi)\}_{\phi\in\Phi}$.*

Proof. By Plancherel Theorem and Weil's Formula we have

$$\|f\|^2 = \|\widehat{f}\|^2 = \int_{\widehat{G}} |\widehat{f}(\xi)|^2 d\nu(\xi) = \int_{H^\perp} \sum_{[\eta]\in\widehat{G}/H^\perp} |\widehat{f}(\xi+\eta)|^2 d\nu(\xi).$$

Let $\widetilde{Y}$ be a SG-dual frame for $\widetilde{X}$ thus for a.e. $\xi \in H^\perp$,

$$\sum_{\phi\in\Phi}\langle \digamma f(\xi), \digamma\phi(\xi)\rangle\overline{\langle \digamma g(\xi), \digamma R\phi(\xi)\rangle} = \langle \digamma f(\xi), \digamma g(\xi)\rangle.$$

We must show that

$$\sum_{(\phi,[s])\in\mathscr{I}} \langle f, \tau_{[s]}\phi\rangle\overline{\langle g, \tau_{[s]}R\phi\rangle} = \langle f, g\rangle.$$

For $f \in V_\Phi$,

$$\begin{aligned}
&\sum_{(\phi,[s])\in\mathscr{I}} \langle f, \tau_{[s]}\phi\rangle\overline{\langle g, \tau_{[s]}R\phi\rangle} \\
&= \sum_{(\phi,[s])\in\mathscr{I}} \left(\int_{\widehat{G}} \widehat{f}(\xi)\omega_{[s]}(\xi)\overline{\widehat{\phi(\xi)}}d\nu(\xi)\int_{\widehat{G}} \overline{\widehat{g}(\xi)}\overline{\omega_{[s]}}(\xi)\widehat{R\phi(\xi)}d\nu(\xi)\right) \\
&= \sum_{(\phi,[s])\in\mathscr{I}} \left(\int_{H^\perp}\sum_{[\eta]\in\widehat{G}/H^\perp} \widehat{f}(\xi+\eta)\omega_{[s]}(\xi)\overline{\widehat{\phi(\xi+\eta)}}d\nu(\xi)\right) \\
&\qquad \times \int_{H^\perp}\sum_{[\eta]\in\widehat{G}/H^\perp} \overline{\widehat{g}(\xi+\eta)}\overline{\omega_{[s]}}(\xi)\widehat{R\phi(\xi+\eta)}d\nu(\xi)\Bigg) \\
&= \sum_{\phi\in\Phi}\left(\int_{H^\perp} \langle \digamma f(\xi), \digamma\phi(\xi)\rangle\omega_{[s]}(\xi)d\nu(\xi)\right. \\
&\qquad \times \left.\overline{\int_{H^\perp} \langle \digamma g(\xi), \digamma R\phi(\xi)\rangle\overline{\omega_{[s]}(\xi)}d\nu(\xi)}\right)
\end{aligned}$$

$$= \sum_{\phi \in \Phi} \langle \widehat{\mathcal{F}f(\xi), \mathcal{F}\phi}(\xi) \rangle \overline{\langle \widehat{\mathcal{F}g(\xi), \mathcal{F}R\phi(\xi)} \rangle},$$

the Parseval's identity [14] implies that

$$\sum_{\phi \in \Phi} \langle \widehat{\mathcal{F}f(\xi), \mathcal{F}\phi}(\xi) \rangle \overline{\langle \widehat{\mathcal{F}g(\xi), \mathcal{F}R\phi(\xi)} \rangle} = \langle \mathcal{F}f(\xi), \mathcal{F}g(\xi) \rangle_{L^2_*}$$

$$= \int_{H^\perp} \langle \mathcal{F}f(\xi), \mathcal{F}g(\xi) \rangle_{l^2_\perp} d\nu(\xi).$$

By Weil's Formula and Plancherel Theorem we have

$$\int_{H^\perp} \langle \mathcal{F}f(\xi), \mathcal{F}g(\xi) \rangle d\nu(\xi) = \int_{H^\perp} \sum_{[\eta] \in \widehat{G}/H^\perp} \widehat{f(\xi + \eta)} \overline{\widehat{g(\xi + \eta)}} d\nu(\xi)$$

$$= \int_{\widehat{G}} \widehat{f(\xi)} \overline{\widehat{g(\xi)}} d\nu(\xi)$$

$$= \langle \widehat{f}, \widehat{g} \rangle_{L^2(\widehat{G})}$$

$$= \langle f, g \rangle_{L^2(G)}.$$

Therefore, Y is a SG-dual frame for X.
For the converse, let Y be SG-dual frame for X, then

$$\sum_{(\phi, [s]) \in \mathcal{I}} \langle f, \tau_{[s]}\phi \rangle \overline{\langle g, \tau_{[s]}R\phi \rangle} = \langle f, g \rangle.$$

We must show that for all $f, g \in V_\Phi$

$$\sum_{\phi \in \Phi} \langle \mathcal{F}f(\xi), \mathcal{F}\phi(\xi) \rangle \overline{\langle \mathcal{F}g(\xi), \mathcal{F}R\phi(\xi) \rangle} = \langle \mathcal{F}f(\xi, \mathcal{F}g(\xi)) \rangle, \text{ for a.e. } \xi \in H^\perp. \quad (4.4)$$

Let 4.4 fails, then there exists a measurable subset E_1 of $H^\perp$ with $\nu(E_1) > 0$ and $[\eta_1], [\eta_2] \in \widehat{G}/H^\perp$, such that for a.e. $\xi \in E_1$,

$$\mathcal{F}(\xi) = \left(\sum_{\phi \in \Phi} \langle P(\xi)e_{[\eta_1]}, \mathcal{F}\phi(\xi) \rangle \overline{\langle P(\xi)e_{[\eta_2]}, \mathcal{F}R\phi(\xi) \rangle} - \langle P(\xi)e_{[\eta_1]}, P(\xi)e_{[\eta_2]} \rangle \right) \neq 0.$$

Thus one of the following inequality holds

$$Re(\mathcal{F}(\xi)) > 0,\ Re(\mathcal{F}(\xi)) < 0,\ Im(\mathcal{F}(\xi)) > 0,\ Im(\mathcal{F}(\xi)) < 0, \quad \text{for } \xi \in E_1.$$

For example, we assume that $Re(\mathcal{F}(\xi)) < 0$. Set

$$P_1(\xi) = \chi_{E_1} P(\xi)e_{[\eta_1]} \text{ and } P_2(\xi) = \chi_{E_1}(\xi)P(\xi)e_{[\eta_2]}.$$

Thus there exist $f, g \in M$ such that $\mathcal{F}f(\xi) = P_1(\xi)$ and $\mathcal{F}g(\xi) = P_2(\xi)$ for a.e. $\xi \in H^\perp$. Since

$$\sum_{\phi \in \Phi} \int_{H^\perp} \langle \widehat{\mathcal{F}f(\xi), \mathcal{F}\phi}(\xi) \rangle \overline{\langle \widehat{\mathcal{F}g(\xi), \mathcal{F}R\phi(\xi)} \rangle} d\nu(\xi) = \int_{H^\perp} \langle \mathcal{F}f(\xi), \mathcal{F}g(\xi) \rangle d\nu(\xi),$$

then

$$0 = Re\left\{ \int_{H^\perp} \left(\sum_{\phi\in\Phi} \langle P(\xi)e_{[\eta_1]}, {}_F\phi(\xi)\rangle \overline{\langle P(\xi)e_{[\eta_2]}, {}_F R\phi(\xi)\rangle} - \langle P(\xi)e_{[\eta_1]}, P(\xi)e_{[\eta_2]}\rangle \right) d\nu(\xi) \right\}$$

$$= \int_{E_1} Re(\mathscr{F}(\xi)) d\nu(\xi),$$

that is a contradiction with $Re(\mathscr{F}(\xi)) < 0$. And the other cases are similar to it. □

By the pervious theorem and Theorem 3.4, the following corollary is true.

Corollary 4.6. *Retain the assumption of Theorem* 4.5 *then Y is a SG-dual frame of type I for X if and only if the collection $\widetilde{Y} = \{{}_F R\phi(\xi)\}_{\phi\in\Phi}$ is a SG-dual frame of type I for $\widetilde{X} = \{{}_F\phi(\xi)\}_{\phi\in\Phi}$.*

To prove the next corollary we use Theorem 4.5 and Lemma 4.4 imply that (3)

Corollary 4.7. *Retain the assumption of Theorem* 4.5 *then Y is a SG-dual frame of type II for X if and only if the collection $\widetilde{Y} = \{{}_F R\phi(\xi)\}_{\phi\in\Phi}$ is a SG-dual frame of type II for $\widetilde{X} = \{{}_F\phi(\xi)\}_{\phi\in\Phi}$.*

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