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Research Article

# Numerical Simulation for a Differential Difference Equation With an Interior Layer

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**Abstract.** This paper addresses the solution of a differential-difference type equation having an interior layer behaviour. An approach is suggested to solve this equation using a numerical integration scheme and linear interpolation. Taylor expansions are utilized to handle the shift arguments. In order to solve the discretized equation, the tridiagonal solver is applied. The approach is analyzed for convergence. Numerical examples are demonstrated to validate the scheme. Maximum errors in the solution, in contrast to the other methods, are organized to explain the approach. The layer profile in the solutions of the examples is illustrated in graphs.

**Keywords.** Differential-difference equation, Numerical integration, Layer behaviour, Convergence

**Mathematics Subject Classification (2020).** 65L10, 65L11, 65L12



## 1. Introduction

Differential difference equations are those in which the growth of a present parameters is inconsistently dependent on a particular history. This means that a physical systems rate of change depends on both its present and past history. The layer behavior differential difference equations have been extensively used in control theory for a number of years. Subsequently, these equations play an important part in predator-prey models [14], thermo-elasticity [2], population dynamics [10], models of the red blood cell system [13], and models of neuronal variability [22].

Bender and Orszag [1], El'sgol'ts and Norkin [5], Mickens [15], Driver [4], Kokotović *et al*. [9], Miller *et al*. [16], O'Malley, Jr. [17] are the authors who have produced books explaining various methods for solving a singularly perturbed differential-difference equations (SPDDEs). In [6, 7], Kadalbajoo and Sharma created an fitted finite difference approach to solve SPDDEs, in which the solution of the problem displays layer profile and arising in a mathematical model of neuronal variability. In [11], Lange and Miura developed an asymptotic analysis for a class of SPDDEs with negative and positive shifts. In [12], the authors concentrate on problems with solutions that display layer behaviour at either one of the boundaries or both of the boundary. The Laplace transforms used to the investigation of the layer equation produce new and interesting findings. Patidar and Sharma [18] designed non-standard fitted finite difference methods based on the methods given in [15] for SPDDEs with negative and positive shifts. Rai and Sharma [19] developed numerical schemes using some modifications in El-Mistikawy-Werle exponential finite difference scheme. Sirisha *et al*. [21] devised a mixed difference scheme to solve the SPDDEs. Salama and Al-Amery [20] constructed a mixed asymptotic solution for SPDDE using the composite expansion method. This work deals with constant shifts, which are not affected by the perturbation arguments. Swamy *et al*. [23] constructed a computational method of order four to solve SPDDE with mixed arguments. In [24], Woldaregay and Duressa developed a numerical approach for SPDDEs is solved using a fitted exponentially spline method.

## 2. Statement of the Problem

Consider a SPDDE with layer behaviour consisting a small delay and advanced terms of the form:

$$\varepsilon z''(\varphi)+\alpha(\varphi)z'(\varphi)+\beta(\varphi)\,z(\varphi-\delta)+R(\varphi)\,z(\varphi)+V(\varphi)\,z(\varphi+\eta)=F(\varphi) \tag{2.1}$$

on $(0,1)$, with the boundary conditions

$$\begin{cases} z(\varphi)=\mu(\varphi) & \text{on } -\delta\le\varphi\le 0,\\ z(\varphi)=\rho(\varphi) & \text{on } 1\le\varphi\le 1+\eta, \end{cases} \tag{2.2}$$

where $0<\varepsilon\ll 1$ is a perturbation argument, $\alpha(\varphi)$, $\beta(\varphi)$, $R(\varphi)$, $V(\varphi)$, $L(\varphi)$, $\mu(\varphi)$ and $\rho(\varphi)$ are smooth functions and $0<\delta=o(\varepsilon)$ is the delay argument and $0<\eta=o(\varepsilon)$ is the positive shift arguments. If $(\alpha(\varphi)-\delta\beta(\varphi)+\eta V(\varphi))>0$, the solution of eq. (2.1) with eq. (2.2) displays the layer appear at the left end of the region and if $(\alpha(\varphi)-\delta\beta(\varphi)+\eta V(\varphi))<0$ then the layer appears at the right-end of the domain. If $A(\varphi)=0$, then the solution displays an oscillatory or two layers depends on the scenario while $B(\varphi)+R(\varphi)+V(\varphi)$ is positive or negative. Due to the differentiability of the solution $z(\varphi)$ in eq. (2.1), we can use Taylor series to expand the terms $z(\varphi-\delta)$ and $z(\varphi-\eta)$, then we get as following

$$\begin{cases} z(\varphi-\delta)\approx z(\varphi)-\delta z'(\varphi)+\frac{\delta^2}{2}z''(\varphi),\\ z(\varphi+\eta)\approx z(\varphi)+\eta z'(\varphi)+\frac{\eta^2}{2}z''(\varphi). \end{cases} \tag{2.3}$$

Using eq. (2.3) in eq. (2.1), we get

$$\varepsilon z''(\varphi)+\mathcal{H}(\varphi)\, z'(\varphi)+Q(\varphi)\, z(\varphi)=F(\varphi), \quad 0\le\varphi\le 1. \tag{2.4}$$

Eq. (2.4) is a convection-diffusion problem. Here

$$\mathcal{H}(\varphi)=\frac{\alpha(\varphi)-\delta\beta(\varphi)+\eta V(\varphi)}{\left(1+\beta\frac{\delta^2}{2\varepsilon}+V\frac{\eta^2}{2\varepsilon}\right)}, \quad Q(\varphi)=\frac{\beta(\varphi)+R(\varphi)+V(\varphi)}{\left(1+\beta\frac{\delta^2}{2\varepsilon}+V\frac{\eta^2}{2\varepsilon}\right)}, \quad F(\varphi)=\frac{L(\varphi)}{\left(1+\beta\frac{\delta^2}{2\varepsilon}+V\frac{\eta^2}{2\varepsilon}\right)}.$$

We solve eq. (2.4), with boundary constraints

$$z(0)=\mu(0) \text{ and } z(1)=\rho(1), \tag{2.5}$$

where the solution of the problem eq. (2.4) with eq. (2.5) is taken as the approximation solution to the problem of eq. (2.1)-eq. (2.2). Let $\mathcal{H}(\varphi)$ vanish at some $l_i\in(-1,1)$. Let $N_i=[l_i-\xi, l_i-\xi]$ be a neighborhood of the critical point $l_i$ such that it does not comprise any more turning point. Additionally, the assumption is also

$$|\mathcal{H}'(\varphi)|\ge\left|\frac{\mathcal{H}'(l_i)}{2}\right|\in N_i.$$

The transformation $\varphi=\xi^{-1}(u-l_i)$ reduces the study of the layer profile of $z(\varphi)$ appear a given turning point $l_i$ to the scenario such that if $\mathcal{H}(\varphi)$ has an unique zero at $\varphi=0$. Thus, we address the solution of eq. (2.4)-eq. (2.5) under the following hypothesis

(i) $\mathcal{H}(\varphi)\in C^2[-1,1]$, $F(\varphi)$ and $Q(\varphi)\in C^1[-1,1]$.

(ii) $Q(\varphi)\ge Q_0>0$ on $[-1,1]$, where $Q_0$ is a positive constant.

(iii) $\mathcal{H}(\varphi)$ has simple zero at $\varphi=0$ and no other zeros in $[-1,1]$.

(iv) $|\mathcal{H}'(\varphi)|\ge\left|\frac{\mathcal{H}'(0)}{2}\right|$ for $-1\le\varphi\le 1$.

(v) $\sigma=\frac{Q(0)}{\mathcal{H}'(0)}$, and $\sigma_l,\sigma_s$ be positive constants such that $\sigma_l\le 1\le\sigma_s$ and $\sigma_l\le|\sigma|\le\sigma_s$.

For a given function $g(\varphi)\in C^k[-1,1]$, let $g_k=\sum_{i=0}^{k}\max_{-1\le u\le 1}|g^{(i)}|$, where $g^{(i)}$ denote $i$ih derivative of $g(\varphi)$.

## 3. Analytical Results

**Lemma 3.1.** *Let $\pi(\varphi)$ be any Piecewise continuous function satisfying $\pi(-1)\ge 0$ and $\pi(1)\ge 0$. Then, $L\pi(\varphi)\ge 0\ \forall\ \varphi\in(-1,1)$ indicates that $\pi(\varphi)\ge 0\ \forall\ \varphi\in[-1,1]$.*

*Proof.* Assume $\varphi^*$ be such that $\pi(\varphi^*)=\min_{-1\le\varphi\le 1}\pi(\varphi)$. Let us assume that $\pi(\varphi^*)\le 0$. Clearly, $\varphi^*\notin(-1,1)$. Since $\varphi^*$ is the point of minimum. Consequently, $\pi'(\varphi^*)=0$ and $\pi''(\varphi^*)\ge 0$. Now

$$\begin{aligned} L\pi\varphi^* &= \varepsilon z''(\varphi^*)+\mathcal{H}(\varphi^*)z'(\varphi^*)+Q(\varphi^*)z(\varphi^*) \\ &< 0 \end{aligned}$$

which is completely inconsistent.

This occurs $\pi(\varphi^*)\ge 0$ and since $\varphi^*$ is chosen arbitrarily therefore $\pi(\varphi)\ge 0$, $\forall\ \varphi\in[-1,1]$. □

**Lemma 3.2.** *If $z(u)$ is the solution of equations* (2.1)-(2.2) *then*

$$\|z\|_0 \le \frac{\|\mathcal{F}\|_0}{Q_0} + \max(|\mu(-1)|, |\rho(1)|).$$

*Proof.* Define

$$\pi^{\pm}(\varphi) = \frac{\|\mathcal{F}\|_0}{Q_0} + \max(|\mu(-1)|, |\rho(1)|) + z(\varphi).$$

Consequently, it gives

$$\begin{aligned}
\pi^{\pm}(-1) &= \frac{\|\mathcal{F}\|_0}{Q_0} + \max(|\mu(-1)|, |\rho(1)|) + z(-1) \\
&= \frac{\|\mathcal{F}\|_0}{b_0} + \max(|\mu(-1)|, |\rho(1)|) \pm \mu(-1) \ge 0, \\
\pi^{\pm}(1) &= \frac{\|\mathcal{F}\|_0}{b_0} + \max(|\mu(-1)|, |\rho(1)|) + z(1) \\
&= \frac{\|\mathcal{F}\|_0}{Q_0} + \max(|\mu(-1)|, |\rho(1)|) \pm \rho(1) \ge 0, \\
L\pi^{\pm}(\varphi) &= C_\varepsilon(\pi^{\pm}(\varphi))'' + \mathcal{H}(\varphi)(\pi^{\pm}(\varphi))' + Q(\varphi)(\pi^{\pm}(\varphi)) \\
&= Q(\varphi)\left(\frac{\|\mathcal{F}\|_0}{Q_0} + \max(|\mu(-1)|, |\rho(1)|) + Lz(\varphi)\right) \\
&= Q(\varphi)\left(\frac{\|\mathcal{F}\|_0}{Q_0} + \max(|\mu(-1)|, |\rho(1)|) + F(\varphi)\right) \\
&= \left(\|\mathcal{F}\|_0 \pm F(\varphi) + Q(\varphi)\max(|\mu(-1)|, |\rho(1)|)\right) \ge 0 \quad (\text{since } Q(\varphi) \ge Q_0 \ge 0).
\end{aligned}$$

Therefore, using maximum principle, we get $\pi^{\pm}(\varphi) \ge 0$ for $\varphi \in [-1,1]$ which is the necessary bound on the approach of the problem's eq. (2.1)-eq. (2.2). □

**Lemma 3.3** ([21])**.** *Let $z(\varphi)$ be the solution to the eq.* (2.1)-(2.2) *and $\alpha(\varphi), \beta(\varphi), F(\varphi) \in C^j[-1,0]$, $j > 0$, are sufficiently smooth functions in* $[-1,1]$. *Then, there exit positive constant $C$ and $\eta$ such that*

$$|D^i z(\varphi)| \le C \quad \forall\ \varphi \in [-1,1].$$

**Theorem 3.1** ([8])**.** *[8] Let $z(\varphi)$ be the solution to the problem* (2.4)-(2.5) *and $\mathcal{H}(\varphi), Q(\varphi), F(\varphi) \in C^j[-1,0)$, $j > 0$, $|\mathcal{H}(\varphi)| \ge v$ ($v$ is a positive constant) are sufficiently smooth functions in* $[-1,1]$. *Then, there exit positive constant $C$ and $\eta$ such that*

$$|D^i z(\varphi)| \le C\left(1 + C_\varepsilon^{\ -i} \exp\left(\frac{v\varphi}{C_\varepsilon}\right)\right), \quad \text{for } i = 1,2,\dots,j+1,\ \varphi \in [-1,0)$$

*and*

$$|D^s z(\varphi)| \le C\left(1 + C_\varepsilon^{\ -s} \exp\left(-\frac{v\varphi}{C_\varepsilon}\right)\right), \quad \text{for } s = 1,2,\dots,j+1,\ \varphi \in [0,1].$$

## 4. Description of the Method

Partition the domain $[-1,1]$ into $N$ evenly spaced domains of grid size $h=\frac{2}{N}$ and with grid points $u_i=-1+ih$, $i=0,1,\dots,N$. If we define $\frac{N}{2}=l$. So, decompose the region $[-1,1]$ into two subranges $[\varphi_{i-1},\varphi_i]$, $\forall\ i=1,2,\dots,n-1$ and $[\varphi_i,\varphi_{i+1}]$, $i=n+1,n+2,\dots,N-1$. For interior layer issue, in the region $[\varphi_{i-1},\varphi_i]$, $\forall\ i=1,2,\dots,n-1$. Layer appear at right end position and in $[\varphi_i,\varphi_{i+1}]$, $\forall\ i=n+1,n+2,\dots,N-1$. Layer appear at left end points. To that aim, we provide the scheme approach for both right-end layer in $[-1,0]$ and left-end layer in $[0,1]$ scenarios. Taylor' s expansion is used in the vicinity of the point $\varphi$, then

$$z'(\varphi-\varepsilon)\approx z'(\varphi)-\varepsilon z''(\varphi)$$

indicates

$$\varepsilon z''(\varphi)=z'(\varphi)-z'(\varphi-\varepsilon). \tag{4.1}$$

As a result, eq. (2.4) is modify by the first order difference equation, which includes a tiny deviating parameter as a perturbation parameter

$$z'(\varphi)=z'(\varphi-\varepsilon)-\mathcal{H}(\varphi)z'(\varphi)-Q(\varphi)z(\varphi)+F(\varphi). \tag{4.2}$$

Integrating eq. (4.2) with regard to $\varphi$ from $\varphi_i$ to $\varphi_{i+1}$, we derive

$$\begin{aligned}\int_{\varphi_i}^{\varphi_{i+1}} z'(\varphi)d\varphi &= \int_{\varphi_i}^{\varphi_{i+1}} (z'(\varphi-C_\varepsilon)-\mathcal{H}(\varphi)z'(\varphi)-Q(\varphi)z(\varphi)+F(\varphi))d\varphi,\\ z(\varphi_{i+1})-z(\varphi_i) &= \int_{\varphi_i}^{\varphi_{i+1}} z'(\varphi-C_\varepsilon)\,d\varphi-\int_{u_i}^{u_{i+1}}\mathcal{H}(\varphi)z'(\varphi)d\varphi\\ &\quad -\int_{\varphi_i}^{\varphi_{i+1}} Q(\varphi)z(\varphi)\,d\varphi+\int_{\varphi_i}^{\varphi_{i+1}}F(\varphi)\,d\varphi.\end{aligned} \tag{4.3}$$

We have used the Gauss two-point formula to determine

$$\int_{-1}^{1}\mathcal{G}(\varphi)\,d\varphi=\mathcal{G}\left(\frac{1}{\sqrt{3}}\right)+\mathcal{G}\left(\frac{-1}{\sqrt{3}}\right).$$

The Gaussian quadrature two-point formula becomes for every piecewise continuous functions $\mathcal{G}$ in an unique region $[\varphi_i,\varphi_{i+1}]$,

$$\int_{\varphi_i}^{\varphi_{i+1}}\mathcal{G}(\varphi)\,d\varphi=\frac{h}{2}(\mathcal{G}(\varphi_i+r)+\mathcal{G}(\varphi_{i+1}-r)), \tag{4.4}$$

where $r=\frac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2}$. Using eq. (4.4) in eq. (4.3), we get

$$\begin{aligned}z_{i+1}-z_i &= z(\varphi_{\ i+1}-C_\varepsilon)-z(\varphi_i-C_\varepsilon)-\mathcal{H}(\varphi_{i+1})z(\varphi_{i+1})+\mathcal{H}(\varphi_i)z(\varphi_i)\\ &\quad+\frac{h}{2}[\mathcal{H}'(\varphi_{i+1}-r)z(\varphi_{i+1}-r)+\mathcal{H}'(\varphi_i+r)z(\varphi_i+r)]\\ &\quad-\frac{h}{2}[Q(\varphi_{i+1}-r)z(\varphi_{i+1}-r)+Q(\varphi_i+r)z(\varphi_i+r)]\\ &\quad+\frac{h}{2}[F(\varphi_{i+1}-r)+F(\varphi_i+r)].\end{aligned} \tag{4.5}$$

Implementing the linear approximation for $z(\varphi_{i+1}-C_\varepsilon)$, $z(\varphi_i-C_\varepsilon)$, $z(\varphi_i-r)$, $z(\varphi_{i+1}-r)$, eq. (4.5) reduces to

$$\begin{aligned}&\left\{\frac{C_\varepsilon}{h}+\mathcal{H}'(\varphi_i+r)\frac{r}{2}-Q(\varphi_i+r)\frac{r}{2}\right\}z_{i-1}\\&+\left\{\frac{-2C_\varepsilon}{h}-\mathcal{H}(\varphi_i)-\mathcal{H}'(\varphi_{i+1}+r)\frac{r}{2}-\mathcal{H}'(\varphi_i+r)h_1+Q(\varphi_{i+1}+r)\left(\frac{r}{2}\right)+Q(\varphi_i+r)h_1\right\}z_i\\&+\left\{\frac{C_\varepsilon}{h}-\mathcal{H}'(\varphi_{i+1}-r)\left(\frac{h-r}{2}\right)+\mathcal{H}(\varphi_{i+1})+Q(\varphi_i+r)\left(\frac{h-r}{2}\right)\right\}z_{i+1}\\&=\frac{h}{2}\{F(\varphi_{i+1}-r)+F(\varphi_i+r)\}.\end{aligned}\tag{4.6}$$

Eq. (4.6) can be represented as the following in a recurrence relation

$$\mathcal{A}_iz_{i-1}+\mathcal{D}_iz_i+\mathcal{X}_iz_{i+1}=F_i,\quad i=1,2,3,\dots,\frac{n}{2}-1,\tag{4.7}$$

where

$$\mathcal{A}_i=\left\{\frac{C_\varepsilon}{h}+\mathcal{H}'(\varphi_i+r)\frac{r}{2}-Q(\varphi_{i+1}+r)\frac{r}{2}\right\},$$

$$\mathcal{D}_i=\frac{-2C_\varepsilon}{h}-\mathcal{H}(\varphi_i)-\mathcal{H}'(\varphi_{i+1}+r)\frac{r}{2}-\mathcal{H}'(\varphi_i+r)h_1+Q(\varphi_{i+1}+r)\left(\frac{r}{2}\right)+Q(\varphi_i+r)h_1,$$

$$\mathcal{X}_i=\frac{C_\varepsilon}{h}-\mathcal{H}'(\varphi_{i+1}-r)h_2+\mathcal{H}(\varphi_{i+1})+Q(\varphi_{i+1}-r)h_2,$$

$$F_i=\frac{h}{2}[F(\varphi_{i+1}-r)+F(\varphi_i+r)],$$

here $h_1=\left(\frac{h+r}{2}\right)$, $h_2=\left(\frac{h-r}{2}\right)$.

Furthermore, we examine our approach to the problem of eq. (2.4) with the left-end boundary layer of the fundamental region in the domain $[0,1]$. Taylor series expansion of $z'(\varphi+C_\varepsilon)$ obtain

$$z'(\varphi+C_\varepsilon)\approx z'(\varphi)+C_\varepsilon z''(\varphi)$$

indicates

$$C_\varepsilon z''(\varphi)\approx z'(\varphi)+C_\varepsilon)-z'(\varphi).\tag{4.8}$$

Using eq. (4.8) in eq. (2.4) is simplified to

$$z'(\varphi)=z'(\varphi+C_\varepsilon)+\mathcal{H}(\varphi)z'(\varphi)+Q(\varphi)z(\varphi)-F(\varphi).\tag{4.9}$$

Integrating eq. (4.9) on $[\varphi_{i-1},\varphi_i]$ we get

$$z(\varphi_i)-z(\varphi_{i-1})=\int_{\varphi_{i-1}}^{\varphi_i}z'(\varphi+C_\varepsilon)\,d+\int_{\varphi_{i-1}}^{\varphi_i}\mathcal{H}(\varphi)z'(\varphi)d\varphi+\int_{\varphi_{i-1}}^{\varphi_i}Q(\varphi)z(\varphi)\,d\varphi-\int_{\varphi_{i-1}}^{\varphi_i}F(\varphi)\,d\varphi.$$

For every continuous and piecewise function $\mathcal{G}$ in the region $[\varphi_{i-1},\varphi_i]$ we obtain the Gaussian quadrature two-point formula.

$$\int_{\varphi_{i-1}}^{\varphi_i}\mathcal{G}(\varphi)\,d\varphi\;=\frac{h}{2}(\mathcal{G}(\varphi_{i-1}-r)+\mathcal{G}(\varphi_i+r)).\tag{4.10}$$

Using eq. (4.10), in the eq. (4.9) becomes

$$z(u_i)-z(u_{i-1})=z(\varphi_{i-1}-C_\varepsilon)-z(\varphi_i+C_\varepsilon)+\mathcal{H}(\varphi_{i-1})z(\varphi_{i-1})-\mathcal{H}(\varphi_i)z(\varphi_i)$$

$$-\frac{h}{2}[\mathcal{H}'(\varphi_{i-1}-r)z(\varphi_{i-1}-r)+\mathcal{H}'(\varphi_i+r)z(\varphi_i+rt)]$$
$$+\frac{h}{2}[Q(\varphi_{i-1}-r)z(\varphi_{i-1}-r)+Q(\varphi_i+r)z(\varphi_i+r)]$$
$$-\frac{h}{2}[F(\varphi_{i-1}-r)+F(\varphi_i+r)]. \tag{4.11}$$

Linear approximation utilised for $z(\varphi_{i-1}-C_\varepsilon)$, $z\varphi_i+C_\varepsilon$, $z(\varphi_{i-1}-r)$, $z(\varphi_i+r)$, in eq. (4.11), then we have

$$\left\{\frac{C_\varepsilon}{h}-\mathcal{H}(\varphi_{i-1})-\mathcal{H}'(\varphi_{i-1}+r)h_1+Q(\varphi_{i-1}+r)h_1\right\}z_{i-1}$$
$$+\left\{\frac{-2C_\varepsilon}{h}+\mathcal{H}(\varphi_i)-\mathcal{H}'(\varphi_i-r)h_2-\mathcal{H}'(\varphi_{i-1}+r)\left(\frac{r}{2}\right)+Q(\varphi_i-r)h_2+Q(\varphi_{i-1}+r)\left(\frac{r}{2}\right)\right\}z_i$$
$$+\left\{\frac{C_\varepsilon}{h}+\mathcal{H}'(\varphi_i-r)\left(\frac{r}{2}\right)-Q(\varphi_i-r)\left(\frac{r}{2}\right)\right\}z_{i+1}$$
$$=\frac{h}{2}\{F(\varphi_i-r)+F(\varphi_{i-1}+r)\}. \tag{4.12}$$

Here $h_1=\left(\frac{h-r}{2}\right)$, $h_2=\left(\frac{h+r}{2}\right)$.

The tridiagonal system of eq. (4.12) is

$$\mathcal{A}_i z_{i-1}+\mathcal{D}_i z_i+\mathcal{X}_i z_{i+1}=F_i, \quad \forall\ i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\dots,n-1, \tag{4.13}$$

where

$$\mathcal{A}_i=\left\{\frac{C_\varepsilon}{h}-\mathcal{H}(\varphi_{i-1})-\mathcal{H}'(\varphi_{i-1}+r)\left(\frac{h-r}{2}\right)+Q(\varphi_{i-1}+r)\left(\frac{h-r}{2}\right)\right\}$$
$$\mathcal{D}_i=\left\{\frac{-2C_\varepsilon}{h}+\mathcal{H}(\varphi_i)-\mathcal{H}'(\varphi_i+r)h_2-\mathcal{H}'(\varphi_{i-1}+r)\left(\frac{r}{2}\right)+Q(\varphi_i-r)h_2+Q(\varphi_{i-1}+r)\left(\frac{r}{2}\right)\right\}$$
$$\mathcal{X}_i=\left\{\frac{C_\varepsilon}{h}+\mathcal{H}'(\varphi_i-r)\left(\frac{r}{2}\right)-Q(\varphi_i-r)\left(\frac{r}{2}\right)\right\}$$
$$F_i=\frac{h}{2}[F(\varphi_i-r)+F(\varphi_{i-1}+r)].$$

Utilizing eq. (4.7) in $[\varphi_{i-1},\varphi_i]$, $\forall\ i=n+1,n+2,\dots,N-1$ and eq. (4.13) in $[\varphi_i,\varphi_{i+1}]$, $\forall\ i=1,2,\dots,n-1$. We obtain a system of $(N-2)$ equations with $(N+1)$ unknowns. From the given constraints in eq. (2.4) we find two new equations. Hence, in $[-1,t_m]$ we use the finite difference technique eq. (4.13) for $i=n+1,n+2,\dots,N-1$ and in $[t_m,1]$ the scheme eq. (4.7) for $i=1,2,\dots,n-1$ is utilized to get the solution.

From eq. (2.4) holds for $i=m$ at $\varphi=\varphi_m$,

$$C_\varepsilon z''(\varphi_m)+Q(\varphi_m)z(t_m)=F(\varphi_m). \tag{4.14}$$

We use the methods of averaging differences eq. (4.7) and eq. (4.13) because there is an interior layer at $\varphi=\varphi_m$. Consequently, the difference equation Eq. (4.14) becomes

$$A_m z_{m-1}+B_{mi}z_m+C_m z_{m+1}=F_m, \quad \text{for } i=m. \tag{4.15}$$

Here

$$A_m=\frac{\mathcal{A}_{iL}+\mathcal{A}_{iR}}{2},\ \ B_m=\frac{\mathcal{D}_{iL}+\mathcal{D}_{iR}}{2},\ \ C_m=\frac{\mathcal{X}_{iL}+\mathcal{X}_{iR}}{2}\ \text{ and }\ F_m=\frac{F_{iL}+F_{iR}}{2}.$$

Now, we apply the Thomas algorithm to calculate the matrix of equations (4.7), (4.13), and (4.15).

## 5. Analysis of Convergence

Incorporating the boundary constraints, the matrix system of eq. (4.7) in the form is obtained by

$$(X+Y)z+U+\tau(h)=0, \tag{5.1}$$

where

$$X=\left[\frac{C_\varepsilon}{h},\frac{-2C_\varepsilon}{h},\frac{C_\varepsilon}{h}\right]=\begin{bmatrix}\frac{-2C_\varepsilon}{h} & \frac{C_\varepsilon}{h} & 0 & 0 & \dots & 0\\ \frac{C_\varepsilon}{h} & \frac{-2C_\varepsilon}{h} & \frac{C_\varepsilon}{h} & 0 & \dots & 0\\ 0 & \frac{C_\varepsilon}{h} & \frac{-2C_\varepsilon}{h} & \frac{C_\varepsilon}{h} & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & 0 & \frac{C_\varepsilon}{h} & \frac{-2C_\varepsilon}{h}\end{bmatrix}$$

and

$$Y=[\theta_i,\vartheta_i,w_i]=\begin{bmatrix}\theta_1 & w_1 & 0 & 0 & \dots & 0\\ \theta_2 & \vartheta_2 & w_2 & 0 & \dots & 0\\ 0 & \theta_3 & \vartheta_3 & w_3 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & 0 & \theta_{N-1} & \vartheta_{N-1}\end{bmatrix},$$

where

$$\theta_i=\frac{r}{2}(\mathcal{H}(\varphi_i+r)-Q(\varphi_i+r)),$$

$$\vartheta_i=\left\{-\mathcal{H}(\varphi_i)-\mathcal{H}'(\varphi_{i+1}-r)\frac{r}{2}-\mathcal{H}'(\varphi_i+r)\left(\frac{h+r}{2}\right)+Q(\varphi_{i+1}-r)\frac{r}{2}+Q(\varphi_i+r)\left(\frac{h+r}{2}\right)\right\},$$

$$w_i=\left\{\mathcal{H}(\varphi_{i+1})-\mathcal{H}'(\varphi_{i+1}-r)\left(\frac{h+r}{2}\right)+Q(\varphi_{i+1}-r)\left(\frac{h-r}{2}\right)\right\},$$

$$Q=\left\{p_1+\left(\frac{C_\varepsilon}{h}+z_1\right)\mu_0,p_2,p_3,\dots,p_{N-2},p_N-1+\left(\frac{C_\varepsilon}{h}+w_{N-1}\right)\rho_1\right\},$$

where $p_i=\frac{h}{2}\{F(u_{i+1}-r)+F(u_i+r)\}$, $\forall\ i=1,2,\dots,N-1$.

According to the approach, the local truncated error $\tau(h)$ is equals $h^2$, i.e., $\tau(h)=O(h^2)$. Let $w=[z_1,z_2,\dots,z_{N-1}]^T$, $\tau(h)=[T_1,T_2,\dots,T_{N-1}]^T$, $\widetilde{O}=[0,0,\dots,0]^T$ are correlated vectors of eq. (5.1). Let $W=[W_1,W_2,\dots,W_{N-1}]^T\cong W$ which satisfies the equation

$$(X+Y)W+U=0. \tag{5.2}$$

Let $l_i=z_i-W_i$, $i=1,2,\dots,N-1$ be the discretization error because of this $L=[l_1,l_2,\dots,l_{N-1}]^T=z-W$. Subtract eq. (5.1)-eq. (5.2). Thus, the error equation is obtained.

$$(X+Y)L=\tau(h). \tag{5.3}$$

Let $|\mathcal{H}(\varepsilon)|\le\widetilde{c}_1$ and $|Q(\varphi)|\le\widetilde{c}_2$ where $\widetilde{c}_1$, $\widetilde{c}_2$ is a positive constant if $\mathcal{H}_{i,j}$ take the position of $(i,j)$th elements in $\mathcal{H}$, then

$$|\mathcal{H}_{i,i+1}|=|w_i|\le\left(\mathcal{H}(\varphi_{i+1})+\mathcal{H}'(\varphi_{i+1}-r)\left(\frac{h+r}{2}\right)+Q(\varphi_{i+1}-r)\left(\frac{h-r}{2}\right)\right)\widetilde{c}_1,\quad i=1,\dots,N-2,$$

$$|\mathcal{H}_{i,i-1}| = |\theta_i| \leq \frac{r}{2}[\mathcal{H}'(\varphi_i + r) - Q(\varphi_i + r)]\tilde{c}_2, \quad \forall\ i = 2,3,\dots,N-1$$

with a small sufficient value of $h$, we obtain

$$|\mathcal{H}_{i\ i+1}| < C_\varepsilon, \quad i = 1,2,\dots,N-2 \tag{5.4}$$

and

$$|\mathcal{H}_{i\ i-1}| < C_\varepsilon, \quad i = 2,3,\dots,N-1. \tag{5.5}$$

Hence, $(X+Y)$ is irreducible. Using the sum of the components of the $i$th row of $(X+Y)$ be $\widetilde{S}_i$, then we have

$$\widetilde{S}_1 = \sum_{j=1}^{N-1} \widetilde{M}_{ij} = \{h(\mathcal{H}(\varphi_{i+1}) - \mathcal{H}(\varphi_i)) + C_\varepsilon - h(\mathcal{H}'(\varphi_i + r)h_1 + Q(\varphi_i + r)h_1)\}$$
$$+ \left\{\frac{h}{2}(\mathcal{H}(\varphi_{i+1} - r) - \mathcal{H}'(\varphi_{i+1} - r))\right\}, \quad \text{where } h_1 = \left(\frac{h+r}{2}\right) \text{ for } i = 1,$$

$$S_i = \sum_{j=1}^{N-1} M_{ij} = \mathcal{H}(\varphi_{i+1} - \mathcal{H}(\varphi_i) + \frac{h}{2}[Q(\varphi_i + r) + Q(\varphi_{i+1} - r) - (\mathcal{H}(\varphi_i + r) + \mathcal{H}'(\varphi_{i+1} - r))], \quad i = N-1,$$

$$\widetilde{S}_{N-1} = \sum_{j=1}^{N-1} \widetilde{M}_{N-1\ j} = h(-\mathcal{H}(\varphi_i)) - C_\varepsilon - h\left(\mathcal{H}'(\varphi_{i+1} - r)\frac{r}{2} - \mathcal{H}'(\varphi_i + r)\frac{r}{2}\right)\left(Q(\varphi_{i+1} - r)\frac{r}{2} + Q(\varphi_i + r)\frac{h}{2}\right),$$
$$i = 2,3,\dots,N-2.$$

Let $\widetilde{C}_i = \min\{|\mathcal{H}(\varphi)|\}$ and $\widetilde{C}_1^* = \max\{|\mathcal{H}(\varphi)|\}$ since $0 < \varepsilon \leq 1$ and $\varepsilon \propto o(h)$ results indicate that for sufficient small $h$, $(X+Y)$, is monotonic. Hence $(X+Y)^{-1}$ exists and $(X+Y)^{-1} \geq 0$ thus eq. (5.3) we obtain

$$\|L\| \leq \|(X+Y)^{-1}\|\|T\|. \tag{5.6}$$

For sufficient small $h$, we have

$$\tilde{s}_i > h(\mathcal{H}(\varphi_{i+1}) - \mathcal{H}(\varphi_i)), \quad \text{for } i = 1,$$
$$\tilde{s}_i > h(\mathcal{H}(\varphi_{i+1}) - \mathcal{H}(\varphi_i)), \quad \text{for } i = N-1,$$
$$\tilde{s}_i > -\mathcal{H}(\varphi_i)h, \qquad \text{for } i = 2,3,\dots,N-2.$$

Let $(X+Y)^{-1}$ be the $(i,k)$th components of $(X+Y)^{-1}$ and $(X+Y)^{-1} = \max\sum_{k=1}^{N-1}(X+Y)^{-1}_{(i\ k)}$ and

$$\|\tau(h)\| = \max \tau(h). \tag{5.7}$$

Since $(X+Y)^{-1}_{(i\ k)} \geq 0$ and $\sum_{k=1}^{N-1}(X+Y)^{-1}_{(i\ k)}\widetilde{S}_K = 1$ for $i = 1,2,\dots,N-1$.

$$(X+Y)^{-1}_{(i\ k)} \leq \frac{1}{\tilde{s}_K} \leq \frac{1}{h\widetilde{C}}, \tag{5.8}$$

$$(X+Y)^{-1}_{(i\ N-1)} \leq \frac{1}{\tilde{s}_{N-1}} \leq \frac{1}{h\widetilde{C}}. \tag{5.9}$$

Furthermore,

$$\sum_{k=2}^{N-2}(X+Y)^{-1}_{(i\ k)} \leq \frac{1}{\min \tilde{s}_k} \leq \frac{1}{h\widetilde{C}}, \quad \forall\ i = 1,2,\dots,N-2. \tag{5.10}$$

Using the eq. (5.7)-eq. (5.10), from eq. (5.6), we have $L \leq O(h)$. Hence, the suggested approach is first order convergent.

## 6. Numerical Experiments

The maximum absolute errors (MAEs) in the solution of the following examples are determine by double mesh principle

$$L_\varepsilon^N = \max_{0\le i\le N} |z_i^N - z_{2i}^{2N}|.$$

**Example 6.1.**

$$-\varepsilon z''(\varphi)+2(1-2\varphi)z'(\varphi)+4z(\varphi)+2z(\varphi-\delta)+z(\varphi+\eta)=0, \quad \varphi\in(0,1)$$
$$\text{with } z(\varphi)=1 \text{ on } -\delta\le\varphi\le 0,\ z(\varphi)=1 \text{ on } 1\le\varphi\le 1+\eta.$$

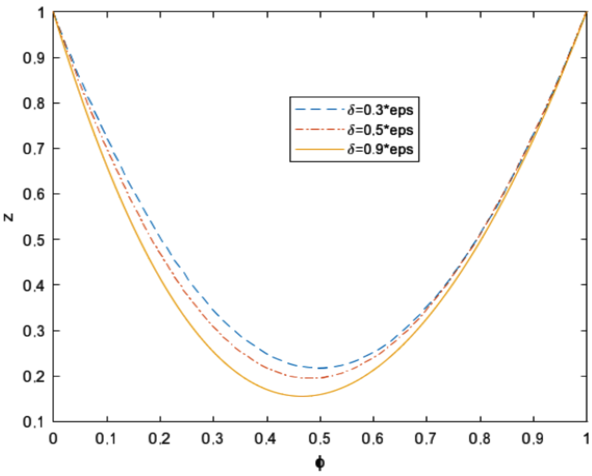


**Figure 1.** The layer profile for Example 6.1 with $\varepsilon=2^{-2}$, $\eta=0.5\varepsilon$

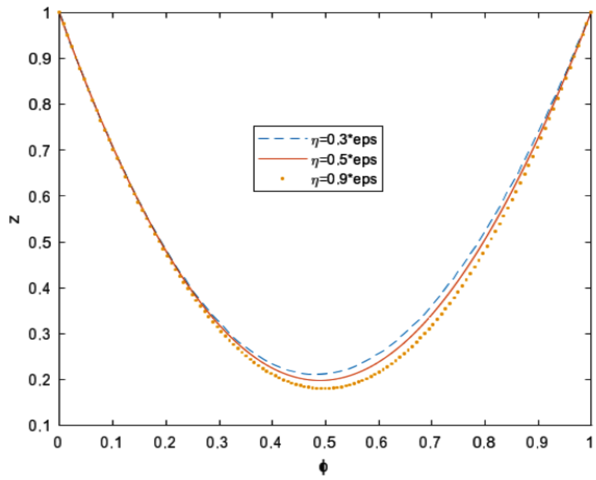


**Figure 2.** The layer profile for Example 6.1 with $\varepsilon=2^{-2}$, $\delta=0.5\varepsilon$

**Example 6.2.**

$$-\varepsilon z''(\varphi)+2(1-2\varphi)z'(\varphi)+4z(\varphi)+2z(\varphi-\delta)+z(\varphi+\eta)=4(1-4\varphi), \quad \varphi\in(0,1)$$
$$\text{with } z(\varphi)=1 \text{ on } -\delta\le\varphi\le 0,\ z(\varphi)=1 \text{ on } 1\le\varphi\le 1+\eta.$$

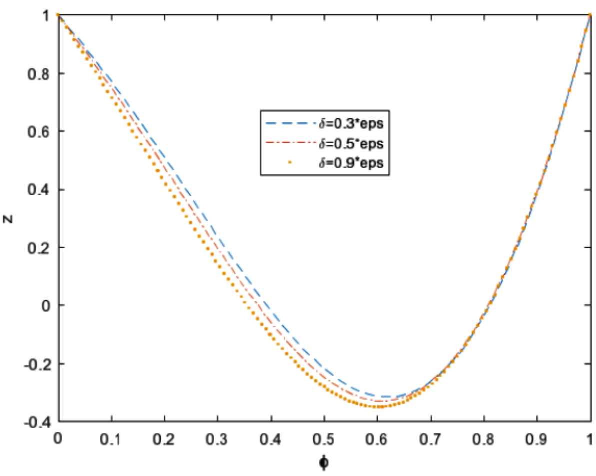


**Figure 3.** The layer profile for Example 6.2 with $\varepsilon=2^{-2}$, $\eta=0.5\varepsilon$

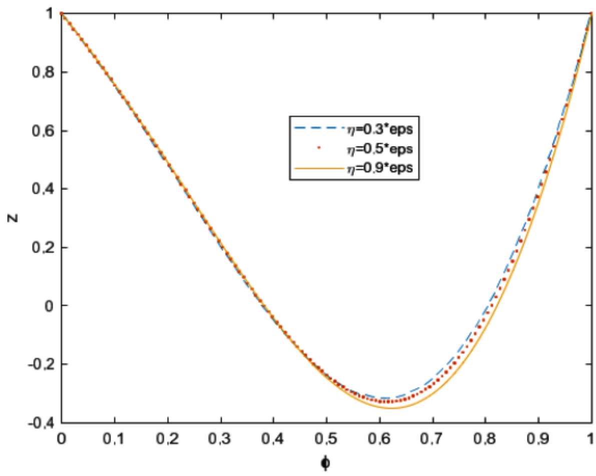


**Figure 4.** The layer profile for Example 6.2 with $\varepsilon=2^{-2}$, $\delta=0.5\varepsilon$

## 7. Conclusion

We have discussed a numerical scheme for the solution of SPDDEs with mixed shifts. The domain is divided into two subintervals since the problem under consideration involves interior layer behavior. In each subinterval, the numerical scheme is constructed using integration and linear interpolation to get the solution.The suggested numerical approach is analyzed for convergence.

In order to evaluate the efficiency of the suggested approach several numerical experiments are conducted. The MAEs in the solution of examples are tabulated in the form of tables in comparison to the method given in [19]. The impact of small delay and advance on the interior layer solution is demonstrated by plotting the graphs (Figures 1–4). It is observed that the when $\eta$ is increasing for a fixed delay the width of the interior layer decreases, whereas it increases when $\delta$ increases for a fixed $\eta$.

**Table 1.** MAEs in Example 6.1 for $\eta = 0.5 * \varepsilon$, $\delta = 0.9 * \varepsilon$

| $\varepsilon \downarrow$ \ $N \rightarrow$ | 32 | 64 | 128 | 256 | 512 | 1024 |
|---|---|---|---|---|---|---|
| Suggested method: | | | | | | |
| $10^{-1}$ | 1.4128e-03 | 5.6633e-04 | 1.0715e-04 | 2.1513e-05 | 6.5480e-06 | 1.7221e-06 |
| $10^{-2}$ | 1.6805e-03 | 5.4316e-04 | 2.0253e-04 | 8.5823e-05 | 3.9131e-05 | 1.0650e-05 |
| $10^{-3}$ | 1.8219e-03 | 4.8134e-04 | 1.5776e-04 | 5.3158e-05 | 2.0838e-05 | 9.0937e-06 |
| $10^{-4}$ | 2.0398e-03 | 5.8169e-04 | 1.5393e-04 | 4.4192e-05 | 1.4408e-05 | 5.1303e-06 |
| $10^{-5}$ | 2.0694e-03 | 6.1271e-04 | 1.7888e-04 | 4.9959e-05 | 1.2881e-05 | 4.0366e-06 |
| $10^{-6}$ | 2.0725e-03 | 6.1627e-04 | 1.8289e-04 | 5.3965e-05 | 1.5595e-05 | 4.2433e-06 |
| $10^{-7}$ | 2.0728e-03 | 6.1663e-04 | 1.8332e-04 | 5.4461e-05 | 1.6142e-05 | 4.7418e-06 |
| $10^{-8}$ | 2.0728e-03 | 6.1667e-04 | 1.8336e-04 | 5.4512e-05 | 1.6202e-05 | 4.8108e-06 |
| Results in [19] | | | | | | |
| $10^{-1}$ | 1.4450e-03 | 4.2460e-04 | 1.1560e-04 | 3.0190e-05 | 7.7160e-06 | 1.9510e-06 |
| $10^{-2}$ | 3.7790e-03 | 1.7120e-03 | 6.5890e-04 | 2.1420e-04 | 6.1880e-05 | 1.6690e-05 |
| $10^{-3}$ | 3.9870e-03 | 1.9670e-03 | 9.7760e-04 | 4.8520e-04 | 2.2850e-04 | 9.2590e-05 |
| $10^{-4}$ | 4.0050e-03 | 1.9740e-03 | 9.8130e-04 | 4.8930e-04 | 2.4430e-04 | 1.2210e-04 |
| $10^{-5}$ | 4.0070e-03 | 1.9750e-03 | 9.8170e-04 | 4.8950e-04 | 2.4440e-04 | 1.2210e-04 |
| $10^{-6}$ | 4.0070e-03 | 1.9750e-03 | 9.8170e-04 | 4.8950e-04 | 2.4440e-04 | 1.2210e-04 |
| $10^{-7}$ | 4.0070e-03 | 1.9750e-03 | 9.8170e-04 | 4.8950e-04 | 2.4440e-04 | 1.2210e-04 |
| $10^{-8}$ | 4.0070e-03 | 1.9750e-03 | 9.8170e-04 | 4.8950e-04 | 2.4440e-04 | 1.2210e-04 |

**Table 2.** MAEs in Example 6.1 for $\varepsilon = 1/64$

| $N \rightarrow$ | 32 | 64 | 128 | 256 | 512 | 1024 |
|---|---|---|---|---|---|---|
| Suggested method: | | | | | | |
| $\delta \downarrow$ | $\eta = 0.5 * \varepsilon$ | | | | | |
| $0.0\varepsilon$ | 1.7387e-03 | 5.8438e-04 | 2.3115e-04 | 1.0078e-04 | 4.6846e-05 | 2.2562e-05 |
| $0.3\varepsilon$ | 1.7267e-03 | 5.8133e-04 | 2.2861e-04 | 9.9606e-05 | 4.6239e-05 | 2.2259e-05 |
| $0.5\varepsilon$ | 1.7193e-03 | 5.7606e-04 | 2.2723e-04 | 9.8871e-05 | 4.5899e-05 | 2.2091e-05 |
| $0.7\varepsilon$ | 1.7044e-03 | 5.6756e-04 | 2.2434e-04 | 9.7412e-05 | 4.5221e-05 | 2.1760e-05 |
| $0.9\varepsilon$ | 1.6818e-03 | 5.5684e-04 | 2.1997e-04 | 9.5312e-05 | 4.4222e-05 | 2.1268e-05 |
| $\eta \downarrow$ | $\delta = 0.5 * \varepsilon$ | | | | | |
| $0.0\varepsilon$ | 1.7338e-03 | 5.7955e-04 | 2.3023e-04 | 1.0027e-04 | 4.6646e-05 | 2.2472e-05 |
| $0.3\varepsilon$ | 1.7276e-03 | 5.7855e-04 | 2.2895e-04 | 9.9682e-05 | 4.6323e-05 | 2.2302e-05 |
| $0.5\varepsilon$ | 1.7193e-03 | 5.7606e-04 | 2.2723e-04 | 9.8871e-05 | 4.5899e-05 | 2.2091e-05 |
| $0.7\varepsilon$ | 1.7077e-03 | 5.7212e-04 | 2.2482e-04 | 9.7724e-05 | 4.5310e-05 | 2.1800e-05 |
| $0.9\varepsilon$ | 1.6927e-03 | 5.6674e-04 | 2.2172e-04 | 9.6241e-05 | 4.4575e-05 | 2.1431e-05 |

**Table 3.** MAEs in Example 6.2 for $\eta = 0.9 * \varepsilon$, $\delta = 0.5 * \varepsilon$

| $N \rightarrow$ / $\varepsilon \downarrow$ | 32 | 64 | 128 | **256** | 512 | 1024 |
|---|---|---|---|---|---|---|
| Suggested method: | | | | | | |
| $10^{-1}$ | 6.8208e-03 | 3.3046e-03 | 1.0440e-03 | 2.1831e-04 | 4.0858e-05 | 2.0419e-05 |
| $10^{-2}$ | 3.6377e-03 | 1.7228e-03 | 8.1624e-04 | 4.0194e-04 | 2.0058e-04 | 1.0053e-04 |
| $10^{-3}$ | 4.8774e-03 | 1.2265e-03 | 3.3472e-04 | 1.6649e-04 | 8.1156e-05 | 4.0992e-05 |
| $10^{-4}$ | 5.0871e-03 | 1.5083e-03 | 4.2362e-04 | 1.0624e-04 | 3.6151e-05 | 1.7093e-05 |
| $10^{-5}$ | 5.1043e-03 | 1.5294e-03 | 4.5375e-04 | 1.3233e-04 | 3.5455e-05 | 9.1834e-06 |
| $10^{-6}$ | 5.1060e-03 | 1.5313e-03 | 4.5606e-04 | 1.3544e-04 | 3.9989e-05 | 1.1487e-05 |
| $10^{-7}$ | 5.1062e-03 | 1.5315e-03 | 4.5628e-04 | 1.3570e-04 | 4.0324e-05 | 1.1957e-05 |
| $10^{-8}$ | 5.1062e-03 | 1.5316e-03 | 4.5630e-04 | 1.3573e-04 | 4.0355e-05 | 1.1995e-05 |
| Results in [19] | | | | | | |
| $10^{-1}$ | 1.8430e-02 | 5.4620e-03 | 1.4940e-03 | 3.9130e-04 | 1.0021e-04 | 4.2021e-05 |
| $10^{-2}$ | 4.8550e-02 | 2.2060e-02 | 8.5150e-03 | 2.7740e-03 | 8.0360e-04 | 2.1680e-04 |
| $10^{-3}$ | 5.1450e-02 | 2.5560e-02 | 1.2690e-02 | 6.2970e-03 | 2.9671e-03 | 1.2030e-03 |
| $10^{-4}$ | 5.1710e-02 | 2.5620e-02 | 1.2750e-02 | 6.3610e-03 | 3.1760e-03 | 1.5870e-03 |
| $10^{-5}$ | 5.1710e-02 | 2.5620e-02 | 1.2750e-02 | 6.3610e-03 | 3.1781e-03 | 1.5881e-03 |
| $10^{-6}$ | 5.1710e-02 | 2.5620e-02 | 1.2750e-02 | 6.3610e-03 | 3.1780e-03 | 1.5881e-03 |
| $10^{-7}$ | 5.1710e-02 | 2.5620e-02 | 1.2750e-02 | 6.3610e-03 | 3.1780e-03 | 1.5881e-03 |
| $10^{-8}$ | 5.1710e-02 | 2.5620e-02 | 1.2750e-02 | 6.3610e-03 | 3.1780e-03 | 1.5881e-03 |

**Table 4.** MAEs in Example 6.2 for $\varepsilon = 1/64$

| $N \rightarrow$ | 32 | 64 | 128 | 256 | 512 | 1024 |
|---|---|---|---|---|---|---|
| Suggested method: | | | | | | |
| $\delta \downarrow$ | $\eta = 0.5 * \varepsilon$ | | | | | |
| $0.0\varepsilon$ | 3.3553e-03 | 1.7757e-03 | 8.6396e-04 | 4.3957e-04 | 2.2082e-04 | 1.1093e-04 |
| $0.3\varepsilon$ | 3.5493e-03 | 1.8529e-03 | 8.9912e-04 | 4.5478e-04 | 2.2799e-04 | 1.1443e-04 |
| $0.5\varepsilon$ | 3.6650e-03 | 1.8956e-03 | 9.1830e-04 | 4.6270e-04 | 2.3179e-04 | 1.1620e-04 |
| $0.7\varepsilon$ | 3.7687e-03 | 1.9308e-03 | 9.3380e-04 | 4.6874e-04 | 2.3465e-04 | 1.1750e-04 |
| $0.9\varepsilon$ | 3.8593e-03 | 1.9580e-03 | 9.4540e-04 | 4.7278e-04 | 2.3652e-04 | 1.1835e-04 |
| $\eta \downarrow$ | $\delta = 0.5 * \varepsilon$ | | | | | |
| $0.0\varepsilon$ | 3.8568e-03 | 1.9726e-03 | 9.5449e-04 | 4.7863e-04 | 2.3959e-04 | 1.1995e-04 |
| $0.3\varepsilon$ | 3.7457e-03 | 1.9288e-03 | 9.3397e-04 | 4.6969e-04 | 2.3522e-04 | 1.1784e-04 |
| $0.5\varepsilon$ | 3.6650e-03 | 1.8956e-03 | 9.1830e-04 | 4.6270e-04 | 2.3179e-04 | 1.1620e-04 |
| $0.7\varepsilon$ | 3.5794e-03 | 1.8593e-03 | 9.0110e-04 | 4.5494e-04 | 2.2797e-04 | 1.1435e-04 |
| $0.9\varepsilon$ | 3.4890e-03 | 1.8200e-03 | 8.8243e-04 | 4.4641e-04 | 2.2377e-04 | 1.1232e-04 |

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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