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Research Article

Approximate Method for Solving System of Linear Fredholm Fractional Integro-Differential Equations Using Least Squares Method and Lauguerre Polynomials

Parameshwari M. Goud*[1], J. A. Nanware[2], T. L. Holambe[3] and N. B. Jadhav[4]

[1] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, Maharashtra, India*
[2] *Department of PG Studies and Research in Mathematics, Shrikrishna Mahavidyalaya, Gunjoti, Osmanabad 413606, Maharashtra, India*
[3] *Department of Mathematics, Late Shankarrao Gutte Arts, Commerce and Science College, Dharmapuri, Tq. Parli (V), Beed, Maharashtra, India*
[4] *Department of Mathematics, Yashwantrao Chavan Mahavidyalaya, Tuljapur, Osmanabad, Maharashtra, India*
***Corresponding author:** pmgoud21@gmail.com

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Abstract. An attempt is made to develop approximate method to get approximate solution of system of linear Fredholm fractional integro-differential equations using Least squares and Lauguerre polynomial method. The system of linear Fredholm fractional integro-differential equations is reduced to a system of linear equations using Lauguerre polynomials. The method is implemented to obtain approximate solution of the system of Fredholm fractional integro-differential equations. The solutions are simulated using Scilab.

Keywords. System of linear Fredholm fractional integro-differential equations, Least square method, Lauguerre polynomials, Caputo derivative

Mathematics Subject Classification (2020). 26A33, 34A08

1. Introduction

The complex systems with memory are modeled using fractional calculus. In 1823, Abel solved an integral equation associated with Tautochrone problem via fractional calculus (Podlubny [8]). Many physical phenomenon in engineering and science are modeled as *fractional integro-differential equations* (FIDEs). Recently researchers paid a great deal of attention towards the development of analytical and numerical approaches for the system of fractional integro-differential equations [2, 3, 9]. Mahdy and Mohamed [3], and Mahdy [4] studied numerical solution of fractional integro-differential equation using Hermite polynomials and method of Least squares. Chebyshev spectral method was implemented by Zedan *et al*. [10] to find the solution of system of fractional integro-differential equations and Abel's integral equations. Analytical method based on homotopy analysis method is proposed by Zurigat *et al*. [11] to find the solution of system of linear and nonlinear fractional integro-differential equations. Authors have implemented several methods to obtain analytical as well as approximate solution of linear and nonlinear fractional integro-differential equations [6, 7]. Momani and Qaralleh *et al*. [5] developed an efficient tool to obtain solution of system of fractional integro-differential equations by Adomian polynomials. Saeed and Sdeq [9] introduced homotopy perturbation method to solve system of fractional integro-differential equations. Aforementioned work motivates us to study approximate method for the solution of system of linear Fredholm fractional integro-differential equations. In this paper we develop approximate method to obtain the solution of system of linear fractional integro-differential equations using Least square and Lauguerre polynomial method.

Consider the following system of linear fractional Fredholm integro-differential equation in Caputo sense:

$$ {}^{c}D^{\xi} v_i(x) = F_i(x) + \int_0^1 K_i(x,t)\left[\sum_{k=1}^{n} \delta_{ik} v_k(t)\right] dt, \quad 0 \le x, t \le 1, \tag{1.1} $$

$$ v_i^{(j)}(x_0) = v_{ij}, \quad i = j = 1, 2, \dots, n, \tag{1.2} $$

where $F_i(x)$ and $K_i(x,t)$ are provided functions, and real variables x and t have a range of $[0,1]$. ${}^{c}D^{\xi} v_i(x)$ denotes the ξth Caputo fractional derivative of $v_i(x)$. The rest of the paper is arranged as follows:

In Section 2, we explore the basic definitions and properties of fractional calculus. In Section 3, approximate method is developed for the problem under investigation. In Section 4, proposed method is illustrated and approximate solution is obtained. In the last section conclusion of the work is given.

2. Basic Definitions

In this section we explore the definitions and basic properties of fractional calculus.

Definition 2.1 ([5])**.** Define the classes:

$$ C_\mu = \{v(x) \mid v(x) = x^q v_1(x), x > 0, q > \mu, v_1(x) \in C[o,\infty)\}, $$
$$ C_\mu^n = \{v(x) \mid v^n(x) \in C_\mu, x > 0, n \in N\}. $$

Definition 2.2 ([8]). The Riemann-Liouville fractional integral operator of order $\xi \geq 0$ is defined as:

$$J^{\xi}v(x) = \frac{1}{\Gamma(\xi)}\int_0^x (x-t)^{(\xi-1)}v(t)dt, \quad x > 0.$$

Definition 2.3 ([8]). The fractional derivative of $v(x)$ in the Caputo sense of order ξ is defined as:

$${}^cD^{\xi}v(x) = \frac{1}{\Gamma(n-\xi)}\int_0^x \frac{v^{(n)}(t)}{(x-t)^{\xi+1-n}}dt,$$

where $n-1 < \xi \leq n$, $n \in N$, $x > 0$.

Properties of Caputo fractional derivative and fractional integrals are given below:

- ${}^cD^{\xi}C = 0$, C is a constant,
- $J^{\xi}J^{\nu}v(x) = J^{\xi+\nu}v(x)$, $\xi, \nu > 0$, $\nu \in C_{\mu}$, $\mu > 0$,
- $J^{\xi}\, {}^cD^{\xi}v(x) = v(x) - \sum\limits_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k}$, $x > 0$, $m-1 < \xi \leq m$,
- $J^{\xi}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\xi)}x^{\beta+\xi}$, $\xi > 0$, $\beta > -1$, $x > 0$,
- ${}^cD^{\xi}J^{\xi}v(x) = v(x)$, $x > 0$, $m-1 < \xi \leq m$,
- ${}^cD^{\xi}x^{\beta} = -\begin{cases} 0, & \beta \in N_0,\ \beta < \lceil\xi\rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\xi+1)}x^{\beta-\xi}, & \beta \in N_0 \text{and } \beta \geq \lceil\xi\rceil, \end{cases}$

where $\lceil\xi\rceil$ denote the smallest integer greater than or equal to ξ and $N_0 = \{0, 1, 2, \dots\}$.

Definition 2.4 ([1]). The Lauguerre polynomials of order m is defined by,

$$L_m(x) = \sum_{p=0}^{m}(-1)^p\frac{m!}{(m-p)!(p!)^2}x^p.$$

Recurrence relation of Lauguerre polynomials is given by,

$$(m+1)L_{m+1}(x) = (2m+1-x)L_m(x) - mL_{m-1}(x), \quad (m \geq 1).$$

The Lauguerre polynomials are,

- $L_0(x) = 1$,
- $L_1(x) = 1-x$,
- $L_2(x) = \frac{x^2}{2} - 2x + 1$,
- $L_3(x) = \frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1$.

The Lauguerre number, is the value of the Lauguerre polynomials at zero argument. Thus,

$$L_m(0) = 1. \tag{2.1}$$

3. Analysis of Approximate Method

In this section, we develop approximate method to obtain the approximate solution of system of linear fractional integro-differential equations using Least square and Lauguerre polynomials.

We define the unknown function $v_i(x)$ as

$$v_i(x) = \sum_{j=0}^{m}\beta_j^i L_j(x), \quad 0 \leq x \leq 1, \tag{3.1}$$

where $L_j(x)$ is Lauguerre polynomial and β^i_j are constants, $i = 1,2,\dots,n$ and $j = 0,1,\dots,m$.

Putting equation (3.1) into equation (1.1), we obtain

$$^cD^{\xi}\left[\sum_{j=0}^{m}\beta^i_jL_j(x)\right] = F_i(x)+\int_0^1 K_i x,t)\left[\sum_{k=1}^{n}\delta_{ik}\left(\sum_{j=0}^{m}\beta^i_jL_j(t)\right)\right]dt.$$

The residual equation obtained is as follows:

$$R_i(x,\beta^i_0,\beta^i_1,\dots,\beta^i_m) = \sum_{j=0}^{m}\beta^i_j\,{}^cD^{\xi}L_j(x)-\int_0^1 K_i(x,t)\left[\sum_{k=1}^{n}\delta_{ik}\left(\sum_{j=0}^{m}\beta^i_jL_j(t)\right)\right]dt-F_i(x).$$

Let

$$S(\beta^i_0,\beta^i_1,\dots,\beta^i_m) = \int_0^1[R_i(x,\beta^i_0,\beta^i_1,\dots,\beta^i_m)]^2W(x)dx,$$

where $W(x)$ is positive weight function defined on $[0,1]$. Assume that $W(x) = 1$. Thus

$$S(\beta^i_0,\beta^i_1,\dots,\beta^i_m) = \int_0^1\left\{\sum_{j=0}^{m}\beta^i_j\,{}^cD^{\xi}L_j(x)-\int_0^1 K_i x,t)\left[\sum_{k=1}^{n}\delta_{ik}\left(\sum_{j=0}^{m}\beta^i_jL_j(t)\right)\right]dt-F_i(x)\right\}^2dx. \quad (3.2)$$

We obtain the value of β^i_j, by finding the minimum value of S_i as:

$$\frac{\partial S_i}{\partial\beta^i_j} = 0, \quad j = 0,1,\dots,m. \quad (3.3)$$

Applying (3.3) on (3.2) we obtain,

$$\int_0^1\left\{\sum_{j=0}^{m}\beta^i_j\,{}^cD^{\xi}L_j(x)-\int_0^1 K_i(x,t)\left[\sum_{k=1}^{n}\delta_{ik}\left(\sum_{j=0}^{m}\beta^i_jL_j(t)\right)\right]dt-F_i(x)\right\}$$
$$*\left\{{}^cD^{\xi}L_j(x)-\int_0^1 K_i(x,t)\left[\sum_{k=1}^{n}\delta_{ik}\left(\sum_{j=0}^{m}L_j(t)\right)\right]dt\right\}dx = 0, \quad (3.4)$$

$$\int_0^1\{[R_i(x,\beta^i_j)-F_i(x)]*H^i_j\}dx = 0, \quad (3.5)$$

where

$$R_i(x,\beta^i_j) = \sum_{j=0}^{m}\beta^i_j\,{}^cD^{\xi}L_j(x)-\int_0^1 K_i(x,t)\left[\sum_{k=1}^{n}\delta_{ik}\sum_{j=0}^{m}\beta^i_jL_j(t)\right]dt,$$

$$H^i_j = {}^cD^{\xi}L_j(x)-\int_0^1 K_i(x,t)\left[\sum_{k=1}^{n}\delta_{ik}\left(\sum_{j=0}^{m}L_j(t)\right)\right]dt.$$

By evaluating the aforementioned equation, we can construct an algebraic system $A\beta^i_j = B$ with unknown coefficients β^i_j, where

$$A = \begin{pmatrix} \int_0^1 R_i(x,\beta^i_0)H^i_0dx & \int_0^1 R_i(x,\beta^i_1)H^i_0dx & \cdots & \int_0^1 R_i(x,\beta^i_m)H^i_0dx \\ \int_0^1 R_i(x,\beta^i_0)H^i_1dx & \int_0^1 R_i(x,\beta^i_1)H^i_1dx & \cdots & \int_0^1 R_i(x,\beta^i_m)H^i_1dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 R_i(x,\beta^i_0)H^i_ndx & \int_0^1 R_i(x,\beta^i_1)H^i_ndx & \cdots & \int_0^1 R_i(x,\beta^i_m)H^i_ndx \end{pmatrix}, \quad B = \begin{pmatrix} \int_0^1 F_i(x)H^i_0dx \\ \int_0^1 F_i(x)H^i_1dx \\ \vdots \\ \int_0^1 F_i(x)H^i_ndx \end{pmatrix}.$$

We find the unknown coefficients and an approximate solution of equation (1.1) by solving the system of equations mentioned above.

The absolute error is given by,

$$\text{Absolute error} = |v_i(x) - v_{im}(x)|, \quad 0 \le x \le 1,$$

where $v_i(x)$ is the exact solution and $v_{im}(x)$ is the approximate solution.

4. Illustrations

For the sake of demonstrating the developed method, we consider the following examples.

Example 4.1. Consider the system of Fredholm fractional integro-differential equations,

$$\left.\begin{aligned} D^{\frac{2}{3}}v_1(x) &= \frac{3x^{\frac{1}{3}}\sqrt{(3)}\Gamma(2/3)}{2\pi} - \frac{x}{6} + \int_0^1 2xt[v_1(t)+v_2(t)]dt, \\ D^{\frac{2}{3}}v_2(x) &= \frac{9x^{\frac{4}{3}}\sqrt{(3)}\Gamma(2/3)}{4\pi} + \frac{5x^3}{6} + \int_0^1 x^3[v_1(t)-v_2(t)]dt \end{aligned}\right\} \tag{4.1}$$

subject to $v_1(0) = -1$, $v_2(0) = 0$.

Firstly, by taking the approximation of the solution $v_i(x)$ with $m = 3$ as,

$$\left.\begin{aligned} v_1(x) &= \sum_{j=0}^{3} \beta_j^1 L_j(x), \quad v_1(t) = \sum_{j=0}^{3} \beta_j^1 L_j(t), \\ v_2(x) &= \sum_{j=0}^{3} \beta_j^2 L_j(x), \quad v_2(t) = \sum_{j=0}^{3} \beta_j^2 L_j(t), \end{aligned}\right\} \tag{4.2}$$

where $L_j(x)$ is the Lauguerre polynomials and β_j^1, β_j^2 are unknown constants.

By substituting equation (4.2) into equation (4.1) we obtain,

$$D^{\frac{2}{3}}\left[\sum_{j=0}^{3}\beta_j^1 L_j(x)\right] = \frac{3x^{\frac{1}{3}}\sqrt{(3)}\Gamma(2/3)}{2\pi} - \frac{x}{6} + \int_0^1 2xt\left[\sum_{i=0}^{3}\beta_j^1 L_j(t) + \sum_{j=0}^{3}\beta_j^2 L_j(t)\right]dt,$$

$$D^{\frac{2}{3}}\left[\sum_{j=0}^{3}\beta_j^2 L_j(x)\right] = \frac{9x^{\frac{4}{3}}\sqrt{(3)}\Gamma(2/3)}{4\pi} + \frac{5x^3}{6} + \int_0^1 x^3\left[\sum_{j=0}^{3}\beta_j^1 L_j(t) - \sum_{j=0}^{3}\beta_j^2 L_j(t)\right]dt.$$

Solving the above equations we get,

$$\beta_0^1[-x] + \beta_1^1\left[-\frac{x^{1/3}}{\Gamma(4/3)} - \frac{x}{3}\right] + \beta_2^1\left[\frac{x^{4/3}}{\Gamma(7/3)} - \frac{2x^{1/3}}{\Gamma(4/3)} + \frac{x}{12}\right] + \beta_3^1\left[-\frac{x^{7/3}}{\Gamma(10/3)} + \frac{3x^{4/3}}{\Gamma(7/3)} - \frac{3x^{1/3}}{\Gamma(4/3)} + \frac{19x}{60}\right]$$

$$+\beta_0^2[-x] + \beta_1^2\left[\frac{-x}{3}\right] + \beta_2^2\left[\frac{x}{12}\right] + \beta_3^2\left[\frac{19x}{60}\right] - \frac{3x^{\frac{1}{3}}\sqrt{(3)}\Gamma(2/3)}{2\pi} + \frac{x}{6} = 0,$$

$$\beta_0^1[-x^3] + \beta_1^1\left[\frac{-x^3}{2}\right] + \beta_2^1\left[-\frac{x^3}{6}\right] + \beta_3^1\left[\frac{x^3}{24}\right] + \beta_0^2[x^3] + \beta_1^2\left[-\frac{x^{1/3}}{\Gamma(4/3)} + \frac{x^3}{2}\right] + \beta_2^2\left[\frac{x^{4/3}}{\Gamma(7/3)} - \frac{2x^{1/3}}{\Gamma(4/3)} + \frac{x^3}{6}\right]$$

$$+\beta_3^2\left[-\frac{x^{7/3}}{\Gamma(10/3)} + \frac{3x^{4/3}}{\Gamma(7/3)} - \frac{3x^{1/3}}{\Gamma(4/3)} - \frac{x^3}{24}\right] - \frac{9x^{\frac{4}{3}}\sqrt{(3)}\Gamma(2/3)}{4\pi} - \frac{5x^3}{6} = 0.$$

Hence, the residual equation is obtained as,

$$R_1(x,\beta_0^1,\beta_1^1,\beta_2^1,\beta_3^1) = \beta_0^1[-x] + \beta_1^1\left[-\frac{x^{1/3}}{\Gamma(4/3)} - \frac{x}{3}\right] + \beta_2^1\left[\frac{x^{4/3}}{\Gamma(7/3)} - \frac{2x^{1/3}}{\Gamma(4/3)} + \frac{x}{12}\right]$$

$$+\beta_3^1\left[-\frac{x^{7/3}}{\Gamma(10/3)} + \frac{3x^{4/3}}{\Gamma(7/3)} - \frac{3x^{1/3}}{\Gamma(4/3)} + \frac{19x}{60}\right] + \beta_0^2[-x] + \beta_1^2\left[\frac{-x}{3}\right]$$

$$+\beta_2^2\left[\frac{x}{12}\right]+\beta_3^2\left[\frac{19x}{60}\right]-\frac{3x^{\frac{1}{3}}\sqrt{(3)}\Gamma(2/3)}{2\pi}+\frac{x}{6}=0,$$

$$R_2(x,\beta_0^2,\beta_1^2,\beta_2^2,\beta_3^2)=\beta_0^1[-x^3]+\beta_1^1\left[\frac{-x^3}{2}\right]+\beta_2^1\left[-\frac{x^3}{6}\right]+\beta_3^1\left[\frac{x^3}{24}\right]+\beta_0^2\left[x^3\right]+\beta_1^2\left[-\frac{x^{1/3}}{\Gamma(4/3)}+\frac{x^3}{2}\right]$$
$$+\beta_2^2\left[\frac{x^{4/3}}{\Gamma(7/3)}-\frac{2x^{1/3}}{\Gamma(4/3)}+\frac{x^3}{6}\right]+\beta_3^2\left[-\frac{x^{7/3}}{\Gamma(10/3)}+\frac{3x^{4/3}}{\Gamma(7/3)}-\frac{3x^{1/3}}{\Gamma(4/3)}-\frac{x^3}{24}\right]$$
$$-\frac{9x^{\frac{4}{3}}\sqrt{(3)}\Gamma(2/3)}{4\pi}-\frac{5x^3}{6}=0.$$

Let

$$\left.\begin{aligned}S(x,\beta_0^1,\beta_1^1,\beta_2^1,\beta_3^1)&=\int_0^1[R_1(x,\beta_0^1,\beta_1^1,\beta_2^1,\beta_3^1)]^2dx,\\ S(x,\beta_0^2,\beta_1^2,\beta_2^2,\beta_3^2)&=\int_0^1[R_2(x,\beta_0^2,\beta_1^2,\beta_2^2,\beta_3^2)]^2dx.\end{aligned}\right\}. \tag{4.3}$$

In order to minimize the value of S, we set it

$$\frac{\partial S}{\partial\beta_j^1}=0,\ \frac{\partial S}{\partial\beta_j^2}=0,\quad j=0,1,2. \tag{4.4}$$

By solving the equation (4.4) we get system of equations having the unknown constants $\beta_0^1,\beta_1^1,\beta_2^1,\beta_3^1$ and $\beta_0^2,\beta_1^2,\beta_2^2,\beta_3^2$.

$$\beta_0^1(0.3333333)+\beta_1^1(0.5910453)+\beta_2^1(0.6801252)+\beta_3^1(0.6614163)$$
$$+\beta_0^2(0.3333333)+\beta_1^2(0.1111111)+\beta_2^2(-0.0277778)+\beta_3^2(-0.1055556)$$
$$=-0.4243787,$$
$$\beta_0^1(0.5910453)+\beta_1^1(1.1094269)+\beta_2^1(1.338878)+\beta_3^1(1.3776177)$$
$$+\beta_0^2(0.5910453)+\beta_1^2(0.1970151)+\beta_2^2(-0.0492538)+\beta_3^2(-0.1871644)$$
$$=-0.8139043,$$
$$\beta_0^1(0.6801252)+\beta_1^1(1.338878)+\beta_2^1(1.6756363)+\beta_3^1(1.7932209)$$
$$+\beta_0^2(0.68012521)+\beta_1^2(0.2267084)+\beta_2^2(-0.0566771)+\beta_3^2(-0.215373)$$
$$=-0.9988154,$$
$$\beta_0^1(-0.1428571)+\beta_1^1(-0.0714286)+\beta_2^1(-0.0238095)+\beta_3^1(0.0059524)$$
$$+\beta_0^2(0.1428571)+\beta_1^2(-0.1869975)+\beta_2^2(-0.3355643)+\beta_3^2(-0.3656298)$$
$$=0.4340045,$$
$$\beta_0^1(0.1869975)+\beta_1^1(0.0934988)+\beta_2^1(0.0311663)+\beta_3^1(-0.0077916)$$
$$+\beta_0^2(-0.1869975)+\beta_1^2(0.5297219)+\beta_2^2(0.941311)+\beta_3^2(1.1370776)$$
$$=-0.7037595,$$
$$\beta_0^1(0.3355643)+\beta_1^1(0.1677821)+\beta_2^1(0.0559274)+\beta_3^1(-0.0139818)$$
$$+\beta_0^2(-0.3355643)+\beta_1^2(0.941311)+\beta_2^2(1.6754823)+\beta_3^2(2.0265438)$$
$$=-1.2531901.$$

Using initial conditions $v_1(0) = 0$, and $v_2(0) = 0$ we obtain,

$$\beta_0^1 + \beta_1^1 + \beta_2^1 + \beta_3^1 = -1,$$
$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 0.$$

Solving the above simultaneous equations we obtain,

$$\beta_0^1 = -0.000004, \quad \beta_0^2 = 1.9999013,$$
$$\beta_1^1 = -1, \quad \beta_1^2 = -3.9996441,$$
$$\beta_2^1 = 0.000001, \quad \beta_2^2 = 1.9995636,$$
$$\beta_3^1 = -0.0000007, \quad \beta_3^2 = 0.0001792.$$

Substituting in equation (4.2) we get the approximate solution:

$$v_1(x) = -0.000004 - (1-x) + 0.000001\left(\frac{x^2}{2} - 2x + 1\right) - 0.0000007\left(\frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1\right),$$

$$v_2(x) = 1.9999013 - 3.9996441(1-x) + 1.9995636\left(\frac{x^2}{2} - 2x + 1\right) + 0.0001792\left(\frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1\right).$$

Table 1 compares the approximate solution to the exact solution and also displays the absolute error. Figure 1 and Figure 2 shows the comparison between the exact and approximate solutions to the system of FIDEs (4.1).

Table 1. Numerical results of Example 4.1

x	Exact solution $v_1(x) = x - 1$	Approximate solution $v_1(x)$	Exact solution $v_2(x) = x^2$	Approximate solution $v_2(x)$	Absolute error $v_1(x)$	Absolute error $v_2(x)$
0.1	−0.9	−0.9000037	0.01	0.0099984	0.0000037	0.0000016
0.2	−0.8	−0.8000037	0.04	0.0399976	0.0000037	0.0000024
0.3	−0.7	−0.7000037	0.09	0.0899975	0.0000037	0.0000025
0.4	−0.6	−0.6000037	0.16	0.1599979	0.0000037	0.0000021
0.5	−0.5	−0.5000038	0.25	0.2499986	0.0000038	0.0000014
0.6	−0.4	−0.4000038	0.36	0.3599993	0.0000038	0.0000007
0.7	−0.3	−0.3000039	0.49	0.4900001	0.0000039	0.0000001
0.8	−0.2	−0.2000039	0.64	0.6400005	0.0000039	0.0000005
0.9	−0.1	−0.100004	0.81	0.8100006	0.00004	0.0000006

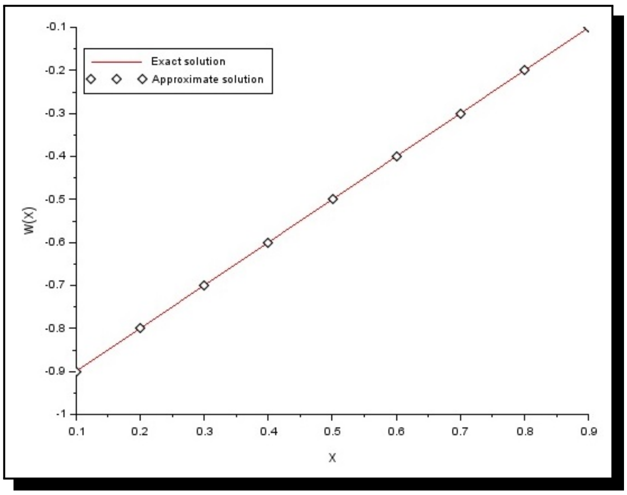

Figure 1. Comparison of Exact and approximate solution of $v_1(x)$

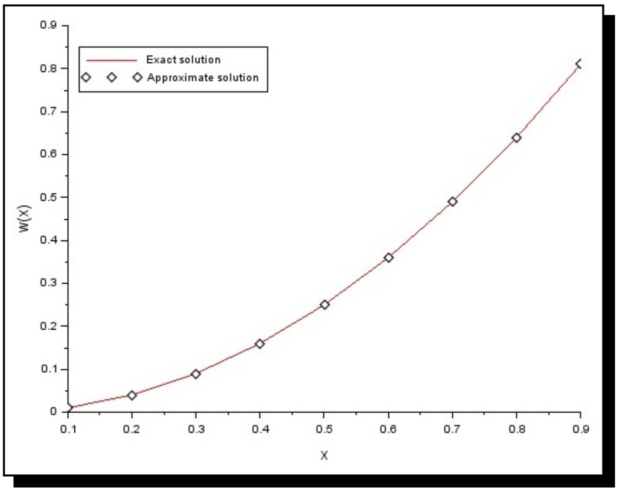

Figure 2. Comparison of Exact and approximate solution of $v_2(x)$

Example 4.2. Consider the system of Fredholm fractional integro-differential equations

$$\left.\begin{aligned} D^{\frac{3}{4}}v_1(x) &= \frac{4x^{\frac{1}{4}}}{\Gamma(1/4)} - \frac{128x^{\frac{9}{4}}}{15\Gamma(1/4)} - \frac{1}{20} - \frac{x}{12} + \int_0^1 (x+t)[v_1(t)+v_2(t)]dt, \\ D^{\frac{3}{4}}v_2(x) &= -\frac{4x^{\frac{1}{4}}}{\Gamma(1/4)} + \frac{32x^{\frac{5}{4}}}{5\Gamma(1/4)} - \frac{2\sqrt(x)}{15} + \int_0^1 \sqrt(x)t^2[v_1(t)-v_2(t)]dt \end{aligned}\right\} \tag{4.5}$$

subject to $v_1(0) = 0$, $v_2(0) = 0$.

Solving the above the equations we get the approximate solution

$$v_1(x) = -4.9986906 + 16.995394(1-x) - 17.994508\left(\frac{x^2}{2} - 2x + 1\right) + 5.9978042\left(\frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1\right),$$

$$v_2(x) = 1.0033553 - 3.0118758(1-x) + 2.0141378\left(\frac{x^2}{2} - 2x + 1\right) - 0.0056173\left(\frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1\right).$$

Table 2 compares the approximate solution to the exact solution and also displays the absolute error. Figure 3 and Figure 4 shows the comparison between the exact and approximate solutions to the system of FIDEs (4.5).

Example 4.3. Consider the system of Fredholm fractional integro-differential equations

$$\left.\begin{aligned} D^{\frac{4}{5}}v_1(x) &= \frac{83x}{80} - \frac{25x^{\frac{6}{5}}}{3\Gamma(1/5)} + \frac{125x^{\frac{11}{5}}}{11\Gamma(1/5)} + \int_0^1 2xt[v_1(t)-v_2(t)]dt, \\ D^{\frac{4}{5}}v_2(x) &= \frac{-67}{160} - \frac{13x}{24} + \frac{125x^{\frac{6}{5}}}{8\Gamma(1/5)} + \int_0^1 (x+t)[v_1(t)+v_2(t)]dt \end{aligned}\right\} \tag{4.6}$$

subject to $v_1(0) = 0$, $v_2(0) = 0$.

Solving the above the equations we get the approximate solution

$$\begin{aligned} v_1(x) = 4.0079097 - 14.028107(1-x) + 16.033595\left(\frac{x^2}{2} - 2x + 1\right) \\ - 6.0133971\left(\frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1\right), \end{aligned}$$

$$v_2(x) = 3.7518711 - 7.5066401(1-x) + 3.7579394\left(\frac{x^2}{2} - 2x + 1\right)$$
$$- 0.0031704\left(\frac{-x^3}{6} + \frac{3x^2}{2} - 3x + 1\right).$$

Table 3 compares the approximate solution to the exact solution and also displays the absolute error. Figure 5 and Figure 6 shows the comparison between the exact and approximate solutions to the system of FIDEs (4.6).

Table 2. Numerical results of Example 4.2

x	Exact solution $v_1(x) = x-1$	Approximate solution $v_1(x)$	Exact solution $v_2(x) = x^2$	Approximate solution $v_2(x)$	Absolute error $v_1(x)$	Absolute error $v_2(x)$
0.1	0.099	0.0990154	−0.09	−0.0899674	0.0000154	0.0000326
0.2	0.192	0.1920225	−0.16	−0.1599564	0.0000225	0.0000436
0.3	0.273	0.273023	−0.21	−0.2099612	0.000023	0.0000388
0.4	0.336	0.3360191	−0.24	−0.2399764	0.0000191	0.0000236
0.5	0.375	0.3750131	−0.25	−0.2499962	0.0000131	0.000038
0.6	0.384	0.3840071	−0.24	−0.2400151	0.0000071	0.0000151
0.7	0.357	0.3570033	−0.21	−0.2100274	0.0000033	0.0000274
0.8	0.288	0.288004	−0.16	−0.1600275	0.000004	0.0000275
0.9	0.171	0.1710112	−0.09	−0.0900098	0.0000112	0.0000098

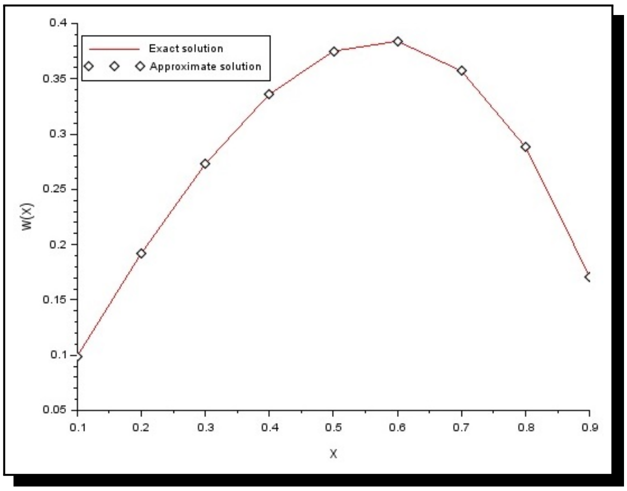

Figure 3. Comparison of exact and approximate solution of $v_1(x)$

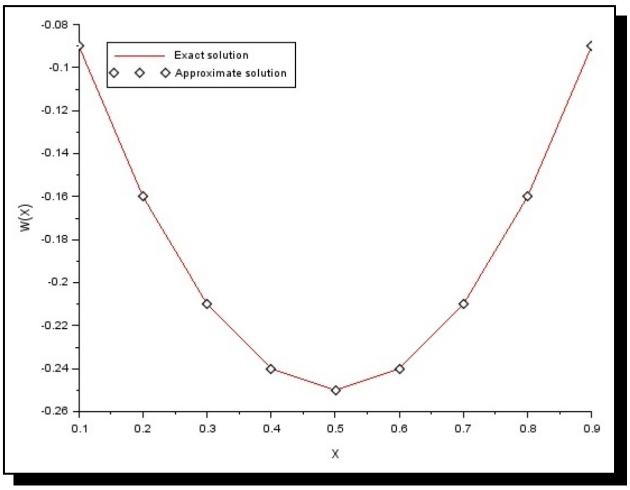

Figure 4. Comparison of exact and approximate solution of $v_2(x)$

Table 3. Numerical results of Example 4.3

x	Exact solution $v_1(x) = x^3 - x^2$	Approximate solution $v_1(x)$	Exact solution $v_2(x) = \frac{x^2}{8}$	Approximate solution $v_2(x)$	Absolute error $v_1(x)$	Absolute error $v_2(x)$
0.1	−0.009	−0.0089193	0.01875	0.0187699	0.0000807	0.0000199
0.2	−0.032	−0.0318918	0.075	0.0750273	0.0001082	0.0000273
0.3	−0.063	−0.0629035	0.16875	0.1687753	0.0000965	0.0000253
0.4	−0.096	−0.0959409	0.3	0.3000171	0.0000591	0.0000171
0.5	−0.125	−0.1249907	0.46875	0.4687558	0.0000093	0.0000058
0.6	−0.144	−0.1440395	0.675	0.6749947	0.0000395	0.0000053
0.7	−0.147	−0.1470738	0.91875	0.9187369	0.0000738	0.0000131
0.8	−0.128	−0.1280804	1.2	1.1999856	0.0000804	0.0000144
0.9	−0.081	−0.0810457	1.51875	1.5187439	0.0000457	0.0000061

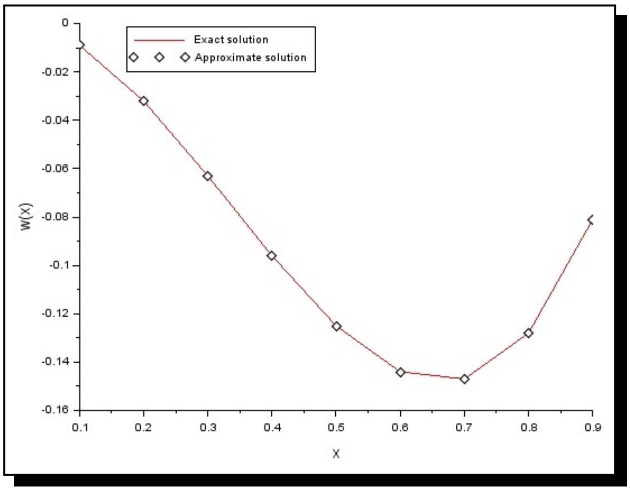

Figure 5. Comparison of exact and approximate solution of $v_1(x)$

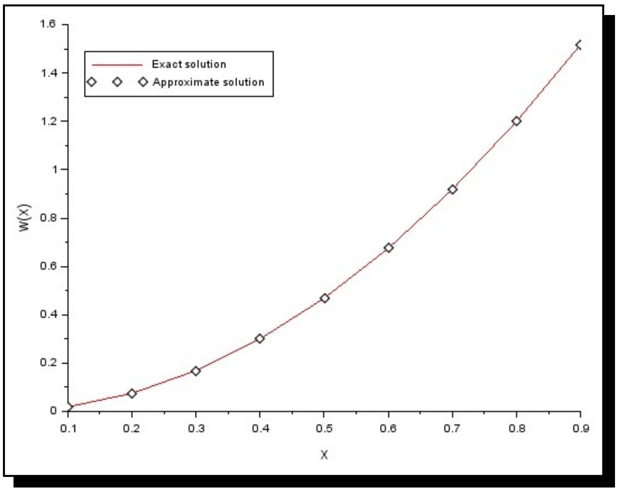

Figure 6. Comparison of exact and approximate solution of $v_2(x)$

5. Conclusion

The least square method with Lauguerre polynomials has been successfully developed to obtain the approximate solution of the system of linear Fredholm FIDEs. The solutions for different values of x are given in tables. The obtained solution for system of Fredholm FIDEs in the examples are simulated using SCILAB-6.0.2 software and compared with the exact solution.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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