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Research Article

Gegenbauer Series for Numerical Solution of Fredholm Integral Equations of the Second Kind

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Abstract. In this work, to solve the integral equation we rely on the technique of orthogonal polynomials, as some authors have shown in the past [11, 12]. However, this work, which simplifies the integral equation in the form of a matrix where it corresponds to a set of linear algebraic equations. Here we rely on the approximation of a series called the Gegenbauer series, which leads us to a rough and effective solution where the error obtained is small compared to the results obtained by some authors.

Keywords. Fredholm integral equations, Gegenbauer polynomials, Numerical solutions

Mathematics Subject Classification (2020). 45B05, 33C45

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1. Introduction

Many mathematical formulas for physical phenomena are based on Fredholm integral equations, so integral equations must be used to find some difficulties in biology or mechanics [3]. He accomplished a lot of work in developing and analyzing numerical methods to solve linear integral equations. The subject of the presented paper is applying the Gegenbauer method for solving linear Fredholm integral equations [3, 4].

2. Problem Formulation

Now consider the following linear Fredholm integral equations of the second kind

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \tag{1}$$

where $u(x)$ are the unknown function, $K(x,t)$ and the function $f(x)$ are given.

The solution of equation (1) is expressed as the truncated Gegenbauer series.

$$u(x) = \sum_{k=0}^{n} C_k G_k, \tag{2}$$

where G_k is the Gegenbauer polynomial and of degree k or in the matrix form [2]

$$[u(x)] = G_x C \tag{3}$$

where

$$G_x = [G_0(x)\ G_1(x)\ \dots\ G_n(x)], \quad C = [C_0\ C_1\ \dots\ C_n],$$

$C_k,\ k = 0,1,\dots,n$ are coefficients to be determined.

3. Method of Solution

To obtain the solution of equation (1) in the form of expression (2) we can first deduce the following matrix approximations corresponding to the Gegenbauer series expansions of the $f(x)$, $k(x,t)$ and $u(t)$.

The function $f(x)$ be approximated by Gegenbauer series

$$f(x) = \sum_{0}^{n} f_k G_k(x). \tag{4}$$

Then, we can put series (4) in the matrix form

$$[f(x)] = G_x F \tag{5}$$

and

$$F = [G_0(x) G_1(x) \dots G_n(x)]^T.$$

We now consider the kernel function $k(x,t)$. If the function $k(x,t)$ can be approximated by double Gegenbauer series of degree n in both x and t of the form [11]

$$k(x,t) = \sum_{k=0}^{n} \sum_{r=0}^{n} K_{k,r} G_k(x) G_r(t). \tag{6}$$

Then, we can put series (6) in the matrix form:

$$[k(x,t)] = G_x K G_t^T, \tag{7}$$

where

$$G_t = [G_0(t)\ G_1(t)\ \dots\ G_n(t)]$$

and

$$k = \begin{pmatrix} k_{00} & \cdots & k_{0n} \\ \vdots & \ddots & \vdots \\ k_{n0} & \cdots & k_{nn} \end{pmatrix}.$$

On the other hand, for the unknown function $u(t)$ in integrand, we write from expression (2) and (3)

$$[u(t)] = G_t C. \tag{8}$$

Substituting the matrix forms (3), (5), (7) and (8) corresponding to the functions $u(x)$, $f(x)$, $k(x,t)$ and $u(t)$, respectively, into equation (1), and the simplifying the result equation we have the matrix equation

$$G_x C = G_x \left[F + \lambda k \int_a^b (G_t^T G_t C) dt \right].$$

Then

$$C = \left[F + \lambda k \int_a^b (G_t^T G_t C) dt \right] = F + \lambda k N C,$$

where

$$N = \int_a^b (G_t^T G_t) dt.$$

Then, we can $C - \lambda k N C = F$. We obtain

$$(I - \lambda k N) C = F, \tag{9}$$

$$N = \int_a^b (G_t^T G_t) dt = [n_{ij}], \quad i,j = 0,1,\dots,n$$

and I is the matrix unit, and the element of the matrix N given by [5,11]

$$n_{ij} = \int_a^b (G_{it} G_{jt}) dt.$$

In equation (9), if $D(\lambda) = |I - \lambda k N| \neq 0$ the matrix inverse exists, we get

$$C = (I - \lambda k N)^{-1} F, \quad \lambda \neq 0. \tag{10}$$

The unknown coefficients C_k, $k = 0,1,\dots,n$ are determined by equation (10), and the integral equation (1) has a unique solution this solution given by the truncated Gegenbauer series.

4. Gegenbauer Polynomials

The Gegenbauer polynomials can be computed via the 10-term recurrence relation

$$(n+1)G(n+1,\alpha,x) = 2(n+\alpha)G(n,\alpha,x) - (n+2\alpha-1)G(n-1,\alpha,x),$$

where the Gegenbauer polynomials are not orthonormal, and the recurrence relation is weakly unstable for $\alpha > 0$, and the instability increases as α increases.

Nothing that, the Gegenbauer polynomials G_n is polynomials with rational coefficients

$$G_0 = 1$$

$$G_1 = 2x$$

$$G_2 = 2x^2 - 1$$

$$G_3 = \frac{8}{3}x^3 - 2x$$

$$G_4 = 4x^4 - 4x^2 + \frac{1}{2}$$

$$G_5 = \frac{32}{5}x^5 - 8x^3 + 2x$$

$$\vdots$$

$$G_{10} = \frac{512}{5}x^{10} - 256x^8 + 224x^6 - 80x^4 + 10x^2 - \frac{1}{5}$$

5. Error Analysis

We can easily check the accuracy of the method. Since the truncated Gegenbauer series in (2) is an approximate solution of equation (1), it must be approximately satisfied this equation.

Then for each $x_i \in [-1.1]$,

$$e(x_i) = \left| u(x_i) - f(x_i) - \lambda \int_{-1}^{1} k(x,t)u(t)dt \right| \to 0$$

is prescribed, then the truncation limit n is increased until the difference $e(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}. On the other hand, the error function can be estimated by

$$e(x) = \left| u(x) - f(x) - \lambda \int_{-1}^{1} k(x,t)u(t)dt \right|.$$

6. Illustrating Examples

For numerical verification of the above method we consider the following examples:

Example 6.1. We consider the following Fredholm integral equation

$$u(x) = 2 - e^x + \int_{-1}^{1} e^{-t}u(t)dt,$$

where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = -e^x$.

Table 1 shows the exact and the approximate solutions of the equation in Example 6.1 in some arbitrary points, the error for $n = 8$ is calculated and compared with the example treated in [12].

Table 1

x	Exact Sol.	Approx. Sol.	Error G_n	Error in [12]
0	−1.000	−0.999	7.493e-7	7.4e-7
0.1	−1.105	−1.105	7.493e-7	
0.2	−1.221	−1.221	7.493e-7	7.4e-7
0.3	−1.350	−1.350	7.494e-7	
0.4	−1.492	−1.492	7.501e-7	7.5e-7
0.5	−1.649	−1.649	7.550e-7	
0.6	−1.822	−1.822	7.788e-7	7.7e-7
0.7	−2.014	−2.014	8.688e-7	
0.8	−2.225	−2.225	1.151e-6	1.1e-6
0.9	−2.460	−2.460	1.921e-6	
1	−2.718	−2.718	3.808e-6	3.8e-6

Example 6.2. We consider the following Fredholm integral equation

$$u(x) = -x + e^{2x} + \int_0^1 xe^{-2t}u(t)dt,$$

where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = e^{2x}$.

Table 2 shows the exact and the approximate solutions of the equation in Example 6.2 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in [13].

Table 2

x	Exact Sol.	Approx. Sol.	Error G_n	Error in [13]
0	1.000	1.000	2.220e-16	0.00000000000
0.1	1.221	1.221	2.681e-6	2.68139265e-6
0.2	1.492	1.492	5.363e-6	5.362784215e-6
0.3	1.822	1.822	8.044e-6	8.044082301e-6
0.4	2.226	2.226	1.072e-5	1.072326582e-5
0.5	2.718	2.718	1.338e-5	1.337965059e-5
0.6	3.320	3.320	1.588e-5	1.58817245e-5
0.7	4.055	4.055	1.762e-5	1.762273736e-5
0.8	4.953	4.953	1.637e-5	1.637473125e-5
0.9	6.050	6.050	5.234e-6	5.233654626e-6
1	7.389	7.389	3.458e-5	3.457600943e-5

Example 6.3. We consider the following Fredholm integral equation

$$u(x) = e^{(x+2)} - 2\int_0^1 e^{(x+t)}u(t)dt$$

where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = e^x$.

Table 3 shows the exact and the approximate solutions in [1, 10] of the equation in Example 6.3 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in [1, 10].

Table 3

x	Exact Sol.	Approx. Sol.	Error in [1, 10]	Error G_n
0	1.000	1.000	0.999999998	7.493e-7
0.1	1.105	1.105	1.105170916	7.493e-7
0.2	1.221	1.221	1.221402756	7.493e-7
0.3	1.350	1.350	1.349858803	7.494e-7
0.4	1.492	1.492	1.491824693	7.501e-7
0.5	1.649	1.649	1.648721264	7.550e-7
0.6	1.822	1.822	1.822118793	7.788e-7
0.7	2.014	2.014	2.013752696	8.688e-7
0.8	2.225	2.225	2.225540911	1.151e-6
0.9	2.460	2.460	2.459603079	1.921e-6
1	2.718	2.718	2.718281765	3.808e-6

Example 6.4. We consider the following Fredholm integral equation

$$u(x) = e^x - 1 + \int_0^1 e^{(-t)} u(t) dt$$

where the function $f(x)$ is chosen so that the exact solution is given by $u(x) = e^x$.

Table 4 shows the exact and the approximate solutions of the equation in Example 4 in some arbitrary points, the error for $n = 10$ is calculated and compared with the example treated in [9].

Table 4

x	Exact Sol.	Approx. Sol.	Error G_n	Error in [9]
0	1.000	1.000	1.794e-6	0.210E-5
0.1	1.105	1.105	1.794e-6	0.200E-5
0.2	1.221	1.221	1.794e-6	0.200E-5
0.3	1.350	1.350	1.794e-6	0.300E-5
0.4	1.492	1.492	1.794e-6	0.300E-5
0.5	1.649	1.649	1.797e-6	0.200E-5
0.6	1.822	1.822	1.809e-6	0.200E-5
0.7	2.014	2.014	1.864e-6	0.200E-5
0.8	2.225	2.225	2.058e-6	0.100E-5
0.9	2.460	2.460	2.651e-6	0.100E-5
1	2.718	2.718	4.248e-6	0.000000

7. Conclusion

We used the Gegenbauer polynomial method to find the approximate solution of the Fredholm linear integral equation of the second type. Polynomials are widely used in roads. We noticed that the degree the n polynomial score, the better the results and the effective method and this is what we have seen in the examples we dealt with where the second example was compared with [12]. We can also see that good rounding. When comparing the approximate solution with the exact solution, the polynomial Gegenbauer method is efficient and convenient compared to some methods [9, 10, 12, 13].

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] A. Altürk, Numerical solution of linear and nonlinear Fredholm integral equations by using weighted mean-value theorem, *SpringerPlus* **5** (2016), Article number: 1962, DOI: 10.1186/s40064-016-3645-8.

[2] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 6th Edition, Elsevier Academic Press, London (2005), URL: https://msashigri.files.wordpress.com/2016/11/methods-of-mathemacial-for-physicists.pdf.

[3] E. Babolian, H. R. Marzban and M. Salmani, Using triangular orthogonal functions for solving Fredholm integral equations of the second kind, *Applied Mathematics and Computation* **201**(1-2) (2008), 452 – 464, DOI: 10.1016/j.amc.2007.12.034.

[4] N. Bildik, A. Konuralp and S. Yalçınbaş, Comparison of Legendre polynomial approximation and variational iteration method for the solutions of general linear Fredholm integro-differential equations, *Computers & Mathematics with Applications* **59** (2010), 1909 – 1917, DOI: 10.1016/j.camwa.2009.06.022.

[5] L. Fox and I. Parker, *Chebyshev Polynomials in Numerical Analysis*, Cambridge University Press, ix + 205 pages (1968), DOI: 10.2307/3613223.

[6] R. P. Kanwal and K. C. Liu, A Taylor expansion approach for solving integral equations, *International Journal of Mathematical Education in Science and Technology* **20** (1989), 411 – 414, DOI: 10.1080/0020739890200310.

[7] K. Maleknejad and M. Karami, Using the WPG method for solving integral equations of the second kind, *Applied Mathematics and Computation* **166** (2005), 123 – 130, DOI: 10.1016/j.amc.2004.04.109.

[8] K. Maleknejad, K. Nouri and M. Yousefi, Discussion on convergence of Legendre polynomial for numerical solution of integral equations, *Applied Mathematics and Computation* **193**(1) (2007), 335 – 339, DOI: 10.1016/j.amc.2007.03.062.

[9] G. Mustafa and O. Yalçın, Chebyshev polynomial approximation for solving the second kind linear fredholm integral equation, *Journal of the Institute of Science and Technology* **26**(3) (2010), 203 – 216.

[10] L. P. Alpha, Pell-Lucas series solution for Fredholm integral equations of the second kind, *International Journal of Science and Research* **10**(6) (2021), 1376 – 1383, DOI: 10.21275/mr21617222711.

[11] M. Sezer and S. Dogan, Chebyshev series solutions of Fredholm integral equations, *International Journal of Mathematical Education in Science and Technology* **27** (1996), 649 – 657, DOI: 10.1080/0020739960270503.

[12] S. Yalçınbaş and M. Aynigül, Hermite series solutions of Fredholm integral equations, *Mathematical and Computational Applications* **16**(2) (2011), 497 – 506, DOI: 10.3390/mca16020497.

[13] S. Yalcinbas, M. Aynigul and T. Akkaya Legendre series solutions of Fredholm integral equations, *Mathematical and Computational Applications* **15**(3) (2010), 371 – 381, DOI: 10.3390/mca15030371.