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Research Article

Bipolar Valued Multi Fuzzy Normal Subnear-Ring of a Near-Ring

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Abstract. In this paper, bipolar valued multi fuzzy normal subnear-ring of a near-ring is introduced and some theorems are stated and proved.

Keywords. Bipolar valued fuzzy subset, Bipolar valued multi fuzzy subset, Bipolar valued multi fuzzy subnear-ring, Bipolar valued multi fuzzy normal subnear-ring, Product and strongest bipolar valued multi fuzzy subnear-ring

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1. Introduction

In 1965, Zadeh [9] introduced the notion of a fuzzy subset of a universal set. Zhang [10, 11] introduced an extension of fuzzy sets named bipolar valued fuzzy sets in 1994 and bipolar valued fuzzy set was developed by Lee [3, 4]. Bipolar valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [4]. Anitha *et al*. [1, 2] introduced the bipolar valued fuzzy subgroup. Shyamala and Shanthi [7] have introduced the bipolar valued multi fuzzy subgroups of a group. Yasodara and Sathappan [8] defined the bipolar valued multi fuzzy subsemirings of a semiring. Bipolar valued multi fuzzy subnear-ring of a near-ring has been introduced by Muthukumaran and Anandh [5]. In this paper, the concept of bipolar valued multi fuzzy normal subnear-ring of a near-ring is introduced and established some results.

2. Preliminaries

Definition 2.1 ([10]). A bipolar valued fuzzy set (BVFS) B in X is defined as an object of the form $B = \{\langle x, B^+(u), B^-(u)\rangle / x \in X\}$, where $B^+ : X \to [0,1]$ and $B^- : X \to [-1,0]$. The positive membership degree $B^+(u)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set B and the negative membership degree $B^-(u)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued fuzzy set B.

Definition 2.2 ([8]). A bipolar valued multi fuzzy set (BVMFS) A in X is defined as an object of the form $B = \{\langle x, B_1^+(u), B_2^+(u), \dots, B_n^+(u), B_1^-(u), B_2^-(u), \dots, B_n^-(u)\rangle / x \in X\}$, where $B_i^+ : X \to [0,1]$ and $B_i^- : X \to [-1,0]$, for all i. The positive membership degrees $B_i^+(u)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar valued multi fuzzy set B and the negative membership degrees $B_i^-(u)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar valued multi fuzzy set B.

Definition 2.3. Let $(N, +, \cdot)$ be a near-ring. A BVMFS B of N is said to be a bipolar valued multi fuzzy subnear-ring of N (BVMFSNR) if the following conditions are satisfied, for all i,

(i) $B_i^+(u - v) \geq \min\{B_i^+(u), B_i^+(v)\}$,

(ii) $B_i^+(uv) \geq \min\{B_i^+(u), B_i^+(v)\}$,

(iii) $B_i^-(u - v) \leq \max\{B_i^-(u), B_i^-(v)\}$,

(iv) $B_i^-(uv) \leq \max\{B_i^-(u), B_i^-(v)\}$, for all $u, v \in N$.

Definition 2.4. Let R be a near-ring. A bipolar valued multi fuzzy subnear-ring A of R is said to be a bipolar valued multi fuzzy normal subnear-ring (BVMFNSNR) of R if $A_i^+(x+y) = A_i^+(y+x)$, $A_i^-(x+y) = A_i^-(y+x)$, $A_i^+(xy) = A_i^+(yx)$ and $A_i^-(xy) = A_i^-(yx)$, for all x, y in R and for all i.

Definition 2.5 ([8]). Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^-\rangle$ and $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^-\rangle$ be any two bipolar valued multi fuzzy subsets of sets R and H, respectively. The product of A and B, denoted by $A \times B$, is defined as $A \times B = \{\langle (x,y), (A_1 \times B_1)^+(x,y), (A_2 \times B_2)^+(x,y), \dots, (A_n \times B_n)^+(x,y), (A_1 \times B_1)^-(x,y), (A_2 \times B_2)^-(x,y), \dots, (A_n \times B_n)^-(x,y)\rangle$, for all $x \in R$ and $y \in H\}$, where $(A_i \times B_i)^+(x,y) = \min\{A_i^+(x), B_i^+(y)\}$ and $(A_i \times B_i)^-(x,y) = \max\{A_i^-(x), B_i^-(y)\}$, for all i.

Definition 2.6 ([8]). Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^-\rangle$ be a bipolar valued multi fuzzy subset in a set S, the strongest bipolar valued multi fuzzy relation on S, that is a bipolar valued multi fuzzy relation on A is

$$V = \{\langle (x,y), V_1^+(x,y), V_2^+(x,y), \dots, V_n^+(x,y), V_1^-(x,y), V_2^-(x,y), \dots, V_n^-(x,y)\rangle / x, y \in S\},$$

where $V_i^+(x,y) = \min\{A_i^+(x), A_i^+(y)\}$ and $V_i^-(x,y) = \max\{A_i^-(x), A_i^-(y)\}$, for all $x, y \in S$, and for all i.

Definition 2.7 ([5]). Let $A = \langle A_1^+, A_2^+, \dots, A_n^+, A_1^-, A_2^-, \dots, A_n^-\rangle$ be a bipolar valued multi fuzzy subnear-ring of a near-ring R and $a \in R$. Then the pseudo bipolar valued multi fuzzy coset $(aA)^p = \langle (aA_1^+)^{p_1^+}, (aA_2^+)^{p_2^+}, \dots, (aA_n^+)^{p_n^+}, (aA_1^-)^{p_1^-}, (aA_2^-)^{p_2^-}, \dots, (aA_n^-)^{p_n^-}\rangle$ is defined by $(aA_i^+)^{p_i^+}(x) = p_i^+(a)A_i^+(x)$ and $(aA_i^-)^{p_i^-}(x) = -p_i^-(a)A_i^-(x)$, for all i, for every $x \in R$ and $p \in P$, where P is a collection of bipolar valued multi fuzzy subsets of R.

Definition 2.8. Let $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ be a bipolar valued multi fuzzy normal subnear-ring of a near-ring R. For any $u, v \in R$, define a binary relation $\sim$ on R by $u \sim v \Leftrightarrow B_i^+(u - v) = B_i^+(o)$ and $B_i^-(u - v) = B_i^-(o)$, for all i, where o is the identity element of R. The equivalence class containing u is denoted by B_u. G/B denotes the corresponding quotient set.

3. Properties

Theorem 3.1. *If $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \dots, C_n^+, C_1^-, C_2^-, \dots, C_n^- \rangle$ are two BVMFSNRs with degree n of a near-ring $(N, +, \cdot)$, then their intersection $B \cap C$ is a BVMFSNR of N.*

Theorem 3.2. *If $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \dots, C_n^+, C_1^-, C_2^-, \dots, C_n^- \rangle$ are two BVMFNSNRs with degree n of a near-ring $(N, +, \cdot)$, then their intersection $B \cap C$ is a BVMFNSNR of N.*

Proof. Let $D = B \cap C$. By Theorem 3.1, D is a BVMFSNR of the near-ring N. Let $u, v \in N$. For all i, $D_i^+(u + v) = \min\{B_i^+(u + v), C_i^+(u + v)\} = \min\{B_i^+(v + u), C_i^+(v + u)\} = D_i^+(v + u)$, $\forall$ $u, v \in N$, and $D_i^+(uv) = \min\{B_i^+(uv), C_i^+(uv)\} = \min\{B_i^+(vu), C_i^+(vu)\} = D_i^+(vu)$, $\forall$ $u, v \in N$. Also, $D_i^-(u + v) = \max\{B_i^-(u + v), C_i^-(u + v)\} = \max\{B_i^-(v + u), C_i^-(v + u)\} = D_i^-(v + u)$, $\forall$ $u, v \in N$, and $D_i^-(uv) = \max\{B_i^-(uv), C_i^-(uv)\} = \max\{B_i^-(vu), C_i^-(vu)\} = D_i^-(vu)$, $\forall$ $u, v \in N$. Hence $B \cap C$ is a BVMFNSNR of the near-ring N. □

Theorem 3.3. *The intersection of a family of BVMFNSNRs with degree n of a near-ring $(N, +, \cdot)$ is a BVMFNSNR of N.*

Proof. The proof follows from Theorem 3.3. □

Theorem 3.4 ([5]). *If $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \dots, C_n^+, C_1^-, C_2^-, \dots, C_i^- \rangle$ are any two BVMFSNRs with degree n of the near-rings R_1 and R_2 respectively, then $B \times C$ is a BVMFSNR of $R_1 \times R_2$.*

Theorem 3.5. *If $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \dots, C_n^+, C_1^-, C_2^-, \dots, C_i^- \rangle$ are any two BVMFNSNRs with degree n of the near-rings R_1 and R_2 respectively, then $B \times C$ is a BVMFNSNR of $R_1 \times R_2$.*

Proof. Let $u_1, u_2 \in R_1$ and $v_1, v_2 \in R_2$. That is $(u_1, v_1), (u_2, v_2) \in R_1 \times R_2$. By Theorem 3.4, $B \times C$ is a BVMFSNR of the near-ring $R_1 \times R_2$. For all i, $(B_i \times C_i)^+[(u_1, v_1) + (u_2, v_2)] = (B_i \times C_i)^+(u_1 + u_2, v_1 + v_2) = \min\{B_i^+(u_1 + u_2), C_i^+(v_1 + v_2)\} = \min\{B_i^+(u_2 + u_1), C_i^+(v_2 + v_1)\} = (B_i \times C_i)^+(u_2 + u_1, v_2 + v_1) = (B_i \times C_i)^+[(u_2, v_2) + (u_1, v_1)]$, for all $(u_1, v_1), (u_2, v_2) \in R_1 \times R_2$, and $(B_i \times C_i)^+[(u_1, v_1)(u_2, v_2)] = (B_i \times C_i)^+(u_1u_2, v_1v_2) = \min\{B_i^+(u_1u_2), C_i^+(v_1v_2)\} = \min\{B_i^+(u_2u_1), C_i^+(v_2v_1)\} = (B_i \times C_i)^+(u_2u_1, v_2v_1) = (B_i \times C_i)^+[(u_2, v_2)(u_1, v_1)]$, for all $(u_1, v_1), (u_2, v_2) \in R_1 \times R_2$, also $(B_i \times C_i)^-[(u_1, v_1) + (u_2, v_2)] = (B_i \times C_i)^-(u_1 + u_2, v_1 + v_2) = \max\{B_i^-(u_1 + u_2), C_i^-(v_1 + v_2)\} = \max\{B_i^-(u_2 + u_1), C_i^-(v_2 + v_1)\} = (B_i \times C_i)^-(u_2 + u_1, v_2 + v_1) = (B_i \times C_i)^-[(u_2, v_2) + (u_1, v_1)]$, for all $(u_1, v_1), (u_2, v_2) \in R_1 \times R_2$, and $(B_i \times C_i)^-[(u_1, v_1)(u_2, v_2)] = (B_i \times C_i)^-(u_1u_2, v_1v_2) = \max\{B_i^-(u_1u_2), C_i^-(v_1v_2)\} = \max\{B_i^-(u_2u_1), C_i^-(v_2v_1)\} = (B_i \times C_i)^-(u_2u_1, v_2v_1) = (B_i \times C_i)^-[(u_2, v_2)(u_1, v_1)]$, for all $(u_1, v_1), (u_2, v_2) \in R_1 \times R_2$. Hence $B \times C$ is a BVMFNSNR of the near-ring $R_1 \times R_2$. □

Theorem 3.6 ([5]). *Let $B = \langle B_1^+, B_2^+, \dots, B_i^+, B_1^-, B_2^-, \dots, B_i^-\rangle$ be a BVMFS of a near-ring R and $M = \langle M_1^+, M_2^+, \dots, M_i^+, M_1^-, M_2^-, \dots, M_i^-\rangle$ be the strongest bipolar valued multi fuzzy relation of R. Then B is a BVMFSNR of R if and only if M is a BVMFSNR of $R\times R$.*

Theorem 3.7. *Let $B = \langle B_1^+, B_2^+, \dots, B_i^+, B_1^-, B_2^-, \dots, B_i^-\rangle$ be a BVMFS of a near-ring R and $M = \langle M_1^+, M_2^+, \dots, M_i^+, M_1^-, M_2^-, \dots, M_i^-\rangle$ be the strongest bipolar valued multi fuzzy relation of R. Then B is a BVMFNSNR of R if and only if M is a BVMFNSNR of $R\times R$.*

Proof. By Theorem 3.6, B is a BVMFSNR of R if and only if M is a BVMFSNR of $R\times R$. Let $u = (u_1, u_2), v = (v_1, v_2) \in R\times R$. For all i, $M_i^+(u+v) = M_i^+[(u_1,u_2)+(v_1,v_2)] = M_i^+(u_1+v_1, u_2+v_2) = \min\{B_i^+(u_1+v_1), B_i^+(u_2+v_2)\} = \min\{B_i^+(v_1+u_1), B_i^+(v_2+u_2)\} = M_i^+(v_1+u_1, v_2+u_2) = M_i^+[(v_1,v_2)+(u_1,u_2)] = M_i^+(v+u)$, for all $u,v\in R\times R$, and $M_i^+(uv) = M_i^+[(u_1,u_2)(v_1,v_2)] = M_i^+(u_1v_1, u_2v_2) = \min\{B_i^+(u_1v_1), B_i^+(u_2v_2)\} = \min\{B_i^+(v_1u_1), B_i^+(v_2u_2)\} = M_i^+(v_1u_1, v_2u_2) = M_i^+[(v_1,v_2)(u_1,u_2)] = M_i^+(vu)$, for all $u,v\in R\times R$. Also, $M_i^-(u+v) = M_i^-[(u_1,u_2)+(v_1,v_2)] = M_i^-(u_1+v_1, u_2+v_2) = \max\{B_i^-(u_1+v_1), B_i^-(u_2+v_2)\} = \max\{B_i^-(v_1+u_1), B_i^-(v_2+u_2)\} = M_i^-(v_1+u_1, v_2+u_2) = M_i^-[(v_1,v_2)+(u_1,u_2)] = M_i^-(v+u)$, for all $u,v \in R\times R$, and $M_i^-(uv) = M_i^-[(u_1,u_2)(v_1,v_2)] = M_i^-(u_1v_1, u_2v_2) = \max\{B_i^-(u_1v_1), B_i^-(u_2v_2)\} = \max\{B_i^-(v_1u_1), B_i^-(v_2u_2)\} = M_i^-(v_1u_1, v_2u_2) = M_i^-[(v_1,v_2)(u_1,u_2)] = M_i^-(vu)$, for all $u,v\in R\times R$. Hence M is a BVMFNSNR of $R\times R$. Conversely, assume that M is a BVMFNSNR of $R\times R$. For all i, $\min\{B_i^+(u_1+v_1), B_i^+(u_2+v_2)\} = M_i^+(u_1+v_1, u_2+v_2) = M_i^+[(u_1,u_2)+(v_1,v_2)] = M_i^+(u+v) = M_i^+(v+u) = M_i^+[(v_1,v_2)+(u_1,u_2)] = M_i^+(v_1+u_1, v_2+u_2) = \min\{B_i^+(v_1+u_1), B_i^+(v_2+u_2)\}$. Put $u_2 = o$ and $v_2 = o$, where o is the identity element of R, then $B_i^+(u_1+v_1) = B_i^+(v_1+u_1)$, for all $u_1, v_1\in R$, and $\min\{B_i^+(u_1v_1), B_i^+(u_2v_2)\} = M_i^+(u_1v_1, u_2v_2) = M_i^+[(u_1,u_2)(v_1,v_2)] = M_i^+(uv) = M_i^+(vu) = M_i^+[(v_1,v_2)(u_1,u_2)] = M_i^+(v_1u_1, v_2u_2) = \min\{B_i^+(v_1u_1), B_i^+(v_2u_2)\}$. Put $u_2 = o$ and $v_2 = o$, where o is the identity element of R, then $B_i^+(u_1v_1) = B_i^+(v_1u_1)$, for all $u_1, v_1\in R$. Also, $\max\{B_i^-(u_1+v_1), B_i^-(u_2+v_2)\} = M_i^-(u_1+v_1, u_2+v_2) = M_i^-[(u_1,u_2)+(v_1,v_2)] = M_i^-(u+v) = M_i^-(v+u) = M_i^-[(v_1,v_2)+(u_1,u_2)] = M_i^-(v_1+u_1, v_2+u_2) = \max\{B_i^-(v_1+u_1), B_i^-(v_2+u_2)\}$. Put $u_2 = o$ and $v_2 = o$, where o is the identity element of R, then $B_i^-(u_1+v_1) = B_i^-(v_1+u_1)$, for all $u_1, v_1\in R$, and $\max\{B_i^-(u_1v_1), B_i^-(u_2v_2)\} = M_i^-(u_1v_1, u_2v_2) = M_i^-[(u_1,u_2)(v_1,v_2)] = M_i^-(uv) = M_i^-(vu) = M_i^-[(v_1,v_2)(u_1,u_2)] = M_i^-(v_1u_1, v_2u_2) = \max\{B_i^-(v_1u_1), B_i^-(v_2u_2)\}$. Put $u_2 = o$ and $v_2 = o$, where o is the identity element of R, then $B_i^-(u_1v_1) = B_i^-(v_1u_1)$, for all $u_1, v_1\in R$. Hence B is a BVMFNSNR of R. □

Theorem 3.8 ([5]). *Let $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^-\rangle$ be a BVMFSNR of a near-ring R. Then the pseudo bipolar valued multi fuzzy coset $(aB)^p$ is a BVMFSNR of R, for every $a\in R$ and $p\in P$.*

Theorem 3.9. *Let $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^-\rangle$ be a BVMFNSNR of a near-ring R. Then the pseudo bipolar valued multi fuzzy coset $(aB)^p$ is a BVMFNSNR of R, for every $a\in R$ and $p\in P$.*

Proof. By Theorem 3.8, the pseudo bipolar valued multi fuzzy coset $(aB)^p$ is a BVMFSNR of R, for every $a\in R$ and $p\in P$. Let $u,v\in R$. For all i, $(aB_i^+)^{p_i^+}(u+v) = p_i^+(a)B_i^+(u+v) = p_i^+(a)B_i^+(v+u) = (aB_i^+)^{p_i^+}(v+u)$, for all $u,v\in R$, and $(aB_i^+)^{p_i^+}(uv) = p_i^+(a)B_i^+(uv) = p_i^+(a)B_i^+(vu) = (aB_i^+)^{p_i^+}(vu)$, for all $u,v\in R$. Also $(aB_i^-)^{p_i^-}(u+v) = p_i^-(a)B_i^-(u+v) = p_i^-(a)B_i^-(v+u) = (aB_i^-)^{p_i^-}(v+u)$, for all

$u,v \in R$, and $(aB_i^-)^{p_i^-}(uv) = p_i^-(a)B_i^-(uv) = p_i^-(a)B_i^-(vu) = (aB_i^-)^{p_i^-}(vu)$, for all $u,v \in R$. Hence the pseudo bipolar valued multi fuzzy coset $(aB)^p$ is a BVMFNSNR of R, for every $a \in R$ and $p \in P$. □

Theorem 3.10 ([5]). *Let $B = \langle B_1^+, B_2^+, \ldots, B_n^+, B_1^-, B_2^-, \ldots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \ldots, C_n^+, C_1^-, C_2^-, \ldots, C_n^- \rangle$ be any two bipolar valued multi fuzzy subsets of the near-rings R and H, respectively and $B \times C$ be a BVMFSNR of $R \times H$. Then the following are true:*

(i) *if $B_i^+(u) \le C_i^+(o')$, $\forall\ u \in R$ and $B_i^-(u) \ge C_i^-(o')$, $\forall\ u \in R$, then B is a BVMFSNR of R, where o' is the identity element of H.*

(ii) *if $C_i^+(u) \le B_i^+(o)$, $\forall\ u \in H$ and $C_i^-(u) \ge B_i^-(o)$, $\forall\ u \in H$, then C is a BVMFSNR of H where o is the identity element of R.*

(iii) *either B is a BVMFSNR of R or C is a BVMFSNR of H.*

Theorem 3.11. *Let $B = \langle B_1^+, B_2^+, \ldots, B_n^+, B_1^-, B_2^-, \ldots, B_n^- \rangle$ and $C = \langle C_1^+, C_2^+, \ldots, C_n^+, C_1^-, C_2^-, \ldots, C_n^- \rangle$ be any two bipolar valued multi fuzzy subsets of the near-rings R and H, respectively and $B \times C$ be a BVMFNSNR of $R \times H$. Then the following are true:*

(i) *if $B_i^+(u) \le C_i^+(o')$, $\forall\ u \in R$ and $B_i^-(u) \ge C_i^-(o')$, $\forall\ u \in R$, then B is a BVMFNSNR of R, where o' is the identity element of H.*

(ii) *if $C_i^+(u) \le B_i^+(o)$, $\forall\ u \in H$ and $C_i^-(u) \ge B_i^-(o)$, $\forall\ u \in H$, then C is a BVMFNSNR of H where o is the identity element of R.*

(iii) *either B is a BVMFNSNR of R or C is a BVMFNSNR of H.*

Proof. (i) By Theorem 3.10, B is a BVMFSNR of R. Let $u,v \in R$. That is $(u,o'),(v,o') \in R \times H$. For all i, using $B_i^+(u) \le C_i^+(o')$, $\forall\ u \in R$ and $B_i^-(u) \ge C_i^-(o')$, $\forall\ u \in R$, then $B_i^+(u+v) = \min\{B_i^+(u+v), C_i^+(o'+o')\} = (B_i \times C_i)^+((u+v),(o'+o')) = (B_i \times C_i)^+[(u,o')+(v,o')] = (B_i \times C_i)^+[(v,o')+(u,o')] = (B_i \times C_i)^+((v+u),(o'+o')) = \min\{B_i^+(v+u), C_i^+(o'+o')\} = B_i^+(v+u)$, for all $u,v \in R$, and $B_i^+(uv) = \min\{B_i^+(uv), C_i^+(o'o')\} = (B_i \times C_i)^+((uv),(o'o')) = (B_i \times C_i)^+[(u,o')(v,o')] = (B_i \times C_i)^+[(v,o')(u,o')] = (B_i \times C_i)^+((vu),(o'o')) = \min\{B_i^+(vu), C_i^+(o'o')\} = B_i^+(vu)$, for all $u,v \in R$. Also, $B_i^-(u+v) = \max\{B_i^-(u+v), C_i^-(o'+o')\} = (B_i \times C_i)^-((u+v),(o'+o')) = (B_i \times C_i)^-[(u,o')+(v,o')] = (B_i \times C_i)^-[(v,o')+(u,o')] = (B_i \times C_i)^-((v+u),(o'+o')) = \max\{B_i^-(v+u), C_i^-(o'+o')\} = B_i^-(v+u)$, for all $u,v \in R$, and $B_i^-(uv) = \max\{B_i^-(uv), C_i^-(o'o')\} = (B_i \times C_i)^-((uv),(o'o')) = (B_i \times C_i)^-[(u,o')(v,o')] = (B_i \times C_i)^-[(v,o')(u,o')] = (B_i \times C_i)^-((vu),(o'o')) = \max\{B_i^-(vu), C_i^-(o'o')\} = B_i^-(vu)$, for all $u,v \in R$. Hence B is a BVMFNSNR of R.

(ii) By Theorem 3.10, C is a BVMFSNR of H. For all i, using $C_i^+(u) \le B_i^+(o)$, $\forall\ u \in H$ and $C_i^-(u) \ge B_i^-(o)$, $\forall\ u \in H$, then $C_i^+(u+v) = \min\{C_i^+(u+v), B_i^+(o+o)\} = (B_i \times C_i)^+((o+o),(u+v)) = (B_i \times C_i)^+[(o,u)+(o,v)] = (B_i \times C_i)^+[(o,v)+(o,u)] = (B_i \times C_i)^+((o+o),(v+u)) = \min\{C_i^+(v+u), B_i^+(o+o)\} = C_i^+(v+u)$, for all $u,v \in H$, and $C_i^+(uv) = \min\{C_i^+(uv), B_i^+(oo)\} = (B_i \times C_i)^+((oo),(uv)) = (B_i \times C_i)^+[(o,u)(o,v)] = (B_i \times C_i)^+[(o,v)(o,u)] = (B_i \times C_i)^+((oo),(vu)) = \min\{C_i^+(vu), B_i^+(oo)\} = C_i^+(vu)$, for all $u,v \in H$. Also, $C_i^-(u+v) = \max\{C_i^-(u+v), B_i^-(o+o)\} = (B_i \times C_i)^-((o+o),(u+v)) = (B_i \times C_i)^-[(o,u)+(o,v)] = (B_i \times C_i)^-[(o,v)+(o,u)] = (B_i \times C_i)^-((o+o),(v+u)) = \max\{C_i^-(v+u), B_i^-(o+o)\} = C_i^-(v+u)$, for all $u,v \in H$, and $C_i^-(uv) = \max\{C_i^-(uv), B_i^-(oo)\} = (B_i \times C_i)^-((oo),(uv)) = (B_i \times C_i)^-[(o,u)(o,v)] = (B_i \times C_i)^-[(o,v)(o,u)] = (B_i \times C_i)^-((oo),(vu)) = \max\{C_i^-(vu), B_i^-(oo)\} = C_i^-(vu)$, for all $u,v \in H$. Hence C is a BVMFNSNR of H. Hence (iii) is clear. □

Theorem 3.12. *Let $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ be a bipolar valued multi fuzzy normal subnear-ring of a near-ring R. For any $u, v \in R$, define a binary relation $\sim$ on R by $u \sim v \Leftrightarrow B_i^+(u-v) = B_i^+(o)$ and $B_i^-(u-v) = B_i^-(o)$, for all i, where o is the identity element of R. Then $\sim$ is a congruence of R.*

Proof. The reflexivity and symmetry are obvious. To prove the transitivity, let $u \sim v$ and $v \sim w$. Then $B_i^+(u-v) = B_i^+(v-w) = B_i^+(o)$ and $B_i^-(u-v) = B_i^-(v-w) = B_i^-(o)$. Then $B_i^+(u-w) = B_i^+(u-v+v-w) \geq \min\{B_i^+(u-v), B_i^+(v-w)\} = B_i^+(o)$ implies that $B_i^+(u-w) = B_i^+(o)$ and $B_i^-(u-w) = B_i^-(u-v+v-w) \leq \max\{B_i^-(u-v), B_i^-(v-w)\} = B_i^-(o)$ implies that $B_i^-(u-w) = B_i^-(o)$. Therefore, $\sim$ is transitive. Hence $\sim$ is an equivalence relation. □

Theorem 3.13. *If $B = \langle B_1^+, B_2^+, \dots, B_n^+, B_1^-, B_2^-, \dots, B_n^- \rangle$ be a bipolar valued multi fuzzy normal subnear-ring of a near-ring R, then the quotient G/B is a near-ring with the operations $B_u + B_v = B_{u+v}$ and $B_u B_v = B_{uv}$.*

Proof. Let $B_u, B_v \in G/B$. Then $B_u - B_v = B_u + B_{-v} = B_{u-v} \in G/B$ and $B_u B_v = B_{uv} \in G/B$. Hence the quotient G/B is a group. □

4. Conclusion

Bipolar valued multi fuzzy normal subnear-ring of a near-ring is defined and their properties are proved. We can extend these concepts into many algebraic systems.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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