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Research Article

Restrained Weakly Connected 2-Domination in the Join of Graphs

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Abstract. Let $G=(V(G),E(G))$ be a connected graph. A restrained weakly connected 2-dominating (RWC2D) set in G is a subset $D\subseteq V(G)$ such that every vertex in $V(G)\setminus D$ is dominated by at least two vertices in D and is adjacent to a vertex in $V(G)\setminus D$, and that the subgraph $\langle D\rangle_w$ weakly induced by D is connected. The restrained weakly connected 2-domination number of G, denoted by $\gamma_{r2w}(G)$, is the smallest cardinality of a restrained weakly connected 2-dominating set in G. In this paper, we characterize the RWC2D sets in the join of two graphs G and H, each of which is of order at least 3 and has no isolated vertex, and in the join $K_1\vee F$, where K_1 is the trivial graph and that at least one component of F is of order at least 3. In particular, it is shown that $2\le\gamma_{r2w}(G\vee H)\le 4$ and $\gamma_{r2w}(K_1\vee F)=\min\{1+\gamma_r(F),\gamma_2(F)\}$, where γ_r and γ_2 are the restrained domination and 2-domination parameters, respectively.

Keywords. Restrained weakly connected 2-domination, Join of graphs

Mathematics Subject Classification (2020). 05C69, 05C76

1. Introduction

Let $G=(V(G),E(G))$ be a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The set of neighbors of a vertex $u\in V(G)$ is called the *open neighborhood* of u in G, denoted by $N_G(u)$, and the set $N_G[u]=N_G(u)\cup\{u\}$ is called the *closed neighborhood* of u in G. If $U\subseteq V(G)$, then the *open neighborhood* and the *closed neighborhood* of U are the sets $N_G(U)=\bigcup_{u\in U}N_G(u)$ and $N_G[U]=U\cup N_G(U)$, respectively. The *subgraph weakly induced* by a subset $D \subseteq V(G)$ is the subgraph $\langle D\rangle_w = (N_G[D], E_w)$, where E_w is the set of all edges in G incident with at least one vertex in D.

A set $S \subseteq V(G)$ is a *dominating set* in G if for every $u \in V(G)\setminus S$, there exists $v \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set in G. A dominating set $S \subseteq V(G)$ with $|S| = \gamma(G)$ is called a γ-*set* in G. Moreover, a dominating set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex in $V(G)\setminus S$ is adjacent to another vertex in $V(G)\setminus S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set in G. A restrained dominating set $S \subseteq V(G)$ with $|S| = \gamma_r(G)$ is called a γ_r-*set* in G. The concept of restrained domination was studied by Domke *et al*. [2]. A dominating set $S \subseteq V(G)$ is called *weakly connected dominating set* in G if the subgraph $\langle S\rangle_w = (V(G), E_w)$ weakly induced by S is connected. The *weakly connected domination number* of G, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set in G. A weakly connected dominating set $S \subseteq V(G)$ with $|S| = \gamma_w(G)$ is called a γ_w-*set* in G. The concept of weakly connected domination was investigated in [3]. A set $D \subseteq V(G)$ is a 2-*dominating set* in G if for every $u \in V(G)\setminus D$, $|D \cap N_G(u)| \geq 2$. The 2-*domination number* of G, denoted by $\gamma_2(G)$, is the smallest cardinality of a 2-dominating set in G. A 2-dominating set $S \subseteq V(G)$ with $|S| = \gamma_2(G)$ is called a γ_2-*set* in G. The concept of 2-domination was introduced by Fink and Jacobson [4]. A 2-dominating set $D \subseteq V(G)$ is called a *weakly connected* 2-*dominating* (WC2D) *set* if the subgraph $\langle D\rangle_w$ weakly induced by D is connected. The *weakly connected* 2-*domination number* of G, denoted by $\gamma_{2w}(G)$, is the smallest cardinality of a weakly connected 2-dominating set in G. Any WC2D set $D \subseteq V(G)$ with $|D| = \gamma_{2w}(G)$ is called a γ_{2w}-*set* in G. The concept of weakly connected 2-domination was investigated in [6].

A *restrained weakly connected* 2-*dominating* (RWC2D) *set* in G is a subset D of $V(G)$ such that every vertex in $V(G)\setminus D$ is dominated by at least two vertices in D and is adjacent to a vertex in $V(G)\setminus D$ and that the subgraph $\langle D\rangle_w$ weakly induced by D is connected. The *restrained weakly connected* 2-*domination number* of G, denoted by $\gamma_{r2w}(G)$, is the smallest cardinality of a RWC2D set in G. Any RWC2D set with cardinality equal to $\gamma_{r2w}(G)$ is called a γ_{r2w}-*set* in G. This concept has been previously studied in [5].

In this paper, characterizations of RWC2D sets in the join of two graphs G and H, each of which is of order at least 3 and without isolated vertex, and in the join $K_1 \vee F$, where K_1 is the trivial graph and F is a graph having at least one component of order at least 3, are obtained. As a consequence, bounds or exact values for γ_{r2w} in the join $G \vee H$ and $K_1 \vee F$ are given. In addition, some necessary and sufficient conditions for the join of two graphs to have restrained weakly connected 2-domination numbers equal to 2, 3, and 4 are provided.

Note that the *join* of two graphs, denoted by $G \vee H$, is the graph with vertex set

$$V(G \vee H) = V(G) \dot\cup V(H)$$

and edge set

$$E(G \vee H) = E(G) \dot\cup E(H) \dot\cup \{uv : u \in V(G) \text{ and } v \in V(H)\}.$$

The symbol $\dot{\cup}$ denotes the disjoint union of sets. As an illustration, Figure 1(c) shows the join $P_3 \vee C_4$ of the path P_3 and the cycle C_4.

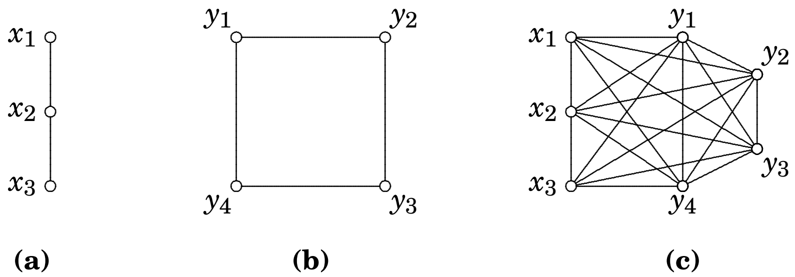

Figure 1. (a) The path P_3; (b) the cycle C_4; and (c) the join $P_3 \vee C_4$

Readers may refer to [1] for other graph theoretic terminologies which are not specifically defined here.

In this paper, we will use the following published results.

Theorem 1.1 ([6])**.** *Let G and H be any nontrivial connected graphs. Then $D \subseteq V(G \vee H)$ is a* WC2D *set in $G \vee H$ if and only if one of the following holds:*

(i) *$D \subseteq V(G)$ and D is a 2-dominating set of G;*

(ii) *$D \subseteq V(H)$ and D is a 2-dominating set of H;*

(iii) *$|D \cap V(G)| = 1$ and $|D \cap V(H)| = 1$ where $D \cap V(G)$ is a dominating set of G and $D \cap V(H)$ is a dominating set of H;*

(iv) *$|D \cap V(G)| = 1$ and $|D \cap V(H)| \geq 2$ where $D \cap V(H)$ is a dominating set of H;*

(v) *$|D \cap V(H)| = 1$ and $|D \cap V(G)| \geq 2$ where $D \cap V(G)$ is a dominating set of G;*

(vi) *$2 \leq |D \cap V(G)| \leq |V(G)|$ and $2 \leq |D \cap V(H)| \leq |V(H)|$.*

Remark 1.2 ([6])**.** Let G and H be any graphs. If D is a nonempty subset of $V(G \vee H)$, then $\langle D \rangle_w$ is connected.

Proposition 1.3 ([5])**.** *Let G be a nontrivial connected graph. Then $2 \leq \gamma_{r2w}(G) \leq |V(G)|$.*

Proposition 1.4 ([5])**.** *If D is a* RWC2D *set in a nontrivial connected graph G, then D contains all vertices of G whose degrees are either 1 or 2.*

2. Main Results

Theorem 2.1. *Let G and H be any graphs without isolated vertices and each of which is of order at least 3. Then $D \subseteq V(G \vee H)$ is a* RWC2D *set in $G \vee H$ if and only if one of the following holds:*

(i) *D is a 2-dominating set in G;*

(ii) *D is a 2-dominating set in H;*

(iii) $|D \cap V(G)| = 1$ *and* $|D \cap V(H)| = 1$ *where* $D \cap V(G)$ *is a dominating set in* G *and* $D \cap V(H)$ *is a dominating set in* H*;*

(iv) $|D \cap V(G)| = 1$ *and* $|D \cap V(H)| \geq 2$ *where* $D \cap V(H) \neq V(H)$ *and* $D \cap V(H)$ *is a dominating set in* H*;*

(v) $|D \cap V(H)| = 1$ *and* $|D \cap V(G)| \geq 2$ *where* $D \cap V(G) \neq V(G)$ *and* $D \cap V(G)$ *is a dominating set in* G*;*

(vi) $2 \leq |D \cap V(G)| < |V(G)|$ *and* $2 \leq |D \cap V(H)| < |V(H)|$*;*

(vii) $D \cap V(G) = V(G)$ *and* $\langle V(H) \setminus (D \cap V(H)) \rangle$ *has no isolated vertex;*

(viii) $D \cap V(H) = V(H)$ *and* $\langle V(G) \setminus (D \cap V(G)) \rangle$ *has no isolated vertex;*

(ix) $D \cap V(G) = V(G)$ *and* $D \cap V(H) = V(H)$*.*

Proof. Suppose $D \subseteq V(G \vee H)$ is a RWC2D set in $G \vee H$. Consider the following cases:

Case 1. $D \cap V(H) = \varnothing$ or $D \cap V(G) = \varnothing$.

Suppose $D \cap V(H) = \varnothing$. Then $D \subseteq V(G)$. Since D is a RWC2D set in $G \vee H$, it follows that D is a 2-dominating set in G. Similarly, if $D \cap V(G) = \varnothing$, then D is a 2-dominating set in H. This proves the necessities for (i) and (ii).

Case 2. $D \cap V(G) \neq \varnothing$ and $D \cap V(H) \neq \varnothing$.

Subcase 2.1. Suppose $D \cap V(G) \subsetneq V(G)$ and $D \cap V(H) \subsetneq V(H)$.

Suppose first that $|D \cap V(G)| = 1$ and $|D \cap V(H)| = 1$. Since D is a WC2D set in $G \vee H$, by Theorem 1.1(iii), $D \cap V(G)$ is a dominating set in G and $D \cap V(H)$ is a dominating set in H, so that the necessity for (iii) holds. Secondly, suppose that $|D \cap V(G)| = 1$ and $2 \leq |D \cap V(H)| < |V(H)|$. Let $u \in V(H) \setminus (D \cap V(H))$. Since D is a 2-dominating set in $G \vee H$, there exists $v \in D \cap V(H)$ such that $uv \in E(H)$. Hence, $D \cap V(H)$ is a dominating set in H. Similarly, if $|D \cap V(H)| = 1$ and $2 \leq |D \cap V(G)| < |V(G)|$, we have $D \cap V(G)$ is a dominating set in G. This proves the necessities for (iv) and (v). The last option of this subcase is when $2 \leq |D \cap V(G)| < |V(G)|$ and $2 \leq |D \cap V(H)| < |V(H)|$ which is the statement in (vi).

Subcase 2.2. Suppose that $D \cap V(G) = V(G)$ and $D \cap V(H) \subsetneq V(H)$, or $D \cap V(H) = V(H)$ and $D \cap V(G) \subsetneq V(G)$. If $D \cap V(G) = V(G)$ and $D \cap V(H) \subsetneq V(H)$. Let $x \in V(H) \setminus (D \cap V(H))$. Since D is a restrained set in $G \vee H$, there exists $y \in V(H) \setminus (D \cap V(H))$ such that $xy \in E(H)$. Since x is arbitrary, it follows that $\langle V(H) \setminus (D \cap V(H)) \rangle$ has no isolated vertex in H. Similarly, if $D \cap V(H) = V(H)$ and $D \cap V(G) \subsetneq V(G)$, then $\langle V(G) \setminus (D \cap V(G)) \rangle$ has no isolated vertex in G. This proves (vii) and (viii).

Subcase 2.3. $D \cap V(G) = V(G)$ and $D \cap V(H) = V(H)$. Then clearly the necessity for (ix) holds.

Conversely, suppose first that D is a 2-dominating set in G. Then by Theorem 1.1(i), D is a WC2D set in $G \vee H$. Let $y \in V(G \vee H) \setminus D$. Then $y \in V(H)$ or $y \in V(G) \setminus (D \cap V(G))$. If $y \in V(H)$, then there exists $z \in V(H)$ such that $yz \in E(H)$ since H is nontrivial graph having no isolated vertex. Thus, $yz \in E(G \vee H)$. On the other hand, if $y \in V(G) \setminus (D \cap V(G))$, then there exists $w \in V(H)$ such that $yw \in E(G \vee H)$. In either scenario, we have seen that D is a restrained dominating set in $G \vee H$. It follows that D is a RWC2D set in $G \vee H$. Similarly, if D is a 2-dominating set in H, then D is a RWC2D set in $G \vee H$. Secondly, suppose $|D \cap V(G)| = 1$ and $|D \cap V(H)| = 1$ where $D \cap V(G)$ and $D \cap V(H)$ are dominating sets in G and H, respectively. By Theorem 1.1(iii), D is a WC2D set in $G \vee H$. Now, let $y \in V(G \vee H) \backslash D$. Then $y \in V(G) \backslash D$ or $y \in V(H) \backslash D$. Suppose that $y \in V(G) \backslash D$. Then by definition of the join of graphs, there exists $z \in V(H) \backslash D$ such that $yz \in E(G \vee H)$. The existence of an element z in $V(H) \backslash D$ is guaranteed since H is nontrivial. Similarly, if $y \in V(H) \backslash D$, there exists $w \in V(G) \backslash D$ such that $yw \in E(G \vee H)$. In either case, D is a restrained dominating set in $G \vee H$. Therefore, D is a RWC2D set in $G \vee H$. Thirdly, suppose that $|D \cap V(G)| = 1$ and $|D \cap V(H)| \geq 2$ where $D \cap V(H) \neq V(H)$ and $D \cap V(H)$ is a dominating set in H. Then by Theorem 1.1(iv), D is a WC2D set in $G \vee H$. Let $y \in V(G \vee H) \backslash D$. If $y \in V(G) \backslash D$, then there exists $z \in V(H) \backslash D$ such that $yz \in E(G \vee H)$. On the other hand, if $y \in V(H) \backslash D$, then there exists $z^* \in V(G) \backslash D$ such that $yz^* \in E(G \vee H)$. Thus, D is a RWC2D set in $G \vee H$. Similarly, if $|D \cap V(H)| = 1$ and $|D \cap V(G)| \geq 2$ where $D \cap V(G) \neq V(G)$ and $D \cap V(G)$ is a dominating set in G, then D is a RWC2D set in $G \vee H$. Fourthly, suppose $2 \leq |D \cap V(G)| < |V(G)|$ and $2 \leq |D \cap V(H)| < |V(H)|$. By Theorem 1.1(vi), we have D is a WC2D set in $G \vee H$. Let $y \in V(G \vee H) \backslash D$. If $y \in V(G) \backslash D$, then there exists $z \in V(H) \backslash D$ such that $yz \in E(G \vee H)$ by definition of the join of graphs. The existence of y and that of z are guaranteed by the assumption that $D \cap V(G) \subsetneq V(G)$ and $D \cap V(H) \subsetneq V(H)$. If $y \in V(H) \backslash D$, then by using similar argument, D is a restrained dominating set in $G \vee H$. Hence, D is a RWC2D set in $G \vee H$. Fifthly, suppose that $D \cap V(G) = V(G)$ and $\langle V(H) \backslash (D \cap V(H)) \rangle$ has no isolated vertex. Since $D \subseteq V(G \vee H)$, by Remark 1.2, D is a weakly connected set in $G \vee H$. Since $V(G) \subseteq D$ and $|V(G)| \geq 2$, D is a 2-dominating set. Thus, D is a WC2D set in $G \vee H$. Let $y \in V(G \vee H) \backslash D$. Then $y \in V(H) \backslash (D \cap V(H))$. Since $\langle V(H) \backslash (D \cap V(H)) \rangle$ has no isolated vertex, there exists $x^* \in V(H) \backslash (D \cap V(H))$ such that $x^* y \in E(H)$. Thus, we have $x^* y \in E(G \vee H)$. It follows that D is a restrained dominating set in $G \vee H$ and hence, D is a RWC2D set in $G \vee H$. Similarly, if $D \cap V(H) = V(H)$ and $\langle V(G) \backslash (D \cap V(G)) \rangle$ has no isolated vertex, then D is a RWC2D set in $G \vee H$. Lastly, suppose that $D \cap V(G) = V(G)$ and $D \cap V(H) = V(H)$. Then $D = V(G \vee H)$ is obviously a RWC2D set in $G \vee H$. This completes the proof. □

The next corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let G and H be any graphs without isolated vertex and each of which is of order at least 3. Then $2 \leq \gamma_{r2w}(G \vee H) \leq 4$.*

Proof. Let $D \subseteq V(G \vee H)$ be such that $|D \cap V(G)| = 2$ and $|D \cap V(H)| = 2$. Then by Theorem 2.1(vi), $D = (D \cap V(G)) \cup (D \cap V(H))$ is a RWC2D set in $G \vee H$. Thus, $\gamma_{r2w}(G \vee H) \leq |D| = 4$. By Proposition 1.3, $\gamma_{r2w}(G \vee H) \geq 2$. Therefore, $2 \leq \gamma_{r2w}(G \vee H) \leq 4$. □

Corollary 2.3. *Let G and H be any graphs without isolated vertices and each of which is of order at least 3. Then $\gamma_{r2w}(G \vee H) = 2$ if and only if one of the following holds:*

(i) $\gamma_2(G) = 2$;

(ii) $\gamma_2(H) = 2$;

(iii) $\gamma(G) = 1$ *and* $\gamma(H) = 1$.

Proof. Suppose $\gamma_{r2w}(G \vee H) = 2$. Let D be a γ_{r2w}-set in $G \vee H$. By Theorem 2.1, (i), (ii) or (iii), we have $D \subseteq V(G)$ and D is a 2-dominating set in G, or $D \subseteq V(H)$ and D is a 2-dominating set in H, or $|D \cap V(G)| = 1$ and $|D \cap V(H)| = 1$, where $D \cap V(G)$ is a dominating set in G and $D \cap V(H)$ is a dominating set in H, respectively. From these options, we have $\gamma_2(G) = 2$, or $\gamma_2(H) = 2$, or $\gamma(G) = 1$ and $\gamma(H) = 1$.

Conversely, suppose first $\gamma_2(G) = 2$. Let D be a γ_2-set in G. By Theorem 2.1(i), D is a RWC2D set in $G \vee H$. Thus, $\gamma_{r2w}(G \vee H) \le 2$. By Proposition 1.3, $\gamma_{r2w}(G \vee H) \ge 2$. Hence, $\gamma_{r2w}(G \vee H) = 2$. Similarly, if $\gamma_2(H) = 2$, then $\gamma_{r2w}(G \vee H) = 2$. Finally, suppose that $\gamma(G) = 1$ and $\gamma(H) = 1$. Let $\{u\}$ be a dominating set in G and let $\{v\}$ be a dominating in H. Set $D = \{u, v\}$. By Theorem 2.1(iii), D is a RWC2D set in $G \vee H$. Hence, $\gamma_{r2w}(G \vee H) \le |D| = |\{u, v\}| = 2$. Again by Proposition 1.3, $\gamma_{r2w}(G \vee H) \ge 2$. Therefore, $\gamma_{r2w}(G \vee H) = 2$. □

Corollary 2.4. *Let G and H be any graphs without isolated vertices and each of which is of order at least* 3. *Suppose $\gamma_{r2w}(G \vee H) \ne 2$. Then $\gamma_{r2w}(G \vee H) = 3$ if and only if one of the following holds:*

(i) $\gamma_2(G) = 3$;

(ii) $\gamma_2(H) = 3$;

(iii) $\gamma(H) = 2$;

(iv) $\gamma(G) = 2$.

Proof. The assumption that $\gamma_{r2w}(G \vee H) \ne 2$ immediately means that $\gamma_{r2w}(G) \ne 2$ and $\gamma_{r2w}(H) \ne 2$. Suppose $\gamma_{r2w}(G \vee H) = 3$. Let D be a γ_{r2w}-set in $G \vee H$. By Theorem 2.1, (i), (ii), (iv) and (v) there are four possible options, namely, $|D \cap V(G)| = 3$ and $D \cap V(G)$ is a 2-dominating set in G, or $|D \cap V(H)| = 3$ and $D \cap V(H)$ is a 2-dominating set in H, or $|D \cap V(G)| = 1$ and $|D \cap V(H)| = 2$ where $D \cap V(H)$ is a dominating set in H, or $|D \cap V(G)| = 2$ and $|D \cap V(H)| = 1$ where $D \cap V(G)$ is a dominating set in G. This means that $\gamma_2(G) \le 3$, or $\gamma_2(H) \le 3$, or $\gamma(H) \le 2$, or $\gamma(G) \le 2$. Since $\gamma_{r2w}(G) \ne 2$, by Corollary 2.3 we have $\gamma_2(G) \ge 3$, or $\gamma_2(H) \ge 3$, or $\gamma(H) \ge 2$, or $\gamma(G) \ge 2$. Consequently, we have either $\gamma_2(G) = 3$, or $\gamma_2(H) = 3$, or $\gamma(H) = 2$, or $\gamma(G) = 2$.

Conversely, suppose first that $\gamma_2(G) = 3$. Let $D \subseteq V(G)$ be a γ_2-set in G. By Theorem 2.1(i), D is a RWC2D set in $G \vee H$. Hence, we have $\gamma_{r2w}(G \vee H) \le |D| = 3$. Since $\gamma_{r2w}(G \vee H) \ne 2$, we have $\gamma_{r2w}(G \vee H) \ge 3$. Thus, $\gamma_{r2w}(G \vee H) = 3$. Similarly, if $\gamma_2(H) = 3$, then $\gamma_{r2w}(G \vee H) = 3$. Now, suppose that $\gamma(H) = 2$. Let D' be a γ-set in H and let $x \in V(G)$. Set $D^* = D' \cup \{x\}$. By Theorem 2.1(iv), D^* is a RWC2D set in $G \vee H$. Thus, $\gamma_{r2w}(G \vee H) \le |D^*| = |D' \cup \{x\}| = 2 + 1 = 3$. Since by assumption $\gamma_{r2w}(G \vee H) \ne 2$, we have $\gamma_{r2w}(G \vee H) \ge 3$. Thus, $\gamma_{r2w}(G \vee H) = 3$. Similarly, if $\gamma(G) = 2$, then by Theorem 2.1(v) and the fact that $\gamma_{r2w}(G \vee H) \ne 2$, we have $\gamma_{r2w}(G \vee H) = 3$. □

Corollary 2.5. *Let G and H be any graphs without isolated vertices and each of which is of order at least 3. Suppose $\gamma_{r2w}(G\vee H)\neq 2,3$. Then $\gamma_{r2w}(G\vee H)=4$ if and only if one of the following holds:*

(i) $\gamma_2(G)=4$;

(ii) $\gamma_2(H)=4$;

(iii) $\gamma(G)=3$;

(iv) $\gamma(H)=3$;

(v) $|D\cap V(G)|=2$ *and* $|D\cap V(H)|=2$.

Proof. Suppose $\gamma_{r2w}(G\vee H)=4$. Let D be a γ_{r2w}-set in $G\vee H$. By Theorem 2.1(i), (ii), (iv), (v) and (vi), we have the following possible options, namely, $|D\cap V(G)|=4$ and D is a 2-dominating set in G, $|D\cap V(H)|=4$ and D is a 2-dominating set in H, $|D\cap V(G)|=1$ and $|D\cap V(H)|=3$ where $D\cap V(H)\neq V(H)$ and $D\cap V(H)$ is a dominating set in H, $|D\cap V(H)|=1$ and $|D\cap V(G)|=3$ where $D\cap V(G)\neq V(G)$ and $D\cap V(G)$ is a dominating set in G, and $|D\cap V(G)|=2$ and $|D\cap V(H)|=2$, respectively. This means that $\gamma_2(G)\leq 4$, or $\gamma_2(H)\leq 4$, or $\gamma(H)\leq 3$, or $\gamma(G)\leq 3$ or $|D\cap V(G)|=2$ and $|D\cap V(H)|=2$. Since $\gamma_{r2w}(G\vee H)\neq 2,3$ by Corollary 2.3(i) and Corollary 2.4(i), $\gamma_2(G)\neq 2,3$. Thus, $\gamma_2(G)\geq 4$. Hence, $\gamma_2(G)=4$. Similarly, we must have $\gamma_2(H)=4$. Also, since $\gamma_{r2w}(G\vee H)\neq 2,3$ by Corollary 2.4(iii), $\gamma(G)\neq 2$. Thus, $\gamma(G)\geq 3$. Hence, $\gamma(G)=3$. Similarly, it can be shown that $\gamma(H)=3$. Therefore, $\gamma_2(G)=4$, or $\gamma_2(H)=4$, or $\gamma(G)=3$, or $\gamma(H)=3$, or $|D\cap V(G)|=2$ and $|D\cap V(H)|=2$.

Conversely, suppose $\gamma_2(G)=4$. Let D be a γ_2-set in G. By Theorem 2.1(i), D is a RWC2D set in $G\vee H$. Thus, $\gamma_{r2w}(G\vee H)\leq|D|=4$. Since $\gamma_{r2w}(G\vee H)\neq 2,3$, $\gamma_{r2w}(G\vee H)\geq 4$. Hence, $\gamma_{r2w}(G\vee H)=4$. Similarly, if $\gamma_2(H)=4$, then $\gamma_{r2w}(G\vee H)=4$. Suppose $\gamma(H)=3$. Let D' be a γ-set in H. Let $x\in V(G)$. Set $D^*=D'\cup\{x\}$. By Theorem 2.1(iv), D^* is a RWC2D set in $G\vee H$. Thus, we have $\gamma_{r2w}(G\vee H)\leq|D^*|=|D'\cup\{x\}|=3+1=4$. Since $\gamma_{r2w}(G\vee H)\neq 2,3$, $\gamma_{r2w}(G\vee H)\geq 4$. Hence, $\gamma_{r2w}(G\vee H)=4$. Similarly, if $\gamma(G)=3$, then by Theorem 2.1(v) and the fact that $\gamma_{r2w}(G\vee H)\neq 2,3$, we have $\gamma_{r2w}(G\vee H)=4$. Lastly, suppose $D=(D\cap V(G))\cup(D\cap V(H))$ where $|D\cap V(G)|=2=|D\cap V(H)|$. Then $|D|=4$. By Theorem 2.1(vi), D is a RWC2D set in $G\vee H$. Thus, $\gamma_{r2w}(G\vee H)\leq 4$. Since $\gamma_{r2w}(G\vee H)\neq 2,3$, $\gamma_{r2w}(G\vee H)\geq 4$. Hence, $\gamma_{r2w}(G\vee H)=4$. □

The following result is useful to prove the needed characterization in the join $K_1\vee H$.

Theorem 2.6 ([6])**.** *Let $K_1=\langle\{v\}\rangle$ and let H be any graph of order at least 2. Then $D\subseteq V(K_1\vee H)$ is a* WC2D *set in $K_1\vee H$ if and only if one of the following holds:*

(i) $v\in D$ *and* $D\setminus\{v\}$ *is a dominating set of* H.

(ii) $D\subseteq V(H)$ *and* D *is a 2-dominating set of* H.

We need the following definition for the join $K_1\vee H$.

Definition 2.7 ([1])**.** Let $G_1 = (V(G_1),E(G_1))$ and $G_2 = (V(G_2),E(G_2))$ be graphs where $V(G_1)$ and $V(G_2)$ are disjoint. Then the *union* of G_1 and G_2 is the graph $G_1 + G_2 = (V(G_1)\dot{\cup} V(G_2), E(G_1)\dot{\cup} E(G_2))$.

Lemma 2.8. *Let H be a connected graph of order $n \geq 3$. Then there exists a 2-dominating set D in H such that D is a proper subset of $V(H)$. As a consequence, $\gamma_2(H) \leq n-1$.*

Proof. If H is a connected graph of order $n \geq 3$, then there is at least one vertex in H of degree greater than or equal to two. Let $x \in V(H)$ such that $deg_H(x) \geq 2$. Let $D = V(H)\setminus\{x\}$. Then $|D \cap N_H(x)| \geq 2$. Thus, D is a 2-dominating set in H. It follows that $\gamma_2(H) \leq |D| = |V(H)\setminus\{x\}| = n-1$. □

Theorem 2.9. *Let $K_1 = \langle\{v\}\rangle$ and $H = H_1 + H_2 + \ldots + H_p + \left\langle \bigcup_{j=1}^{q}\{u_j\} \right\rangle$ where H_i is a component of H with $|V(H_i)| \geq 3$ for $1 \leq i \leq p$ and u_j is an isolated vertex for $1 \leq j \leq q$. Then $D \subseteq V(K_1 \vee H)$ is a* RWC2D *set in $K_1 \vee H$ if and only if one of the following holds:*

(i) $D = \{v\} \cup \left(\bigcup_{i=1}^{p} S_i\right) \cup \left(\bigcup_{j=1}^{q}\{u_j\}\right)$, *where S_i is a restrained dominating set in H_i for each i.*

(ii) $D = \left(\bigcup_{i=1}^{p} S_i'\right) \cup \left(\bigcup_{j=1}^{q}\{u_j\}\right)$, *where S_i' is a 2-dominating set in H_i for each i and $S_i' \subsetneq V(H_i)$ for some i.*

Proof. Suppose D is a RWC2D set in $K_1 \vee H$. Then D is a WC2D set in $K_1 \vee H$. Consider the following cases:

Case 1. Suppose $v \in D$. Then by Theorem 2.6(i), $D\setminus\{v\}$ is a dominating set in H. By Proposition 1.4, $\bigcup_{j=1}^{q}\{u_j\} \subseteq D\setminus\{v\}$. This means that $\left(\bigcup_{i=1}^{p} S_i\right) \cup \left(\bigcup_{j=1}^{q}\{u_j\}\right) \subseteq D\setminus\{v\}$ where S_i is a dominating set in H_i for each i. Since D is a restrained dominating set and $v \in D$, it follows that S_i is a restrained dominating set in H_i for each i. Hence, $D = \{v\}\left(\bigcup_{i=1}^{p} S_i\right) \cup \left(\bigcup_{j=1}^{q}\{u_j\}\right)$, where S_i is a restrained dominating set in H_i for each i. This proves the necessity for (i).

Case 2. Suppose $v \notin D$. Then by Theorem 2.6(ii), $D \subseteq V(H)$ and D is a 2-dominating set in H. Since $\bigcup_{j=1}^{q}\{u_j\} \subseteq D$ by Proposition 1.4 and D is a restrained dominating set, as a consequence, S_i' is a 2-dominating set of H_i for each i and that $S_i' \subsetneq V(H_i)$ for some i. The existence of a 2-dominating set $S_i' \subsetneq V(H_i)$ is guaranteed in Lemma 2.8. This proves the necessity for (ii).

Conversely, suppose first that $D = \{v\} \cup \left(\bigcup_{i=1}^{p} S_i\right) \cup \left(\bigcup_{j=1}^{q}\{u_j\}\right)$ where S_i is a restrained dominating set in H_i for each i. Then by Theorem 2.6(i), D is a WC2D set in $K_1 \vee H$. Since S_i is a restrained dominating set in H_i for each i, we must have $\bigcup_{i=1}^{p} S_i$ is a restrained dominating set in the union $H_1 + H_2 + \ldots + H_p$. This implies that $\left(\bigcup_{i=1}^{p} S_i\right) \cup \left(\bigcup_{j=1}^{q}\{u_j\}\right)$ is a restrained dominating set in H. It follows that D is a RWC2D set in $K_1 \vee H$. On the other hand, suppose that $D = \left(\bigcup_{i=1}^{p} S'_i\right) \cup \left(\bigcup_{j=1}^{q} \{u_j\}\right)$ where S'_i is a 2-dominating set in H_i for each i with $S'_i \subsetneq V(H_i)$ for some i. Then by Theorem 2.6(ii), D is a WC2D set in $K_1 \vee H$. Since $S'_i \subsetneq V(H_i)$ for some i, it follows that there exists $w \in V(H) \backslash D$ such that $vw \in E(K_1 \vee H)$. Hence, D is a restrained set in $K_1 \vee H$. Therefore, D is a RWC2D set in $K_1 \vee H$. □

Theorem 2.9 is still true whenever some (but not all) components of H are of orders equal to 2. In this case, all the vertices of the components of H which are of order 2 are included in any RWC2D set in $K_1 \vee H$ as ascertained in Proposition 1.4.

Corollary 2.10. *Let $K_1 = \langle\{v\}\rangle$ and $H = H_1 + H_2 + \ldots + H_p + \left\langle \bigcup_{j=1}^{q} \{u_j\} \right\rangle$ where H_i is a component of H with $|V(H_i)| \geq 3$ for $1 \leq i \leq p$ and u_j is an isolated vertex for $1 \leq j \leq q$. Then $\gamma_{r2w}(K_1 \vee H) = \min\{1 + \gamma_r(H), \gamma_2(H)\}$.*

Proof. By Theorem 2.9, $\gamma_{r2w}(K_1 \vee H)$ is the smallest among the values $1 + |S|$ where S is a restrained dominating set in H, and $|S'|$ where S' is a 2-dominating set in H with $S' \subsetneq V(H)$. Note that the existence of a proper subset S' of $V(H)$ is ascertained in Lemma 2.8. As a consequence, $\gamma_{r2w}(K_1 \vee H) = \min\{1 + \gamma_r(H), \gamma_2(H)\}$. □

Example 2.11. Consider the graph of $K_1 \vee H$, where $K_1 = \langle\{v\}\rangle$ and $H = C_4 + \langle\{x, y, z, w\}\rangle$. If $D \subseteq V(K_1 \vee H)$ is a RWC2D set in $K_1 \vee H$, then by Theorem 2.9(i) we can have $v \in D$ and that either $D \setminus \{v\} = V(H)$ or $D \setminus \{v\} \subsetneq V(H)$ is a restrained dominating set in H. The darkened vertices in Figure 2(a) shows a particular RWC2D set where $v \in D$ and $D \setminus \{v\}$ is a restrained dominating subset of $V(H)$. If $v \notin D$, then by Theorem 2.9(ii), $D \setminus \{v\} \subsetneq V(H)$ is a 2-dominating set in H. The darkened vertices in Figure 2(b) shows some γ_{r2w}-set in $K_1 \vee H$ where $v \notin D$. Moreover, $\gamma_r(H) = 6$ and $\gamma_2(H) = 6$. By Corollary 2.10, $\gamma_{r2w}(K_1 \vee H) = \min\{1 + \gamma_r(H), \gamma_2(H)\} = \min\{1 + 6, 6\} = 6$.

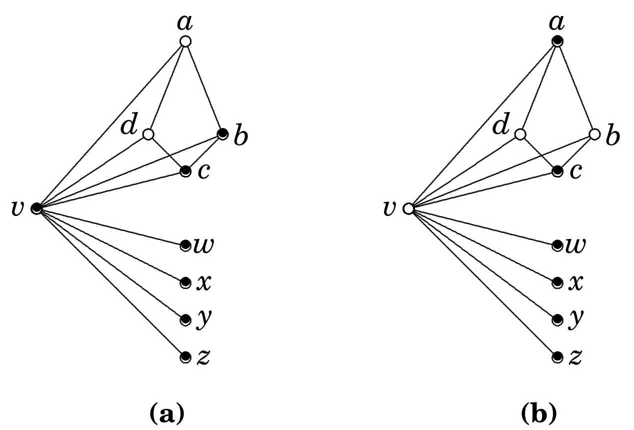

Figure 2. The graph of $K_1 \vee H$, where $K_1 = \langle\{v\}\rangle$ and $H = C_4 + \langle\{w, x, y, z\}\rangle$ with darkened vertices in some RWC2D sets

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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