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Research Article

Certain Results on (k,μ)-Contact Metric Manifold endowed with Concircular Curvature Tensor

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Abstract. The purpose of this paper is to study concircular curvature tensor on (k,μ)-contact metric manifold. Here, first we consider ϕ-concircularly flat (k,μ)-contact metric manifold. Next, we describe concircularly pseudo-symmetric (k,μ)-contact metric manifold. Later, we study concircularly ϕ-recurrent (k,μ)-contact metric manifold. Finally, we provide the three dimensional example for the existence of non-Sasakian concircularly ϕ-recurrent (k,μ)-contact metric manifold.

Keywords. (k,μ)-Contact metric manifold, Non-Sasakian, Concircular curvature tensor, η-Einstein manifold, Scalar curvature

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1. Introduction

A concircular transformation [11, 18] in an $(2n+1)$-dimensional Riemannian manifold M is a transformation which transforms every geodesic circle of M into a geodesic circle. Here, geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. A concircular transformation is always a conformal transformation [11]. Thus, the geometry of concircular transformation, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [2]). An interesting invariant of a concircular transformation is the concircular curvature tensor $\widetilde{Z}$ defined by [18]:

$$\widetilde{Z}(X,Y)U = R(X,Y)U - \frac{r}{2n(2n+1)}[g(Y,U)X - g(X,U)Y], \tag{1.1}$$

for any vector fields $X,Y,U \in TM$, where R is the curvature tensor and r is the scalar curvature. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In [16], it is proved that in a (k,μ)-contact metric manifold, the concircular curvature tensor satisfies $Z(\xi,X)Y = 0$ if and only if the manifold is flat and 3-dimensional. In [10], the authors proved that in a Kenmotsu manifold the concircular curvature tensor satisfies $Z(\xi,X).Z = 0$ if and only if the manifold is of constant scalar curvature. Moreover, in [8] De *et al.* studied concircular curvature tensor on $N(k)$-contact metric manifold. Here they proved that a ϕ-concircularly flat $N(k)$-contact metric manifold is reduces to Sasakian manifold. Recently in [1], author shown that a ϕ-concircularly flat Kenmotsu manifold with respect to semi-symmetric metric connection is an η-Einstein manifold. The notion of concircular curvature tensor was weakened by many authors ([3, 13, 17]).

The present paper is organized as follows: In Section 2 we recall the basic notions and preliminary results of (k,μ)-contact metric manifolds needed throughout the paper. In Section 3, we have proved that a non-Sasakian ϕ-concircularly flat (k,μ)-contact metric manifold always reduces to $N(k)$-contact metric manifold. In Section 4 we describe concircularly pseudo symmetric (k,μ)-contact metric manifold and it is shown that the manifold is turns in to $N(k)$-contact metric manifold and the manifold is concircularly flat provided $k+\mu h = \frac{r}{2n(2n+1)}$. Infact Section 5 is devoted to the study of concircularly ϕ-recurrent (k,μ)-contact metric manifold and it was proved that the manifold becomes $N(k)$-contact metric manifold. Finally, the last section provides the existence of 3-dimensional non-Sasakian concircularly ϕ-recurrent (k,μ)-contact metric manifold.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Also, a contact manifold admits an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and η is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M \times R$ defined by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where X is tangent to M, t is the coordinate of R and λ a smooth function on $M \times R$. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor $[\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. Let g be the compatible Riemannian metric with almost contact structure (ϕ,ξ,η), that is,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

for all vector fields $X,Y \in \chi(M)$. A manifold M together with this almost contact metric structure is said to be almost contact metric manifold and it is denoted by $M(\phi,\xi,\eta,g)$. An almost contact metric structure reduces to a contact metric structure if $g(X,\phi Y) = d\eta(X,Y)$.

Given a contact metric manifold $M(\phi,\xi,\eta,g)$, we consider a $(1,1)$ tensor field h defined by $h = \frac{1}{2}L_\xi\phi$, where L denotes the Lie differentiation, h is a symmetric operator and satisfies $h\phi = -\phi h$. Again we have $trh = tr\phi h = 0$, and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds [4]:

$$\nabla_X\xi = -\phi X - \phi hX, \quad (\nabla_X\eta)Y = g(X + hX,\phi Y). \tag{2.3}$$

Blair *et al.* [4] introduced the (k,μ)-nullity distribution of a contact metric manifold M and is defined by

$$N(k,\mu): p \to N_p(k,\mu) = \{U \in T_pM \mid R(X,Y)U = (kI + \mu h)R_0(X,Y)U\},$$

for all $X,Y \in TM$, where $(k,\mu) \in R^2$. A contact metric manifold with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. If $k = 1$, $\mu = 0$, then the manifold becomes Sasakian [4]. If $\mu = 0$, the (k,μ)-nullity distribution is reduced to the k-nullity distribution [14]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by:

$$N(k): p \to N_p(k) = \{U \in T_pM \mid R(X,Y)U = kR_0(X,Y)U\},$$

k being constant. If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)$-contact metric manifold. Throughout this paper, we study $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifolds.

In a (k,μ)-contact metric manifold the following relations hold [4,5]:

$$h^2 = (k-1)\phi^2, \tag{2.4}$$

$$R_0(X,Y)U = g(Y,U)X - g(X,U)Y, \tag{2.5}$$

$$R(X,Y)\xi = (kI + \mu h)R_0(X,Y)\xi, \tag{2.6}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.7}$$

$$\begin{aligned} S(X,Y) = {} & [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ & + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.8}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y). \tag{2.9}$$

Definition 2.1. A $(2n+1)$-dimensional (k,μ)-contact metric manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y, where a and b are constants. If $b = 0$, then the manifold M reduces to an Einstein manifold.

In [4], the authors have proved the following result:

Lemma 2.1. *A $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifold which is η-Einstein manifold is an $N(k)$-contact metric manifold.*

3. ϕ-Concircularly Flat (k,μ)-Contact Metric Manifold

A $(2n+1)$-dimensional $(n>1)$ (k,μ)-contact metric manifold M is said to be ϕ-concircularly flat if

$$g(\widetilde{Z}(\phi X,\phi Y)\phi U,\phi W)=0. \tag{3.1}$$

Now from (1.1), we have

$$g(\widetilde{Z}(\phi X,\phi Y)\phi U,\phi W)=g(R(\phi X,\phi Y)\phi U,\phi W) - \frac{r}{2n(2n+1)}[g(\phi Y,\phi U)g(\phi X,\phi W)-g(\phi X,\phi U)g(\phi Y,\phi W)]. \tag{3.2}$$

Let us consider an ϕ-concircularly flat (k,μ)-contact metric manifold, we obtain from (3.2) that

$$g(R(\phi X,\phi Y)\phi U,\phi W)=\frac{r}{2n(2n+1)}[g(\phi Y,\phi U)g(\phi X,\phi W)-g(\phi X,\phi U)g(\phi Y,\phi W)]. \tag{3.3}$$

Let $\{\phi e_1,\phi e_2,\dots,\phi e_{2n},\xi\}$ be an orthonormal ϕ-basis of the tangent space. Now taking $X=W=e_i$ in (3.3) and then by using (2.6), we get

$$S(\phi Y,\phi U)=\left[k+\frac{r(2n-1)}{2n(2n+1)}\right]g(\phi Y,\phi U)+\mu g(h\phi Y,\phi U), \tag{3.4}$$

which on simplification gives

$$S(\phi Y,\phi U)=\frac{\begin{pmatrix} k2n(2n+1)(2n-2+\mu)+\mu[2n^2\mu(2n+\mu)-4n(n-1)(2n+1)] \\ +r(2n-1)(2n-2+\mu)\end{pmatrix}}{4n(n-1)(2n+1)}g(\phi Y,\phi U). \tag{3.5}$$

Replacing Y by ϕY and U by ϕU in (3.5) and then by considering (2.1), gives

$$S(Y,U)=\alpha g(Y,U)+\beta\eta(Y)\eta(U),$$

where

$$\alpha=\frac{k[8n^3-4n^2-2+\mu(4n^2+1)]+\mu[-4n^2+2n+2+n\mu(2n+\mu)]+r(2n-1)(2n-2+\mu)}{4n(n-1)(2n+1)},$$

$$\beta=\frac{k[(n-1)(16n^3-4n)-2n\mu(2n+1)]+\mu[-4n^2+2n+2+n\mu(2n+\mu)]+r(2n-1)(2n-2+\mu)}{4n(n-1)(2n+1)}.$$

Thus we can state the following theorem:

Theorem 3.1. *A $(2n+1)$-dimensional $(n>1)$ non-Sasakian ϕ-concircularly flat (k,μ)-contact metric manifold is always an η-Einstein manifold.*

Now from Theorem 3.1 and Lemma 2.1, we have following:

Corollary 3.2. *If a $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact metric manifold is ϕ-concircularly flat, then the manifold is reduces to $N(k)$-contact metric manifold.*

Taking covariant derivative of (3.5) with respect to V, we get

$$(\nabla_V S)(\phi Y,\phi U)=\frac{dr(X)(2n-1)(2n-2+\mu)}{4n(n-1)(2n+1)}(\nabla_V g)(\phi Y,\phi U). \tag{3.6}$$

If we consider the manifold of constant scalar curvature r, that is, $dr(X)=0$, we have

$$(\nabla_V S)(\phi Y,\phi U)=0.$$

This leads us to the following result:

Theorem 3.3. *A $(2n+1)$-dimensional $(n>1)$ non-Sasakian ϕ-concircularly flat (k,μ)-contact metric manifold endowed with a constant scalar curvature always admits an η-parallel Ricci tensor.*

4. Concircularly Pseudo-Symmetric (k,μ)-Contact Metric Manifold

In [9], Deszcz introduced the idea of pseudo-symmetric manifolds which is given by the condition

$$(R(X,Y).R)(U,V)W=L_R[((X\wedge Y).R)(U,V)W],$$

where L_R is some smooth function on M and $X\wedge Y$ is an endomorphism defined by

$$(X\wedge Y)U=g(Y,U)X-g(X,U)Y.$$

A $(2n+1)$-dimensional $(n>1)$ (k,μ)-contact metric manifold M is said to be concircularly pseudo-symmetric if

$$(R(X,Y).\tilde{Z})(U,V)W=L_{\tilde{Z}}[((X\wedge Y).B)(U,V)W], \tag{4.1}$$

holds on the set $U_{\tilde{Z}}=\{x\in M:\tilde{Z}\neq 0\}$ at x, where $L_{\tilde{Z}}$ is some function on $U_{\tilde{Z}}$ and $\tilde{Z}$ is the concircular curvature tensor. We now have the following result:

Theorem 4.1. *Let M be an $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact metric manifold. If M is concircular pseudo-symmetric, then M is an η-Einstein manifold.*

Proof. Let us consider (k,μ)-contact metric manifold which is concircularly pseudo-symmetric. Now it follows from (4.1) that

$$\begin{aligned}(R(X,\xi).\tilde{Z})(U,V)W=L_{\tilde{Z}}[&((X\wedge\xi)(\tilde{Z}(U,V)W)-\tilde{Z}((X\wedge\xi)U,V)W\\&-\tilde{Z}(U,(X\wedge\xi))W-\tilde{Z}(U,V)(X\wedge\xi)W].\end{aligned} \tag{4.2}$$

Now the left hand side of (4.2) becomes

$$R(X,\xi)\tilde{Z}(U,V)W-\tilde{Z}(R(X,\xi)U,V)W-\tilde{Z}(U,R(X,\xi)V)W-\tilde{Z}(U,V)R(X\wedge\xi)W. \tag{4.3}$$

By virtue of (2.7) in (4.3), we get

$$\begin{aligned}&k[\tilde{\tilde{Z}}(U,V,W,\xi)X-\tilde{\tilde{Z}}(U,V,W,X)\xi-\eta(U)\tilde{Z}(X,V)W+g(X,U)\tilde{Z}(\xi,V)W-\eta(V)\tilde{Z}(U,X)W\\&\quad+g(X,V)\tilde{Z}(U,\xi)W-\eta(W)\tilde{Z}(U,V)X+g(X,W)\tilde{Z}(U,V)\xi]\\&\quad+\mu h[\tilde{\tilde{Z}}(U,V,W,\xi)X-\tilde{\tilde{Z}}(U,V,W,X)\xi-\eta(U)\tilde{Z}(X,V)W+g(X,U)\tilde{Z}(\xi,V)W-\eta(V)\tilde{Z}(U,X)W\\&\quad+g(X,V)\tilde{Z}(U,\xi)W-\eta(W)\tilde{Z}(U,V)X+g(X,W)\tilde{Z}(U,V)\xi].\end{aligned} \tag{4.4}$$

Similarly, right hand side of (4.2) gives

$$\begin{aligned}&L_{\tilde{Z}}[\tilde{\tilde{Z}}(U,V,W,\xi)X-\tilde{\tilde{Z}}(U,V,W,X)\xi-\eta(U)\tilde{Z}(X,V)W+g(X,U)\tilde{Z}(\xi,V)W-\eta(V)\tilde{Z}(U,X)W\\&\quad+g(X,V)\tilde{Z}(U,\xi)W-\eta(W)\tilde{Z}(U,V)X+g(X,W)\tilde{Z}(U,V)\xi].\end{aligned} \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.2) with $V=\xi$, we get

$$(L_{\tilde{Z}}-(k+\mu h))[\tilde{\tilde{Z}}(U,\xi,W,\xi)X-\tilde{\tilde{Z}}(U,\xi,W,X)\xi-\eta(U)\tilde{Z}(X,\xi)W+g(X,U)\tilde{Z}(\xi,\xi)W$$
$$-\eta(\xi)\tilde{Z}(U,X)W+g(X,\xi)\tilde{Z}(U,\xi)W-\eta(W)\tilde{Z}(U,\xi)X+g(X,W)\tilde{Z}(U,\xi)\xi]=0. \tag{4.6}$$

This implies either $L_{\tilde{Z}}=(k+\mu h)$ or

$$g(\tilde{Z}(U,X)W,T)=\left(k+\mu h-\frac{r}{2n(2n+1)}\right)[g(X,W)g(U,T)-g(U,W)g(X,T)]. \tag{4.7}$$

By considering (1.1) in (4.7), gives

$$g(R(U,X)W,T)=k[g(X,W)g(U,T)-g(U,W)g(X,T)]$$
$$+\mu[g(X,W)g(hU,T)-g(U,W)g(hX,T)]. \tag{4.8}$$

Contracting above equation over U and T, we obtain

$$S(X,W)=\frac{r+n[4(1-n)+k(4n-6+2\mu)+\mu(4-\mu)]}{2(n-1+\mu)}g(X,W)$$
$$+\frac{2(1-n)+n(2k+\mu)}{2(n-1+\mu)}\eta(X)\eta(W). \tag{4.9}$$

Thus M is an η-Einstein manifold. □

Now from Theorem 4.1 and Lemma 2.1, we can obtain the following result:

Corollary 4.2. *If a $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact metric manifold is concircular pseudo-symmetric, then the manifold is an $N(k)$-contact metric manifold.*

Further, if we take $k+\mu h=\frac{r}{2n(2n+1)}$ in (4.7), we have

$$\tilde{Z}(U,X)W=0. \tag{4.10}$$

Hence we can state the following result:

Corollary 4.3. *If a $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact metric manifold is concircularly pseudo-symmetric, then the manifold is concircularly flat provided the concircularly pseudo-symmetric manifold is never reduces to concircularly semi-symmetric manifold ($L_{\tilde{Z}}=(k+\mu h)=\frac{r}{2n(2n+1)}\neq 0$).*

5. Concircularly ϕ-Recurrent (k,μ)-Contact Metric Manifold

A $(2n+1)$-dimensional $(n>1)$non-Sasakian (k,μ)-contact metric manifold is said to be concircularly ϕ-recurrent if there exists a non zero 1-form A such that

$$\phi^2((\nabla_W\widetilde{Z})(X,Y)U)=A(W)\widetilde{Z}(X,Y)U, \tag{5.1}$$

for arbitrary vector fields X, Y, U and W.

If the 1-form A vanishes, then the manifold reduces to a concircularly ϕ-symmetric.

By virtue of (2.1), equation (5.1) yields

$$-(\nabla_W\widetilde{Z})(X,Y)U+\eta((\nabla_W\widetilde{Z})(X,Y)U)\xi=A(W)\widetilde{Z}(X,Y)U. \tag{5.2}$$

Taking inner product of above expression along V and then plugging $X = V = e_i$, taking summation over i, $1 \le i \le 2n+1$, we have

$$(\nabla_W S)(Y,U) = \frac{(2n-1)dr(W)}{2n(2n+1)}g(Y,U) - \frac{dr(W)}{2n(2n+1)}\eta(Y)\eta(U) - (kI+\mu h)[g(W,\phi U) + g(hW,\phi U)]\eta(Y) - A(W)[S(Y,U) - \frac{r}{2n+1}g(Y,U)]. \tag{5.3}$$

Replacing U by ξ in (5.3), gives

$$(\nabla_W S)(Y,\xi) = \frac{1}{2n+1}[A(W)(dr(W) - 2nk(2n+1) - r)]\eta(Y). \tag{5.4}$$

Now, we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{5.5}$$

In view of (2.3) and (2.7), we get from (5.5)

$$(\nabla_W S)(Y,\xi) = 2nkg(W+hW,\phi Y) + S(Y,\phi W + \phi hW). \tag{5.6}$$

By virtue of (5.4) and (5.6), we have

$$\frac{1}{2n+1}[A(W)(dr(W) - 2nk(2n+1) - r)]\eta(Y) = 2nkg(W+hW,\phi Y) + S(Y,\phi W + \phi hW). \tag{5.7}$$

Replacing Y by ϕY in (5.7) yields

$$2nkg(\phi W + \phi hW,\phi Y) - S(\phi Y,\phi W + \phi hW) = 0. \tag{5.8}$$

Now the above equation takes the form

$$S(Y,W) + S(Y,hW) = 2nkg(Y,W) + [2nk + 2(2n-2+\mu)]g(Y,hW) + 2(2n-2+\mu)(k-1)g(Y,-W+\eta(W)\xi). \tag{5.9}$$

Now it follows from (2.8) that

$$S(Y,hW) = (2n-2-n\mu)g(Y,hW) - (2n-2+\mu)(k-1)g(Y,W) + (2n-2+\mu)(k-1)g(Y,-W+\eta(W)\xi). \tag{5.10}$$

Hence we get from (5.9) that

$$S(Y,W) = [\mu(1-k) + 2(n-1) + 2k]g(Y,W) + [2(nk+n-1) + \mu(n+2)]g(Y,hW) + (2n-2+\mu)(k-1)\eta(Y)\eta(W). \tag{5.11}$$

Plugging W by hW in (5.11), we get

$$S(Y,hW) = [\mu(1-k)+2(n-1)+2k]g(Y,hW) + [2(nk+n-1)+\mu(n+2)]g(Y,(k-1)\phi^2 W). \tag{5.12}$$

From (5.10) and (5.12), it follows that

$$[\mu(k-1-n)-2k]g(Y,hW) = (k-1)[-2nk-\mu(n+1)]g(Y,W) + (k-1)[2nk+\mu(n+1)]\eta(Y)\eta(W). \tag{5.13}$$

By considering (5.13) in (5.11), we have

$$S(Y,W) = \alpha' g(Y,W) + \beta' \eta(Y)\eta(W), \tag{5.14}$$

where

$$\alpha' = \frac{[\mu(1-k)+2(n-1)+2k][\mu(k-1-n)-2k] + [-2nk-\mu(n+1)](k-1)[-2nk-\mu(n+1)]}{\mu(k-1-n)-2k},$$

$$\beta' = \frac{[2(n-1)+\mu(k-1)][\mu(k-1-n)-2k]+[2(nk+n-1)+\mu(n+2)][2nk+\mu(n+1)(k-1)]}{\mu(k-1-n)-2k}.$$

Thus we can state the following result:

Theorem 5.1. *A $(2n+1)$-dimensional $(n>1)$ non-Sasakian concircularly ϕ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold.*

Hence, from Theorem 5.1 and Lemma 2.1, we can obtain the following result:

Corollary 5.2. *If a $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact metric manifold is concircularly ϕ-recurrent then the manifold is reduces to $N(k)$-contact metric manifold.*

6. Example

We consider the 3-dimensional manifold $M=\{(x_1,x_2,x_3)\in R^3 \mid x_1\neq 0\}$, where (x_1,x_2,x_3) are the standard coordinates in R^3. Let $\{E_1,E_2,E_3\}$ be an linearly independent global frame on M such that

$$[E_1,E_2]=2E_3+\frac{2}{x_1}E_1,\quad [E_2,E_3]=2E_1,\quad [E_3,E_1]=0.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1,$$
$$g(E_1,E_2)=g(E_2,E_3)=g(E_1,E_3)=0.$$

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi E_1=E_2,\quad \phi E_2=-E_1,\quad \phi E_3=0.$$

Moreover

$$hE_1=-E_1,\quad hE_2=E_2,\quad hE_3=0.$$

Thus for $E_3=\xi$, (ϕ,ξ,η,g) defines a contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and by using the Koszul's formula, we can easily obtained the following:

$$\left.\begin{array}{lll} \nabla_{E_1}E_1=-\frac{2}{x_1}E_2, & \nabla_{E_1}E_2=\frac{2}{x_1}E_1, & \nabla_{E_1}E_3=0,\\ \nabla_{E_2}E_1=-2E_3, & \nabla_{E_2}E_2=0, & \nabla_{E_2}E_3=2E_1,\\ \nabla_{E_3}E_1=0, & \nabla_{E_3}E_2=0, & \nabla_{E_3}E_3=0. \end{array}\right\} \tag{6.1}$$

From the above expressions, it can be easily seen that (ϕ,ξ,η,g) is a non-Sasakian (k,μ)-contact metric structure on M with $k=-\frac{2}{x_1}\neq 0$ and $\mu=-\frac{2}{x_1}\neq 0$.

Now by using the relations in (6.1), we can easily found the non-vanishing components of the Riemannian curvature tensor R as follows:

$$R(E_2,E_3)E_1=\frac{4}{x_1}E_2.$$

Since $\{E_1,E_2,E_3\}$ forms a basis of M^3, any vector field $X\in\chi(M)$ can be written as

$$W=c_1E_1+c_2E_2+c_3E_3,$$

where c_i, $i=1,2,3$ are positive real numbers.

Thus the covariant derivative of the Riemannian curvature tensor R is given by

$$(\nabla_W R)(E_2,E_3)E_1 = -\frac{8c_2}{x_1^2}E_2 .$$

Let us define a non-vanishing 1-form A as

$$A(W) = \frac{2c_2}{x_1},$$

at any point $p \in M^3$.

Then in view of (1.1), the component of concircular curvature tensor is given by

$$\widetilde{Z}(E_2,E_3)E_1 = \frac{4}{x_1}E_2 . \tag{6.2}$$

Thus the covariant derivative of (6.2) is as follows:

$$(\nabla_W \widetilde{Z})(E_2,E_3)E_1 = -\frac{8c_2}{x_1^2}E_2$$

and so

$$\phi^2((\nabla_W \widetilde{Z})(E_2,E_3)E_1) = \frac{8c_2}{x_1^2}E_2 = A(W)\widetilde{Z}(E_2,E_3)E_1 .$$

This shows that the existence of concircularly ϕ-recurrent non-Sasakian (k,μ)-contact metric manifold.

7. Conclusion

In this study, we have proved that a $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact metric manifold is ϕ-concircularly flat, concircularly pseudo-symmetric and concircularly ϕ-recurrent, then the manifold always reduces to $N(k)$-contact metric manifold.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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