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Research Article

Common Fixed Point Theorems for Weakly Compatible Maps Satisfying Integral Type Contraction in *G*-Metric Spaces

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Abstract. The main purpose of this manuscript is to prove a common fixed point theorem for two weakly compatible maps satisfying the following integral type contraction in G-metric space:

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)} \varphi(t)dt \le \alpha \int_0^{L(x,y,z)} \varphi(t)dt,$$

for all $x,y,z \in X$, where

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,\mathcal{F}x,\mathcal{F}x), G(gx,\mathcal{F}y,\mathcal{F}y), G(gy,\mathcal{F}y,\mathcal{F}y), G(gy,\mathcal{F}x,\mathcal{F}x), G(gz,\mathcal{F}z,\mathcal{F}z), G(gz,\mathcal{F}x,\mathcal{F}x), G(gz,\mathcal{F}y,\mathcal{F}y)\}.$$

Also, we have proved common fixed point theorems for the above mentioned weakly compatible self-mappings along with E.A property and (CLR_g) property.

Keywords. Fixed point, G-metric space, Coincidence point, Weakly compatible maps, E.A property, (CLR_g) property

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

In 2006, Mustafa and Sims [7] introduced the concept of G-metric spaces as follows:

Definition 1.1 ([7]). Let X be a non-empty set and $G : X \times X \times X \to \mathbb{R}$ be a function satisfying the following conditions:

($G1$) $G(x,y,z) = 0$ iff $x = y = z$.

($G2$) $0 < G(x,x,z)$ for all $x,z \in X$ with $x \neq z$.

($G3$) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$.

($G4$) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \dots$ (symmetry in all variables).

($G5$) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then, the function G is called a generalized metric or more specifically a G-metric space on X and the pair (X,G) is called G-metric space.

Definition 1.2 ([7]). Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that the sequence $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to\infty} G(x,x_n,x_m) = 0,$$

that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.1 ([7]). *Let (X,G) be a G-metric space. The followings are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x ;*

(ii) *$G(x_n,x_n,x) \to 0$ as $n \to +\infty$;*

(iii) *$G(x_n,x,x) \to 0$ as $n \to +\infty$;*

(iv) *$G(x_n,x_m,x) \to 0$ as $n,m \to +\infty$.*

Definition 1.3 ([7]). Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \geq N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to 0$.

Proposition 1.2 ([7]). *Let (X,G) be a G-metric space. The followings are equivalent:*

(i) *the sequence $\{x_n\}$ is G-Cauchy ;*

(ii) *for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$ for all $n,m \geq N$.*

Proposition 1.3 ([7]). *Let (X,G) be a G-metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.4 ([7]). A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Proposition 1.4 ([7]). *Let (X,G) be a G-metric space. Then, for any $x,y,z,a \in X$, it follows that:*

(i) *if $G(x,y,z) = 0$, then $x = y = z$;*

(ii) *$G(x,y,z) \leq G(x,x,y) + G(x,x,z)$;*

(iii) *$G(x,y,y) \leq 2G(y,x,x)$;*

(iv) *$G(x,y,z) \leq G(x,a,z) + G(a,y,z)$;*

(v) $G(x,y,z) \leq \frac{2}{3}(G(x,y,a)+G(x,a,z)+G(a,y,z))$;

(vi) $G(x,y,z) \leq G(x,a,a)+G(y,a,a)+G(z,a,a)$.

Definition 1.5. A G-metric space (X,G) is called symmetric G-metric space if

$$G(x,y,y) = G(y,x,x), \quad \text{for all } x,y \text{ in } X.$$

Definition 1.6. A coincidence point of a pair of self-maps $\mathcal{F},g : X \to X$ is a point $\mu \in X$ for which $\mathcal{F}\mu = g\mu$.

A common fixed point of a pair of self-mappings$\mathcal{F},g : X \to X$ is a point $\mu \in X$ for which $\mathcal{F}\mu = g\mu = \mu$.

In 1996, Jungck [6] introduced the concept of weakly compatible mappings to study common fixed point theorems:

Definition 1.7 ([6])**.** Let (X,d) be a metric space. A pair of self-maps $\mathcal{F},g : X \to X$ is weakly compatible if they commute at their coincidence points, that is, if there exists $\mu \in X$, such that $Fg\mu = g\mathcal{F}\mu$, where μ is coincidence point of $\mathcal{F}$ and g.

Proposition 1.5 ([2])**.** *Let F and g be weakly compatible self-mappings of a non-empty set X. If F and g have a unique point of coincidence μ, that is $w = \mathcal{F}\mu = g\mu$, then w is the unique common fixed point of $\mathcal{F}$ and g.*

In 2002, Aamri and El Moutawakil [1] introduced the notion of E.A property as follows:

Definition 1.8 ([1])**.** Let (X,d) be a metric space. Two self-maps P and Q on X are said to satisfy the E.A property, if there exists a sequence $\{\mu_n\}$ in X such that

$$\lim_{n\to\infty} P\mu_n = \lim_{n\to\infty} Q\mu_n = t, \quad \text{for some } t \in X.$$

Inspired by Aamri and El Moutawakil [1], we define the property (E.A) in G-metric space as follows:

Definition 1.9 ([1])**.** A pair $(\mathcal{F},g)$ of self-mappings of a G-metric space (X,G) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\{\mathcal{F}x_n\}$ and $\{gx_n\}$ G-converges to $z \in X$, that is,

$$\lim_{n\to\infty} G(\mathcal{F}x_n,\mathcal{F}x_n,z) = \lim_{n\to\infty} G(gx_n,gx_n,z) = 0.$$

In 2011, Sintunavarat *et al.* [9] introduced the notion of (CLR) property as follows:

Definition 1.10 ([9])**.** Let (X,d) be a metric space. Two self-mappings P and Q on X are said to satisfy the (CLR_P) property, if there exists a sequence $\{\mu_n\}$ in X such that,

$$\lim_{n\to\infty} P\mu_n = \lim_{n\to\infty} Q\mu_n = P(t), \quad \text{for some } t \in X.$$

Inspired by Sintunavarat *et al.* [9], we define the property (CLR) with respect to the mapping $\underbrace{g}$ in G-metric space as follows:

Definition 1.11. A pair $(\mathcal{F},g)$ of self mappings of a G-metric space (X,G) is said to satisfy the property (CLR_g) if there exists a sequence $\{x_n\}$ such that $\{\mathcal{F}x_n\}$ and $\{gx_n\}$ G-converges to $z \in X$, that is,

$$\lim_{n\to\infty} G(\mathcal{F}x_n,\mathcal{F}x_n,gz) = \lim_{n\to\infty} G(gx_n,gx_n,gz) = 0.$$

2. Main Results

In 2012, Aydi [3] proved the following result:

Theorem 2.1 ([3, Theorem 3.1])**.** *Let (X,G) be a G-metric space and $\mathcal{F},g : X \to X$ such that*

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)} \varphi(t)dt \le \alpha \int_0^{G(gx,gy,gz)} \varphi(t)dt, \tag{2.1}$$

for all $x,y,z \in X$, where $\alpha \in [0,1)$ and $\varphi : [0,\infty) \to [0,\infty)$ is a lebesgue integrable mapping which is summable, non negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{2.2}$$

Assume that $\mathcal{F}(X) \subset g(X)$ and $g(X)$ is complete subspace of X, then $\mathcal{F}$ and g have a unique point of coincidence in X. Moreover, if $\mathcal{F}$ and g are weakly compatible, then $\mathcal{F}$ and g have a unique common fixed point in X.

Now, we prove our main result which generalizes the result of Aydi [3].

Theorem 2.2. *Let (X,G) be a G-metric space and $\mathcal{F},g : X \to X$ such that*

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)} \varphi(t)dt \le \alpha \int_0^{L(x,y,z)} \varphi(t)dt, \tag{2.3}$$

where

$$\begin{aligned} L(x,y,z) = \max\{&G(gx,gy,gz), G(gx,\mathcal{F}x,\mathcal{F}x), G(gx,\mathcal{F}y,\mathcal{F}y), G(gy,\mathcal{F}y,\mathcal{F}y), \\ &G(gy,\mathcal{F}x,\mathcal{F}x), G(gz,\mathcal{F}z,\mathcal{F}z), G(gz,\mathcal{F}x,\mathcal{F}x), G(gz,\mathcal{F}y,\mathcal{F}y)\}, \end{aligned}$$

for all $x,y,z \in X$ where $\alpha \in [0,1)$ and $\varphi : [0,\infty) \to [0,\infty)$ is a lebesgue integrable mapping which is summable, non negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{2.4}$$

Assume that $\mathcal{F}(X) \subset g(X)$ and $g(X)$ is complete subspace of X, then $\mathcal{F}$ and g have a unique point of coincidence in X. Moreover if $\mathcal{F}$ and g are weakly compatible, then $\mathcal{F}$ and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point of X. Since $\mathcal{F}(X) \subset g(X)$, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$y_n = \mathcal{F}x_n = gx_{n+1}, \quad n = 0,1,2,\dots. \tag{2.5}$$

On putting, $x = x_n$, $y = x_{n+1}$ and $z = x_{n+1}$ in (2.3) and using (2.5), we get

$$\int_0^{G(\mathcal{F}x_n,\mathcal{F}x_{n+1},\mathcal{F}x_{n+1})} \varphi(t)dt \le \alpha \int_0^{L(x_n,x_{n+1},x_{n+1})} \varphi(t)dt, \tag{2.6}$$

where

$$
\begin{aligned}
L(x_n,x_{n+1},x_{n+1}) &= \max\{G(gx_n,gx_{n+1},gx_{n+1}),G(gx_n,\mathfrak{F}x_n,\mathfrak{F}x_n),\\
&\qquad G(gx_n,\mathfrak{F}x_{n+1},\mathfrak{F}x_{n+1}),G(gx_{n+1},\mathfrak{F}x_{n+1},\mathfrak{F}x_{n+1}),\\
&\qquad G(gx_{n+1},\mathfrak{F}x_n,\mathfrak{F}x_n),G(gx_{n+1},\mathfrak{F}x_{n+1},\mathfrak{F}x_{n+1}),\\
&\qquad G(gx_{n+1},\mathfrak{F}x_n,\mathfrak{F}x_n),G(gx_{n+1},\mathfrak{F}x_{n+1},\mathfrak{F}x_{n+1})\}\\
&= \max\{G(y_{n-1},y_n,y_n),G(y_{n-1},y_n,y_n),G(y_{n-1},y_n,y_n),G(y_n,y_{n+1},y_{n+1}),\\
&\qquad G(y_n,y_n,y_n),G(y_n,y_{n+1},y_{n+1}),G(y_n,y_n,y_n),G(y_n,y_{n+1},y_{n+1})\}\\
&= \max\{G(y_{n-1},y_n,y_n),G(y_n,y_{n+1},y_{n+1})\}.
\end{aligned}
$$

If $L(x_n,x_{n+1},x_{n+1}) = G(y_{n-1},y_n,y_n)$, from (2.6), we have

$$
\begin{aligned}
\int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt &= \int_0^{G(\mathfrak{F}x_n,\mathfrak{F}x_{n+1},\mathfrak{F}x_{n+1})}\varphi(t)dt\\
&\le \alpha\int_0^{L(x_n,x_{n+1},x_{n+1})}\varphi(t)dt,\\
\int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt &\le \alpha\int_0^{G(y_{n-1},y_n,y_n)}\varphi(t)dt\\
&\le \alpha^2\int_0^{G(y_{n-2},y_{n-1},y_{n-1})}\varphi(t)dt,\\
&\le \alpha^n\int_0^{G(y_0,y_1,y_1)}\varphi(t)dt,\\
\lim_{n\to\infty}\int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt &= 0,
\end{aligned}
$$

since $\alpha\in[0,1)$, this implies $\lim_{n\to\infty} G(y_n,y_{n+1},y_{n+1}) = 0$.
If $L(x_n,x_{n+1},x_{n+1}) = G(y_n,y_{n+1},y_{n+1})$,from (2.6), we have

$$
\begin{aligned}
\int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt &= \int_0^{G(\mathfrak{F}x_n,\mathfrak{F}x_{n+1},\mathfrak{F}x_{n+1})}\varphi(t)dt\\
&\le \alpha\int_0^{L(x_n,x_{n+1},x_{n+1})}\varphi(t)dt,\\
\int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt &\le \alpha\int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt\\
&< \int_0^{G(y_n,y_{n+1},y_{n+1})}\varphi(t)dt,
\end{aligned}
$$

a contradiction. Hence, we have

$$G(y_n,y_{n+1},y_{n+1}) = 0. \tag{2.7}$$

Now, we shall prove that $\{y_n\}$ is G-Cauchy sequence. Suppose that $\{y_n\}$ is not a G-Cauchy sequence. Then, there exists $\epsilon > 0$ and n, $m > 0$ with $m(\alpha) > n(\alpha) > 2\alpha$ satisfying

$$G(y_{m(\alpha)},y_{m(\alpha)},y_{l(\alpha)}) \ge \epsilon. \tag{2.8}$$

Now, corresponding to $l(\alpha)$, we choose $m(\alpha)$ to be the smallest for which (2.8) holds.

So, we have

$$G(y_{m(\alpha)-1}, y_{m(\alpha)-1}, y_{l(\alpha)}) < \epsilon. \tag{2.9}$$

Now, using (2.7) and the rectangle inequality, we have

$$\begin{aligned}\epsilon &\le G(y_{m(\alpha)}, y_{m(\alpha)}, y_{l(\alpha)})\\ &\le G(y_{m(\alpha)}, y_{m(\alpha)}, y_{m(\alpha)-1}) + G(y_{m(\alpha)-1}, y_{m(\alpha)-1}, y_{l(\alpha)})\\ &< \epsilon + G(y_{m(\alpha)-1}, y_{m(\alpha)-1}, y_{l(\alpha)}).\end{aligned}$$

Letting $\alpha \to \infty$ in the above inequality and using (2.9), we have

$$\lim_{\alpha\to\infty} G(y_{m(\alpha)}, y_{m(\alpha)}, y_{l(\alpha)}) = \epsilon. \tag{2.10}$$

Again, a rectangle inequality gives us

$$\begin{aligned}G(y_{m(\alpha)-1}, y_{m(\alpha)-1}, y_{l(\alpha)-1}) &\le G(y_{m(\alpha)-1}, y_{m(\alpha)-1}, y_{l(\alpha)}) + G(y_{l(\alpha)}, y_{l(\alpha)}, y_{l(\alpha)-1})\\ &< \epsilon + G(y_{l(\alpha)}, y_{l(\alpha)}, y_{l(\alpha)-1}).\end{aligned}$$

Now, using (2.7) and letting limit as $\alpha \to \infty$, we have

$$\lim_{\alpha\to\infty} G(y_{m(\alpha)-1}, y_{m(\alpha)-1}, y_{l(\alpha)-1}) \le \epsilon. \tag{2.11}$$

Similarly, we have

$$G(y_{m(\alpha)-1}, y_{l(\alpha)}, y_{l(\alpha)}) \le \epsilon$$

and

$$G(y_{l(\alpha)-1,} y_{m(\alpha)}, y_{m(\alpha)}) \le \epsilon.$$

On putting, $x = x_{m(\alpha)}$, $y = x_{l(\alpha)}$ and $z = x_{l(\alpha)}$ in (2.3) and using (2.7)-(2.11), we get

$$\int_0^{G(\mathfrak{F}x_{m(\alpha)},\mathfrak{F}x_{l(\alpha)},\mathfrak{F}x_{l(\alpha)})} \varphi(t)dt \le \alpha \int_0^{L(x_{m(\alpha)},x_{l(\alpha)},x_{l(\alpha)})} \varphi(t)dt, \tag{2.12}$$

where

$$\begin{aligned}L(x_{m(\alpha)}, x_{l(\alpha)}, x_{l(\alpha)}) &= \max\{G(gx_{m(\alpha)}, gx_{l(\alpha)}, gx_{l(\alpha)}), G(gx_{m(\alpha)}, \mathfrak{F}x_{m(\alpha)}, \mathfrak{F}x_{m(\alpha)}),\\ &\qquad G(gx_{m(\alpha)}, \mathfrak{F}x_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)}), G(gx_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)}),\\ &\qquad G(gx_{l(\alpha)}, \mathfrak{F}x_{m(\alpha)}, \mathfrak{F}x_{m(\alpha)}), G(gx_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)}),\\ &\qquad G(gx_{l(\alpha)}, \mathfrak{F}x_{m(\alpha)}, \mathfrak{F}x_{m(\alpha)}), G(gx_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)}, \mathfrak{F}x_{l(\alpha)})\}\\ &= \max\{G(y_{m(\alpha)-1}, y_{l(\alpha)-1}, y_{l(\alpha)-1}), G(y_{m(\alpha)-1}, y_{m(\alpha)}, y_{m(\alpha)}),\\ &\qquad G(y_{m(\alpha)-1}, y_{l(\alpha)}, y_{l(\alpha)}), G(y_{l(\alpha)-1}, y_{l(\alpha)}, y_{l(\alpha)}),\\ &\qquad G(y_{l(\alpha)-1}, y_{m(\alpha)}, y_{m(\alpha)}), G(y_{l(\alpha)-1}, y_{l(\alpha)}, y_{l(\alpha)}),\\ &\qquad G(y_{l(\alpha)-1}, y_{m(\alpha)}, y_{m(\alpha)}), G(y_{l(\alpha)-1}, y_{l(\alpha)}, y_{l(\alpha)})\}\\ &= \max\{\epsilon, 0, \epsilon, 0, \epsilon, 0, \epsilon, 0\} \qquad (\text{as } \alpha \to \infty)\\ &= \epsilon\end{aligned}$$

and

$$0 < \int_0^{\epsilon} \varphi(t)dt$$

$$
\begin{aligned}
&= \int_0^{\lim_{\alpha\to\infty} G(y_{m(\alpha)},y_{m(\alpha)},y_{l(\alpha)})} \varphi(t)dt \\
&= \int_0^{\lim_{\alpha\to\infty} G(\mathcal{F}x_{m(\alpha)},\mathcal{F}x_{m(\alpha)},\mathcal{F}x_{l(\alpha)})} \varphi(t)dt \\
&\le \alpha \int_0^{\lim_{\alpha\to\infty} L(y_{m(\alpha)},y_{m(\alpha)},y_{l(\alpha)})} \varphi(t)dt \\
&\le \alpha \int_0^{\epsilon} \varphi(t)dt \\
&< \int_0^{\epsilon} \varphi(t)dt,
\end{aligned}
$$

a contradiction. Since $\alpha \in [0,1)$. Thus, we proved that $\{y_n\}$ is a G-Cauchy sequence. Since gX is complete, we obtain that $\{gx_n\}$ is G-convergent to some $q \in gX$. So, there exists some p in X such that

$$
\begin{aligned}
&gp = q, \\
&\lim_{n\to\infty} y_n = \lim_{n\to\infty} \mathcal{F}x_n = \lim_{n\to\infty} gx_{n+1} = q = gp. \qquad (2.13)
\end{aligned}
$$

By Proposition 1.1, we have

$$\lim_{n\to\infty} G(gx_n,gx_n,gp) = \lim_{n\to\infty} G(gx_n,gp,gp) = 0.$$

Now, we shall prove that $gp = \mathcal{F}p$. Let, if possible $gp \neq \mathcal{F}p$.

On putting, $x = x_n$, $y = p$ and $z = p$ in (2.3) and using (2.13), we get

$$\int_0^{G(\mathcal{F}x_n,\mathcal{F}p,\mathcal{F}p)} \varphi(t)dt \le \alpha \int_0^{L(x_n,p,p)} \varphi(t)dt, \qquad (2.14)$$

where

$$
\begin{aligned}
L(x_n,p,p) = \max\{&G(gx_n,gp,gp), G(gx_n,\mathcal{F}x_n,\mathcal{F}x_n), G(gx_n,\mathcal{F}p,\mathcal{F}p), \\
&G(gp,\mathcal{F}p,\mathcal{F}p), G(gp,\mathcal{F}x_n,\mathcal{F}x_n), G(gp,\mathcal{F}p,\mathcal{F}p), \\
&G(gx_{n+1},\mathcal{F}x_n,\mathcal{F}x_n), G(gx_{n+1},\mathcal{F}x_{n+1},\mathcal{F}x_{n+1})\}.
\end{aligned}
$$

Letting limit as $n \to \infty$, we have

$$
\begin{aligned}
\lim_{n\to\infty} L(x_n,p,p) = \max\{&G(gp,gp,gp), G(gp,gp,gp), G(gp,\mathcal{F}p,\mathcal{F}p), \\
&G(gp,\mathcal{F}p,\mathcal{F}p), G(gp,gp,gp), G(gp,\mathcal{F}p,\mathcal{F}p), \\
&G(gp,\mathcal{F}p,\mathcal{F}p), G(gp,gp,gp)\} \\
= G(gp,\mathcal{F}p,\mathcal{F}p)&
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\le \int_0^{G(gp,\mathcal{F}p,\mathcal{F}p)} \varphi(t)dt \\
&= \int_0^{\lim_{n\to\infty} G(\mathcal{F}x_n,\mathcal{F}p,\mathcal{F}p)} \varphi(t)dt \\
&\le \alpha \int_0^{\lim_{n\to\infty} L(x_n,p,p)} \varphi(t)dt
\end{aligned}
$$

$$\leq \alpha \int_0^{G(gp,\mathcal{F}p,\mathcal{F}p)} \varphi(t)dt$$
$$< \int_0^{G(gp,\mathcal{F}p,\mathcal{F}p)} \varphi(t)dt,$$

a contradiction. Hence $G(gp,\mathcal{F}p,\mathcal{F}p) = 0$, which implies that $gp = \mathcal{F}p = w$ (say).
Now, since $\mathcal{F}$ and g are weakly compatible maps, it follows that

$$\mathcal{F}gp = g\mathcal{F}p, \text{ i.e., } \mathcal{F}w = gw. \tag{2.15}$$

Now, we shall prove that $w = gw$.
On putting, $x = w$, $y = w$ and $z = p$ in (2.3) and using (2.15), we get

$$\int_0^{G(\mathcal{F}w,\mathcal{F}w,\mathcal{F}p)} \varphi(t)dt \leq \alpha \int_0^{L(w,w,p)} \varphi(t)dt, \tag{2.16}$$

where

$$\begin{aligned} L(w,w,p) = \max\{&G(gw,gw,gp),G(gw,\mathcal{F}w,\mathcal{F}w),G(gw,\mathcal{F}w,\mathcal{F}w),G(gw,\mathcal{F}w,\mathcal{F}w),\\ &G(gw,\mathcal{F}w,\mathcal{F}w),G(gw,\mathcal{F}w,\mathcal{F}w),G(gp,\mathcal{F}p,\mathcal{F}p),G(gp,\mathcal{F}w,\mathcal{F}w)\}\\ = \max\{&G(gw,gw,w),0,0,0,0,0,0,G(w,gw,gw)\}\\ = G&(gw,gw,w) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \int_0^{G(gw,gw,w)} \varphi(t)dt \\ &= \int_0^{G(\mathcal{F}w,\mathcal{F}w,\mathcal{F}p)} \varphi(t)dt \\ &\leq \alpha \int_0^{L(w,w,p)} \varphi(t)dt \\ &\leq \alpha \int_0^{G(gw,gw,w)} \varphi(t)dt \\ &< \int_0^{G(gw,gw,w)} \varphi(t)dt, \end{aligned}$$

a contradiction.
Hence $G(gw,gw,w) = 0$, implies that, $gw = w$.
Hence $gw = w = \mathcal{F}w$, which shows that w is a common fixed point of $\mathcal{F}$ and g.
For uniqueness of common fixed point, let a and b be two common fixed points of $\mathcal{F}$ and g.
Now, from (2.3), we can obtain

$$\int_0^{G(\mathcal{F}a,\mathcal{F}a,\mathcal{F}b)} \varphi(t)dt \leq \alpha \int_0^{L(a,a,b)} \varphi(t)dt,$$

where

$$\begin{aligned} L(a,a,b) = \max\{&G(ga,ga,gb),G(ga,\mathcal{F}a,\mathcal{F}a),G(ga,\mathcal{F}a,\mathcal{F}a),G(ga,\mathcal{F}a,\mathcal{F}a)\\ &G(ga,\mathcal{F}a,\mathcal{F}a),G(gb,\mathcal{F}b,\mathcal{F}b),G(gb,\mathcal{F}a,\mathcal{F}a),G(gb,\mathcal{F}a,\mathcal{F}a)\}\\ = G&(a,a,b), \end{aligned}$$

which implies that

$$\int_0^{G(a,a,b)} \varphi(t)dt \le \alpha \int_0^{L(a,a,b)} \varphi(t)dt$$
$$< \int_0^{G(a,a,b)} \varphi(t)dt,$$

a contradiction. Hence $\mathfrak{F}$ and g have a unique common fixed point. This completes the proof of the theorem. □

Theorem 2.3. *Let (X,G) be a G-metric space and $\mathfrak{F}$ and g be self-maps on X satisfying* (2.3) *and* (2.4)*. Let $g(X)$ is a complete subspace of X, and if $\mathfrak{F}$ and g are weakly compatible. Also, $(\mathfrak{F},g)$ satisfying E.A property, then $\mathfrak{F}$ and g have a unique common fixed point in X.*

Proof. Since $\mathfrak{F}$ and g satisfy E.A property, therefore, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathfrak{F}x_n = \lim_{n\to\infty} gx_n = u, \quad \text{for some } u \text{ in } X.$$

Since $g(X)$ is a complete subspace of X, therefore

$$\lim_{n\to\infty} \mathfrak{F}x_n = \lim_{n\to\infty} gx_n = u = ga, \quad \text{for some } a \text{ in } X. \tag{2.17}$$

Now, we claim that $\mathfrak{F}a = ga = u$.

On putting, $x = x_n$, $y = a$ and $z = a$ in (2.3) and using (2.17), we get

$$\int_0^{G(\mathfrak{F}x_n,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt \le \alpha \int_0^{L(x_n,a,a)} \varphi(t)dt, \tag{2.18}$$

where

$$L(x_n,a,a) = \max\{G(gx_n,ga,ga), G(gx_n,\mathfrak{F}x_n,\mathfrak{F}x_n), G(gx_n,\mathfrak{F}a,\mathfrak{F}a), G(ga,\mathfrak{F}a,\mathfrak{F}a),$$
$$G(ga,\mathfrak{F}x_n,\mathfrak{F}x_n), G(ga,\mathfrak{F}a,\mathfrak{F}a), G(ga,\mathfrak{F}x_n,\mathfrak{F}x_n), G(ga,\mathfrak{F}a,\mathfrak{F}a)\}.$$

On letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} L(x_n,a,a) = \max\{0,0,G(ga,\mathfrak{F}a,\mathfrak{F}a),G(ga,\mathfrak{F}a,\mathfrak{F}a),0,G(ga,\mathfrak{F}a,\mathfrak{F}a),0,G(ga,\mathfrak{F}a,\mathfrak{F}a)\}$$
$$= G(ga,\mathfrak{F}a,\mathfrak{F}a)$$

and

$$0 \le \int_0^{G(\underbrace{g}\, a,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt = \int_0^{\lim_{n\to\infty} G(\mathfrak{F}x_n,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt$$
$$\le \alpha \int_0^{\lim_{n\to\infty} L(x_n,a,a)} \varphi(t)dt$$
$$\le \alpha \int_0^{G(ga,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt$$
$$< \int_0^{G(ga,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt,$$

a contradiction.

Hence $G(ga,\mathfrak{F}a,\mathfrak{F}a) = 0$, which implies that, $ga = \mathfrak{F}a = w$ (say).

Now, since $\mathfrak{F}$ and g are weakly compatible maps, it follows that

$$\mathfrak{F}ga = g\mathfrak{F}a, \ \text{ i.e., } \ \mathfrak{F}w = gw. \tag{2.19}$$

On putting, $x = w$, $y = w$ and $z = p$ in (2.3) and using (2.19), we get

$$\int_0^{G(\mathcal{F}w,\mathcal{F}w,\mathcal{F}p)} \varphi(t)dt \leq \alpha \int_0^{L(w,w,p)} \varphi(t)dt, \tag{2.20}$$

where

$$\begin{aligned} L(w,w,p) &= \max\{G(gw,gw,gp), G(gw,\mathcal{F}w,\mathcal{F}w), G(gw,\mathcal{F}w,\mathcal{F}w), G(gw,\mathcal{F}w,\mathcal{F}w), \\ &\qquad G(gw,\mathcal{F}w,\mathcal{F}w), G(gw,\mathcal{F}w,\mathcal{F}w), G(gp,\mathcal{F}p,\mathcal{F}p), G(gp,\mathcal{F}w,\mathcal{F}w)\} \\ &= \max\{G(gw,gw,w),0,0,0,0,0,0,G(w,gw,gw)\} \\ &= G(gw,gw,w) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \int_0^{G(gw,gw,w)} \varphi(t)dt \\ &= \int_0^{G(\mathcal{F}w,\mathcal{F}w,\mathcal{F}p)} \varphi(t)dt \\ &\leq \alpha \int_0^{L(w,w,p)} \varphi(t)dt \\ &\leq \alpha \int_0^{G(gw,gw,w)} \varphi(t)dt \\ &< \int_0^{G(gw,gw,w)} \varphi(t)dt, \end{aligned}$$

a contradiction. Hence $G(gw,gw,w) = 0$, which implies that, $gw = w$.
Hence $gw = w = \mathcal{F}w$, which shows that w is common fixed point of $\mathcal{F}$ and g. We can prove the uniqueness in the similar way as of Theorem 2.2.
This completes the proof of the theorem. □

Theorem 2.4. *Let* (X,G) *be a* G*-metric space and* $\mathcal{F}$ *and* g *be self-maps on* X *satisfying* (2.3) *and* (2.4)*. If* $\mathcal{F}$ *and* g *are weakly compatible and* $(\mathcal{F},g)$ *satisfying* (CLR_g) *property, then* $\mathcal{F}$ *and* g *have a unique common fixed point in* X.

Proof. Since $\mathcal{F}$ and g satisfy (CLR_g) property, therefore, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{F}x_n = \lim_{n\to\infty} gx_n = ga = u, \quad \text{for some } u,a \text{ in } X. \tag{2.21}$$

Now, we claim that $\mathcal{F}a = ga = u$.
On putting, $x = x_n$, $y = a$ and $z = a$ in (2.3) and using (2.21), we get

$$\int_0^{G(\mathcal{F}x_n,\mathcal{F}a,\mathcal{F}a)} \varphi(t)dt \leq \alpha \int_0^{L(x_n,a,a)} \varphi(t)dt, \tag{2.22}$$

where

$$\begin{aligned} L(x_n,a,a) = \max\{&G(gx_n,ga,ga), G(gx_n,\mathcal{F}x_n,\mathcal{F}x_n), G(gx_n,\mathcal{F}a,\mathcal{F}a), G(ga,\mathcal{F}a,\mathcal{F}a), \\ &G(ga,\mathcal{F}x_n,\mathcal{F}x_n), G(ga,\mathcal{F}a,\mathcal{F}a), G(ga,\mathcal{F}x_n,\mathcal{F}x_n), G(ga,\mathcal{F}a,\mathcal{F}a)\}. \end{aligned}$$

Letting limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} L(x_n,a,a) &= \max\{0,0,G(u,\mathfrak{F}a,\mathfrak{F}a),G(u,\mathfrak{F}a,\mathfrak{F}a),0,G(u,\mathfrak{F}a,\mathfrak{F}a),0,G(u,\mathfrak{F}a,\mathfrak{F}a)\} \\ &= G(u,\mathfrak{F}a,\mathfrak{F}a)\end{aligned}$$

and

$$\begin{aligned}0 &\le \int_0^{G(u,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt \\ &= \int_0^{\lim_{n\to\infty} G(\mathfrak{F}x_n,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt \\ &\le \alpha \int_0^{\lim_{n\to\infty} L(x_n,a,a)} \varphi(t)dt \\ &\le \alpha \int_0^{G(u,\mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt \\ &< \int_0^{G(u,\ \mathfrak{F}a,\mathfrak{F}a)} \varphi(t)dt,\end{aligned}$$

a contradiction.

Hence $G(u,\mathfrak{F}a,\mathfrak{F}a) = 0$, which implies that $ga = \mathfrak{F}a = u$ (say).

Now, since $\mathfrak{F}$ and g are weakly compatible maps, it follows that

$$\mathfrak{F}ga = g\mathfrak{F}a, \text{ i.e., } \mathfrak{F}u = gu. \tag{2.23}$$

On putting, $x = u$, $y = u$ and $z = a$ in (2.3) and using (2.23), we get

$$\int_0^{G(\mathfrak{F}u,\mathfrak{F}u,\mathfrak{F}a)} \varphi(t)dt \le \alpha \int_0^{L(u,u,a)} \varphi(t)dt, \tag{2.24}$$

where

$$\begin{aligned}L(u,u,a) = \max\{&G(gu,gu,ga),G(gu,\mathfrak{F}u,\mathfrak{F}u),G(gu,\mathfrak{F}u,\mathfrak{F}u),G(gu,\mathfrak{F}a,\mathfrak{F}a), \\ &G(gu,\mathfrak{F}u,\mathfrak{F}u),G(gu,\mathfrak{F}u,\mathfrak{F}u),G(ga,\mathfrak{F}a,\mathfrak{F}a),G(ga,\mathfrak{F}u,\mathfrak{F}u)\} \\ = G(&gu,gu,u)\end{aligned}$$

and

$$\begin{aligned}0 &\le \int_0^{G(gu,gu,u)} \varphi(t)dt \\ &= \int_0^{G(\mathfrak{F}u,\mathfrak{F}u,\mathfrak{F}a)} \varphi(t)dt \\ &\le \alpha \int_0^{L(u,u,a)} \varphi(t)dt \\ &\le \alpha \int_0^{G(gu,gu,u)} \varphi(t)dt \\ &< \int_0^{G(gu,gu,u)} \varphi(t)dt,\end{aligned}$$

a contradiction.

Hence $G(gu,gu,u)=0$, which implies that $gu=u=\mathcal{F}u$, which shows that u is a common fixed point of $\mathcal{F}$ and g. Uniqueness of the common fixed point is easy consequence of inequality (2.3). This completes the proof of the theorem. □

Example 2.1. Let $X=[0,1]$ and let $G: X\times X\times X\to R^+$ be G-metric defined as follows:

$$G(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\},\quad \text{for all } x,y,z\in X.$$

Then (X,G) is a G-metric space. Define the self-mappings $\mathcal{F}$ and g by

$$\mathcal{F}(x)=\frac{x}{4},\quad g(x)=\frac{x}{2}.$$

Let $\{x_n\}=\{\frac{1}{n}\}_{n\geq 1}$.

Clearly, we have

$$\mathcal{F}X=\left[0,\frac{1}{4}\right)\subseteq\left[0,\frac{1}{2}\right)=gX,$$

$$\lim_{n\to\infty}\mathcal{F}(x_n)=\lim_{n\to\infty}\frac{1}{4n}=0,$$

$$\lim_{n\to\infty}g(x_n)=\lim_{n\to\infty}\frac{1}{2n}=0.$$

Hence, we have

$$\lim_{n\to\infty}\mathcal{F}(x_n)=\lim_{n\to\infty}g(x_n)=0\in X.$$

Therefore, $\mathcal{F}$ and g satisfy the E.A property.

Also, we can see that

$$\lim_{n\to\infty}\mathcal{F}(x_n)=\lim_{n\to\infty}g(x_n)=g(0).$$

This implies $\mathcal{F}$ and g satisfy the (CLR_g) property.

Also, $\mathcal{F}g(0)=g\mathcal{F}(0)=0$, implies that $\mathcal{F}$ and g are weakly compatible.

Now, we check the condition (2.3) of Theorem 2.2, that is

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)}\varphi(t)dt\leq\alpha\int_0^{L(x,y,z)}\varphi(t)dt.$$

Consider $\alpha=\frac{1}{2}$, $\varphi(t)=2t$.

Without loss of generality, take $x\leq y\leq z$.

Now, we have

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)}\varphi(t)dt=\int_0^{\frac{z-x}{4}}2t\,dt=\left(\frac{z-x}{4}\right)^2$$

and

$$\begin{aligned}L(x,y,z)&=\max\{G(gx,gy,gz),G(gx,\mathcal{F}x,\mathcal{F}x),G(gx,\mathcal{F}y,\mathcal{F}y),G(gy,\mathcal{F}y,\mathcal{F}y),\\&\qquad G(gy,\mathcal{F}x,\mathcal{F}x),G(gz,\mathcal{F}z,\mathcal{F}z),G(gz,\mathcal{F}x,\mathcal{F}x),G(gz,\mathcal{F}y,\mathcal{F}y)\}\\&=\max\left\{\frac{z-x}{2},\frac{x}{4},\frac{y-2x}{4},\frac{y}{4},\frac{2y-x}{4},\frac{z}{4},\frac{2z-x}{4},\frac{2z-y}{4}\right\}\\&=\frac{2z-x}{4},\quad\text{for all } x,y,z\in[0,1).\end{aligned}$$

One can easily check that

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)} \varphi(t)dt = \left(\frac{z-x}{4}\right)^2 \le \frac{1}{2}\left(\frac{2z-x}{4}\right)^2 = \alpha \int_0^{L(x,y,z)} \varphi(t)dt.$$

Therefore, all the conditions of Theorems 2.2, 2.3 and 2.4 are satisfied. Hence $\mathcal{F}$ and g have a unique common fixed point. In this example, it is clear that 0 is the unique common fixed point of $\mathcal{F}$ and g.

In 2013, Aydi [4] proved a common fixed point theorem satisfying φ-contraction integral type condition in G-metric spaces for (CLR_g) property:

Theorem 2.5. *Let (X,G) be a G-metric space and the pair $(\mathcal{F},g)$ of self mappings is weakly compatible such that*

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)} \varphi(t)dt \le \phi\left(\int_0^{L(x,y,z)} \varphi(t)dt\right),$$

for all $x,y,z \in X$, $\phi \in \Phi$ and $\varphi : [0,\infty) \to [0,\infty)$ is Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

where

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,\mathcal{F}x,\mathcal{F}x), G(gy,\mathcal{F}y,\mathcal{F}y), G(gz,\mathcal{F}z,\mathcal{F}z)\}.$$

If the pair $(\mathcal{F},g)$ satisfies the (CLR_g) property then $\mathcal{F}$ and g have a unique common fixed point in X.

Let Φ be the set of all functions ϕ such that $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$, for all $t \in (0,\infty)$. If $\phi \in \Phi$, then ϕ is called Φ-mapping. If ϕ is a Φ-mapping then it is obvious that:

(i) $\phi(t) < t$ for all $t \in (0,\infty)$,

(ii) $\phi(0) = 0$.

In rest of the paper, by ϕ we mean a Φ-mapping.

Now, we prove our next result satisfying φ-contraction integral type conditions in G-metric spaces which generalizes the result of Aydi [4].

Theorem 2.6. *Let (X,G) be a G-metric space and the pair $(\mathcal{F},g)$ of self mappings is weakly compatible such that*

$$\int_0^{G(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)} \varphi(t)dt \le \phi\left(\int_0^{L(x,y,z)} \varphi(t)dt\right), \tag{2.25}$$

for all $x,y,z \in X$, $\phi \in \Phi$ and $\varphi : [0,\infty) \to [0,\infty)$ is Lebesgue integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0,$$

where

$$L(x,y,z)=\max\{G(gx,gy,gz),G(gx,\mathcal{F}x,\mathcal{F}x),G(gx,\mathcal{F}y,\mathcal{F}y),G(gy,\mathcal{F}y,\mathcal{F}y)\\ G(gy,\mathcal{F}x,\mathcal{F}x),G(gz,\mathcal{F}z,\mathcal{F}z),G(gz,\mathcal{F}x,\mathcal{F}x),G(gz,\mathcal{F}y,\mathcal{F}y)\}.$$

If the pair $(\mathcal{F},g)$ satisfies the (CLR_g) property then $\mathcal{F}$ and g have a unique common fixed point in X.

Proof. If the pair $(\mathcal{F},g)$ satisfies the (CLR_g) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}\mathcal{F}(x_n)=\lim_{n\to\infty}g(x_n)=g(a),$$

for some $a\in X$.

On putting $x=x_n$, $y=x_n$ and $z=a$ in inequality (2.25), we have

$$\int_0^{G(\mathcal{F}x_n,\mathcal{F}x_n,\mathcal{F}a)}\varphi(t)dt\le\phi\left(\int_0^{L(x_n,x_n,a)}\varphi(t)dt\right), \tag{2.26}$$

where

$$\begin{aligned}L(x_n,y_n,a)=\max\{&G(gx_n,gx_n,ga),G(gx_n,\mathcal{F}x_n,\mathcal{F}x_n),G(gx_n,\mathcal{F}x_n,\mathcal{F}x_n),\\ &G(gx_n,\mathcal{F}x_n,\mathcal{F}x_n),G(gx_n,\mathcal{F}x_n,\mathcal{F}x_n),G(ga,\mathcal{F}a,\mathcal{F}a),\\ &G(ga,\mathcal{F}x_n,\mathcal{F}x_n),G(ga,\mathcal{F}x_n,\mathcal{F}x_n)\}\\ =\max\{&0,0,0,0,0,G(ga,\mathcal{F}a,\mathcal{F}a),0\}\\ =G(&ga,\mathcal{F}a,\mathcal{F}a),\text{ as } n\to\infty.\end{aligned}$$

Now, taking limit as $n\to\infty$ in inequality (2.26), we have

$$\begin{aligned}\int_0^{G(ga,ga,\mathcal{F}a)}\varphi(t)dt&\le\phi\left(\lim_{n\to\infty}\int_0^{L(x_n,x_n,a)}\varphi(t)dt\right)\\ &\le\phi\left(\int_0^{G(ga,\mathcal{F}a,\mathcal{F}a)}\varphi(t)dt\right).\end{aligned}$$

Therefore, we have

$$\int_0^{G(ga,ga,\mathcal{F}a)}\varphi(t)dt<\int_0^{G(ga,\mathcal{F}a,\mathcal{F}a)}\varphi(t)dt. \tag{2.27}$$

Similarly, we can show that

$$\int_0^{G(ga,\mathcal{F}a,\mathcal{F}a)}\varphi(t)dt<\int_0^{G(ga,ga,\mathcal{F}a)}\varphi(t)dt. \tag{2.28}$$

From (2.27) and (2.28), we have

$$\begin{aligned}\int_0^{G(ga,ga,\mathcal{F}a)}\varphi(t)dt&<\int_0^{G(ga,\mathcal{F}a,\mathcal{F}a)}\varphi(t)dt\\ &<\int_0^{G(ga,ga,\mathcal{F}a)}\varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence $\mathcal{F}a=ga$. Suppose that $b=\mathcal{F}a=ga$. Since the pair $(\mathcal{F},g)$ is weakly compatible and $b=\mathcal{F}a=ga$, therefore $\mathcal{F}b=g\mathcal{F}a=\mathcal{F}ga=gb$. Finally, we prove that $b=\mathcal{F}b$.

Let, on contrary $b \neq \mathcal{F}b$, then inequality (2.25) implies that

$$\int_0^{G(\mathcal{F}b,\mathcal{F}b,\mathcal{F}a)} \varphi(t)dt \le \phi\left(\int_0^{L(b,b,a)} \varphi(t)dt\right), \tag{2.29}$$

where

$$\begin{aligned} L(b,b,a) &= \max\{G(gb,gb,ga), G(gb,\mathcal{F}b,\mathcal{F}b), G(gb,\mathcal{F}b,\mathcal{F}b), G(gb,\mathcal{F}b,\mathcal{F}b) \\ &\qquad G(gb,\mathcal{F}b,\mathcal{F}b), G(ga,\mathcal{F}a,\mathcal{F}a), G(ga,\mathcal{F}b,\mathcal{F}b), G(ga,\mathcal{F}b,\mathcal{F}b)\} \\ &= G(\mathcal{F}b,\mathcal{F}b,b). \end{aligned}$$

From (2.29), we have

$$\begin{aligned} \int_0^{G(\mathcal{F}b,\mathcal{F}b,b)} \varphi(t)dt &\le \phi\left(\int_0^{G(\mathcal{F}b,\mathcal{F}b,b)} \varphi(t)dt\right) \\ &< \int_0^{G(\mathcal{F}b,\mathcal{F}b,b)} \varphi(t)dt, \end{aligned}$$

which is a contradiction, hence $b = \mathcal{F}b = gb$.
Therefore, b is a common fixed point of the mappings $\mathcal{F}$ and g.
For uniqueness of common fixed point, let a and b be two common fixed points of $\mathcal{F}$ and g.
From (2.25), we can obtain

$$\int_0^{G(\mathcal{F}a,\mathcal{F}a,\mathcal{F}b)} \varphi(t)dt \le \phi\left(\int_0^{L(a,a,b)} \varphi(t)dt\right),$$

where

$$\begin{aligned} L(a,a,b) &= \max\{G(ga,ga,gb), G(ga,\mathcal{F}a,\mathcal{F}a), G(ga,\mathcal{F}a,\mathcal{F}a), G(ga,\mathcal{F}a,\mathcal{F}a) \\ &\qquad G(ga,\mathcal{F}a,\mathcal{F}a), G(gb,\mathcal{F}b,\mathcal{F}b), G(gb,\mathcal{F}a,\mathcal{F}a), G(gb,\mathcal{F}a,\mathcal{F}a)\} \\ &= G(a,a,b), \end{aligned}$$

which implies that

$$\begin{aligned} \int_0^{G(a,a,b)} \varphi(t)dt &\le \phi\left(\int_0^{L(a,a,b)} \varphi(t)dt\right) \\ &< \int_0^{G(a,a,b)} \varphi(t)dt, \end{aligned}$$

a contradiction. Hence $\mathcal{F}$ and g have unique common fixed point.
This completes the proof of the theorem. □

3. Conclusion

In Theorem 2.2, we have generalized the results proved by Aydi [3] in the setting of G-metric spaces for a pair of weakly compatible maps. Furthermore, Theorem 2.3 is proved for a pair of weakly compatible self-maps along with E.A property and Theorem 2.4 is proved for a pair of weakly compatible self-maps along with (CLR) property to show the existence of unique common fixed point. A suitable example is also provided to prove the validity of our results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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