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Research Article

Shehu Variational Iteration Method For Solve Some Fractional Differential Equations

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Abstract. The idea proposed in the work is to extend the Shehu transform method to resolve the nonlinear fractional partial differential equations by combining them with the variational iteration method (VIM). We apply this technique to solve nonlinear fractional equations as nonlinear time fractional Fokker Planck equation.

Keywords. Caputo fractional derivate, Variational iteration method, Shehu transform, Fokker-Planck equation, Telegraph equation of space

Mathematics Subject Classification (2020). 15A09, 65F05, 65F10, 65Y04

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivates and integrals of arbitrary orders. In the last few decades fractional calculus found many applications in various field of physical science, fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing and many other physical process. Since the physical side is often associated with fractional differential equation as explained in the previous paragraph, many researchers have used existing methods to solve this type of equations. Historically, the origin of the integral transform can be traced back to the work of Pierre-Simon de Laplace in 1780 and Joseph Fourier in 1822. In the recent years, differential and integral equations have been solved using many integral transforms like Sumudu transform [18], the New Integral transform “Elzaki transform” [4], and so on.

The objective of the present study is to combine two powerful methods, variational iteration method [5–7, 15] and Shehu transform [2, 9] to get a faster method to solve nonlinear fraction partial differential equations.

The present paper has been organised as follows: In Section 2, some basic notions about fractional calculus and basic definitions and properties of shehu transform method. In Section 3, we give an analysis of the proposed method. In Section 4, we present some applications and finally the conclusion.

2. Basic Preliminaries

In this section, we present successively the auxiliary basic functions, the fractionary derivative according to the Riemann Liouville approach and the Caputo fractionary derivative.

2.1 Auxiliary Functions

In this section, we present successively the definitions and some properties of the Euler gamma and beta functions, the definitions and some properties of two classical Mittag-Leffler functions.

Definition 2.1 (Euler gamma function)**.** For all complex number such that $z \in \mathbb{C}/\{0,-1,-2,-3,\dots\}$, the Euler gamma function [1, 14], denoted by Γ is defined by

$$\Gamma(z)=\begin{cases}\displaystyle\int_0^{\infty} t^{(z-1)}e^{-t}dt, & \text{if } Re(z)>0,\\ \frac{\Gamma(z+1)}{z}, & \text{if } Re(z)\le 0 \text{ and } z\notin Z_0^-.\end{cases} \tag{2.1}$$

Definition 2.2 (Euler beta function)**.** The Euler beta function [1, 3] is defined by the integral of by the Euler integral of the first kind.

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt, \quad Re(z)>0, R(w)>0. \tag{2.2}$$

Definition 2.3 (Mittag-Leffler function to one parameter)**.** The classical Mittag-Leffler function to one parameter α, is defined by the expression

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad z\in\mathbb{C},\ Re(z)>0. \tag{2.3}$$

Definition 2.4 (Mittag-Leffler function to two parameters)**.** The classical Mittag-Leffler function to two parameters α,β is given by the expression

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z\in\mathbb{C},\ \beta\in\mathbb{C},\ Re(\alpha)>0. \tag{2.4}$$

2.2 Riemann-Liouville Fractional Derivative

The fractional derivative according to the Riemann-Liouville approach [3, 8, 10, 14, 17] is defined from the juxtaposition of the fractional integral previously defined and the integer derivative.

As it was presented above, we start by writing successively the fractional derivatives on the left, then on the right, of complex order α such than $Re(\alpha) \geq 0$.

Definition 2.5 (Riemann-Liouville fractional derivative to left-sided)**.** The fractional derivative to left-sided of Riemann-Liouville of order α according to the Riemann-Liouville approach of function f is defined and denote respectively by

$$(D^{\alpha}_{a+}f)(x) = \left(\frac{d}{dx}\right)^{n} (I^{n-\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} (x-t)^{n-\alpha-1} f(t)dt, \quad x > a. \tag{2.5}$$

Definition 2.6 (Riemann-Liouville fractional derivative to right-sided)**.**

$$(D^{\alpha}_{b-}f)(x) = \left(\frac{-d}{dx}\right)^{n} (I^{n-\alpha}_{-b}f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{dx}\right)^{n} \int_{b}^{x} (t-x)^{n-\alpha-1} f(t)dt, \quad x < b. \tag{2.6}$$

In the two previous definitions, the integer n and the complex α are dependent on each other according to the relation $n = [Re(\alpha)] + 1$ where the notation $[Re(\alpha)]$ designates the integer part of $Re(\alpha)$.

2.3 Caputo Fractional Derivative

In 1967, Michele Caputo [3, 8, 10, 13, 14, 17], introduced a variant of the Riemann-Liouville fractional derivative. It is distinguished from the latter by the nullity of constant functions and also by its applications in many fields of engineering sciences. By keeping the same notations and the same data as before, we give the definitions of the fractional derivatives of Caputo, respectively on the left, on the right using their Riemann-Liouville analogues. We will see later, the importance of this fractional derivative in practical applications.

Definition 2.7 (Derivative of Caputo to left-sided, to right-sided)**.**

$$({}^{c}D^{\alpha}_{a+}f)(x) = \left(D^{\alpha}_{a+}\left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^{k}\right]\right)(x), \tag{2.7}$$

$$({}^{c}D^{\alpha}_{b-}f)(x) = \left(D^{\alpha}_{b-}\left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(b-t)^{k}\right]\right) \tag{2.8}$$

with $n = [Re(\alpha)] + 1$ if $\alpha \notin N_0$ and $n = \alpha$ if $\alpha \in N_0$.

In case $0 < R(e) < 1$, the Caputo fractional derivative functions are defined by the expressions

$$({}^{c}D^{\alpha}_{a+}f)(x) = (D^{\alpha}_{a+}[f(t) - f(a)])(x),$$

$$({}^{c}D^{\alpha}_{b-}f)(x) = (D^{\alpha}_{b-}[f(t) - f(b)])(x).$$

Theorem 2.1. *Let α and n such that $n = [Re(\alpha)] + 1$ and $Re(\alpha) \geq 0$. We assume that $f \in AC^{n}[a,b]$ and that $({}^{c}D^{\alpha}_{a+}f)(x)$ (respectively $({}^{c}D^{\alpha}_{b-}f)(x)$) exist almost everywhere in $[a,b]$. In this conditions, we have*

$$({}^{c}D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} (x-t)^{n-\alpha-1} f^{(n)}(t)dt = (I^{n-\alpha}_{a+})D^{n}f(x), \tag{2.9}$$

$$({}^cD_{b-}^{\alpha}f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1}f^{(n)}(t)dt = (I_{b-}^{n-\alpha})D^n f(x). \tag{2.10}$$

3. Shehu Transform

The Shehu transform was defined by Shehu Maitama [9, 19], in 2019. In this section, we give some basics definitions and properties of this transform.

Definition 3.1 (Shehu transform)**.** The Shehu transform of the function $v(t)$ of exponential order is defined over the set of functions

$$A = \left\{v(t) : \exists\, N, n_1, n_2 > 0,\, N \times \exp\left(\left|\frac{t}{n_j}\right|\right) \text{ if } t \in (-1)^j \times [0,\infty[\right\} \tag{3.1}$$

by the following integral

$$S[v(t)] = V(s,u) = \lim_{x\to\infty}\int_0^x \exp\left(\frac{-st}{u}\right)v(t)dt. \tag{3.2}$$

There is convergence if the limit of the integral exists, and divergence in the opposite case. In case of convergence, the previous expression is written

$$S[v(t)] = V(s,u) = \int_0^\infty \exp\left(\frac{-st}{u}\right)v(t)dt. \tag{3.3}$$

Definition 3.2 (Inverse Shehu transform)**.** The inverse Shehu transform is given by

$$S^{-1}[V(s,u)] = v(t), \quad s,u > 0. \tag{3.4}$$

Definition 3.3 (Equivalent form of the inverse Shehu transform)**.** The inverse Shehu transform can be write also in the following form,

$$v(t) = S^{-1}[V(s,u)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}\exp\left(\frac{st}{u}\right)V(s,u)ds, \tag{3.5}$$

where s and u are Shehu transform variables, and α is a real scalar and the integral in equation is taken along $s = \alpha$ in the complex plane $s = x + iy$.

Theorem 3.1 (The sufficient condition for the existence of Shehu transform)**.** *If the function $v(t)$ is piecewise continues in every finite interval $0 \le t \le \beta$ and of exponential order α for $t > \beta$, then its Shehu transform $V(s,u)$ exists.*

Proof. See [9]. □

Properties 3.1. *In this paragraph, we present the main properties of Shehu transform* [3, 9]*.*

(i) *Property of linearity: the Shehu transform is a linear operator.*

Let λ, μ arbitrary scalars and $u(t), v(t) \in A$. Then $\lambda u(t)$ and $\mu v(t)$ are trivially in set A, previously defined in (3.1)*. In these conditions*

$$S[(\lambda u + \mu v)(t)] = \lambda S[u(t)] + \mu S[v(t)]. \tag{3.6}$$

(ii) *Property of change of scale of Shehu transform.*
Let the function $v(\beta t)$ be in set A, where β is an arbitrary constant. Then
$$[v(\beta t)] = \frac{u}{\beta} V\left(\frac{s}{\beta}, u\right). \tag{3.7}$$

Properties 3.2.

$$v(t) = 1, \qquad S[1] = \frac{u}{s},$$
$$v(t) = t, \qquad S[t] = \frac{u^2}{s^2},$$
$$v(t) = \tfrac{t^n}{\Gamma(n+1)}, \qquad S[v(t)] = \left(\frac{u}{s}\right)^{n+1},$$
$$v(t) = \frac{t^n}{\Gamma(n+1)}, \qquad S[v(t)] = \left(\frac{u}{s}\right)^{n+1}.$$

Theorem 3.2 (Shehu transform of derivative function)**.** *If the function $v^{(n)}(t)$ design the nth derivate of the function $v(t) \in A$ relatively to variable t, then its Shehu transform is defined by*
$$S[v^{(n)}(t)] = \frac{s^n}{u^n} V(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} v^{(k)}(0). \tag{3.8}$$

Proof. See [2, 9]. □

Example 3.1 (Particular order)**.**
$$n = 1 : S[v^{(1)}(t)] = \frac{s}{u} V(s,u) - v(0)$$
$$n = 2 : S[v^{(2)}(t)] = \frac{s^2}{u^2} V(s,u) - \frac{s}{u} v(0) - v^{(1)}(0)$$
$$n = 3 : S[v^{(3)}(t)] = \frac{s^3}{u^3} V(s,u) - \frac{s^2}{u^2} v(0) - \frac{s}{u} v^{(1)}(0) - v^{(2)}(0).$$

Theorem 3.3 (Shehu transform of Caputo fractional derivative)**.** *If $v(t) \in AC^n(a,b)$, then the Shehu transform of its fractional derivative of Caputo is given by*
$$S\left[{}^cD^{\alpha}_{0+} v(t)\right] = \left(\frac{s}{u}\right)^{\alpha} S[v(t)] - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-(k+1)} v^{(k)}(0), \quad n-1 < \alpha \le n,\ n = 1, 2, \dots. \tag{3.9}$$

Proof. See [2, 9]. □

4. Applications of Variational Iteration Shehu Transform Method with Caputo Fractional Derivative

For this section, it seems very important to begin to even briefly recall the iterative variational method to then analytically express the combination of the iterative variational method with the integral transformation of Shehu.

After that, we shall present few numeric examples on solving fractional differential equations.

4.1 Variational Iteration Method (VIM)

Let the following general nonlinear ordinary differential equation

$$Lu(x)+Nu(x)=h(x), \tag{4.1}$$

where L is a linear operator, N is a nonlinear operator and the function $h(x)$ is the second member. According to *variational iterative method* (VIM) [5–7, 11, 12, 15, 16], we can construct the correction functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(Lu_n(t)+N\widetilde{u}_n(t)-h(t))dt, \tag{4.2}$$

where λ is Lagrange multiplier which can be identified optimally, according to the variational theory, u_n is the nth approximate solution, and $\widetilde{u}_n$ is considered as a restricted variation, to mean that $\delta\widetilde{n}=0$.

After identification of Lagrange multiplier, the successive approximations $u_{n+1}(x)$, for $n\geq 0$ of the solution u can be readily obtained.

The exact solution will determined in form

$$u(x)=\lim_{n\to+\infty}u_n(x). \tag{4.3}$$

4.2 Variational Iteration Shehu Transform Method

To illustrate the basic idea of this method, we consider a general nonlinear fractional partial differential non homogeneous equation with initial conditions of the form

$${}^cD_t^\alpha v(x,t)+Rv(x,t)+Nv(x,t)=g(x,t), \tag{4.4}$$

$$v(x,0)=h(x), \tag{4.5}$$

where $t>0$, $x\in\mathbb{R}$, $0<\alpha\leq 1$.

The function ${}^cD_t^\alpha v(x,t)$ is the Caputo fractional derivative of the $v(x,t)$, R is the linear differential operator, N represents the general nonlinear operator, and $g(x,t)$ the second member.

It is a question of constructing, similarly to the iterative variational method, a series of approximate functions of the exact solution of the problem (4.4)-(4.5).

This method is called variational iteration Shehu transform method. We proceed as follows. We begin to apply the Shehu transform on both sides of (4.4) using the different properties listed above of this transform.

Proposition 4.1. *If $v(x,t)$ is solution of problem* (4.4)-(4.5)*, then*

$$v(x,t)=\left[h(x)+S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,t)\right)\right]-S^{-1}\left[\frac{u^\alpha}{s^\alpha}(SRv(x,t)+SNv(x,t))\right]. \tag{4.6}$$

Proof. Let follow problem

$${}^cD_t^\alpha v(x,t)+Rv(x,t)+Nv(x,t)=g(x,t),$$

$$v(x,0)=h(x).$$

We begin to apply the Shehu transform on both sides of equation using the different properties listed above of this transform, then use the Shehu transform of Caputo fractional derivative.

$$S[g(x,t)]=S\left[{}^cD_t^\alpha v(x,t)+Rv(x,t)+Nv(x,t)\right]$$

$$S[g(x,t)]=S\left[{}^cD_t^\alpha v(x,t)\right]+[Rv(x,t)+Nv(x,t)]$$

$$S\left[{}^cD_t^\alpha v(x,t)\right]=S[g(x,t)]-[Rv(x,t)+Nv(x,t)]$$

$$S\left[{}^cD_t^\alpha v(x,t)\right]=\left(\frac{s}{u}\right)^\alpha S[v(x,t)]-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-(k+1)}v^{(k)}(x,0),\quad n-1<\alpha\le n,\ n=1,2\dots$$

$$\left(\frac{s}{u}\right)^\alpha S[v(x,t)]=\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-(k+1)}v^{(k)}(x,0)+S[g(x,t)]-S[Nv(x,t)+Rv(x,t)]$$

By multiplying $\left(\frac{s}{u}\right)^{-\alpha}$ on both sides of last equation, then using applying the Shehu transform inverse on both sides of equation obtained, it results

$$S[v(x,t)]=\left(\frac{s}{u}\right)^{-\alpha}\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-(k+1)}v^{(k)}(x,0)+\left(\frac{s}{u}\right)^{-\alpha}S[g(x,t)]-\left(\frac{s}{u}\right)^{-\alpha}S[Nv(x,t)+Rv(x,t)].$$

To finish proof of proposition, we can studied the case where $0<\alpha\le 1$.

In these conditions,necessarily $n=1$. By substituting the condition $n=1$, in the previous equation, we obtain

$$S[v(x,t)](s,u)=\frac{u}{s}h(x)+\frac{u^\alpha}{s^\alpha}S[g(x,t)]-\frac{u^\alpha}{s^\alpha}S[Rv(x,t)+Nv(x,t)].$$

Finally, we take the Shehu transport inverse from both sides of the above equation

$$v(x,t)=\left[h(x)+S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,t)\right)\right]-S^{-1}\left[\frac{u^\alpha}{s^\alpha}(SRv(x,t)+SNv(x,t))\right].$$

This finish the proof. □

Theorem 4.1. *According a modification of variational iterative method* [6, 7, 19]*, the exact solution of problem* (4.4)-(4.5) *is given in form as a limit of successive approximations* $v_n(x,t), n=0,1,2,\dots$*, in the others words,*

$$v(x,t)=\lim_{n\to+\infty}v_n(x,t) \tag{4.7}$$

with respectively

$$v_n(x,t)=v_{n-1}(x,t)-\int_0^t\left[\frac{\partial}{\partial\tau}v_{n-1}(x,\tau)+\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}S[Rv_{n-1}(x,\tau)+Nv_{n-1}(x,\tau)]\right)\right.$$
$$\left.-\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,\tau)\right)\right]d\tau. \tag{4.8}$$

Proof. Applying $\left(\frac{\partial}{\partial t}\right)$ on both sides of equation below.

$$v(x,t)=\left[h(x)+S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,t)\right)\right]-S^{-1}\left[\frac{u^\alpha}{s^\alpha}(SRv(x,t)+SNv(x,t))\right].$$

We obtain,

$$\frac{\partial}{\partial t}v(x,t)+\frac{\partial}{\partial t}S^{-1}\left[\frac{u^\alpha}{s^\alpha}S((Rv(x,t)+Nv(x,t))\right]-\frac{\partial}{\partial t}S^{-1}\left[\frac{u^\alpha}{s^\alpha}Sg(x,t)\right]=0. \tag{4.9}$$

This method is based on the construct of the following correctional functional for eq. (4.8), where λ is a general Lagrange multiplier which can be identified optimally via the variational theory. Applying variational iteration method with $0<\alpha\le 1$, we obtain,

$$\begin{aligned}v_n(x,t)=v_{n-1}(x,t)+\int_0^t\lambda(t,\tau)\bigg[\frac{\partial}{\partial\tau}v_{n-1}(x,\tau)+\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}S[Rv_{n-1}(x,\tau)+Nv_{n-1}(x,\tau)]\right)\\ -\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,\tau)\right)\bigg]d\tau.\end{aligned} \tag{4.10}$$

In this case, the general Lagrange multiplier is given by the formula [7,19]

$$\lambda(t,\tau)=\frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!},$$

with $m=1$. In this case, $\lambda=-1$ [19].

Finally, the correctional functional is given by

$$\begin{aligned}v_n(x,t)=v_{n-1}(x,t)-\int_0^t\bigg[\frac{\partial}{\partial\tau}v_{n-1}(x,\tau)+\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}S[Rv_{n-1}(x,\tau)+Nv_{n-1}(x,\tau)]\right)\\ -\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,\tau)\right)\bigg]d\tau.\end{aligned}$$

This finish the proof of Theorem 4.1. □

Example 4.1 (First successive approximations)**.** We can consider,

$$v_0(x,t)=v(x,0)=h(x),$$

for start the iterations. We have successively

$$v_1(x,t)=v_0(x,t)-\int_0^t\left[\frac{\partial}{\partial\tau}v_0(x,\tau)+\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}S[Rv_0(x,\tau)+Nv_0(x,\tau)]\right)-\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,\tau)\right)\right]d\tau,$$

$$v_2(x,t)=v_1(x,t)-\int_0^t\left[\frac{\partial}{\partial\tau}v_1(x,\tau)+\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}S[Rv_1(x,\tau)+Nv_1(x,\tau)]\right)-\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,\tau)\right)\right]d\tau,$$

$$v_3(x,t)=v_2(x,t)-\int_0^t\left[\frac{\partial}{\partial\tau}v_2(x,\tau)+\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}S[Rv_2(x\tau)+Nv_2(x,\tau)]\right)-\frac{\partial}{\partial\tau}S^{-1}\left(\frac{u^\alpha}{s^\alpha}Sg(x,\tau)\right)\right]d\tau,$$

and so one.

For $n\in\mathbb{N}$, we continue in this manner to obtain the general recursive relation (4.9). Finally, the approximate solution (possibly exact) is calculated by

$$v(x,t)=\lim_{n\to+\infty}v_n(x,t).$$

4.3 Treatment of Numerical Example

In this part, we apply variational iteration method Shehu transform with the Caputo fractional derivate to solve linear and nonlinear equations. The example that we present is a nonlinear time-fractional Fokker Planck equation described according following model:

Example 4.2.

$$^{c}D_t^{\alpha}v(x,t)=\left(\frac{x}{3}v(x,t)\right)_x-\left(\frac{4}{x}v^2(x,t)\right)_x+(v^2(x,t))_{xx},\quad 0<\alpha\le 1, \tag{4.11}$$

with the initial condition

$$v(x,0)=x^2. \tag{4.12}$$

By applying VIM combined with Shehu transform in the problem (4.11)-(4.12), we obtain the following iteration,

$$v_n(x,t)=v_{n-1}(x,t)-\int_0^t\left[\frac{\partial v_{n-1}}{\partial\tau}-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{x}{3}v_{n-1}\right)_x+\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{4}{x}v_{n-1}^2\right)_x\right.$$
$$\left.-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S(v_{n-1}^2)_{xx}\right]d\tau.$$

In particular cases $n=1,2,3$, we get following iterations,

$$v_1(x,t)=v_0(x,t)-\int_0^t\left[\frac{\partial v_0}{\partial\tau}-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{x}{3}v_0\right)_x+\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{4}{x}v_0^2\right)_x-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S(v_0^2)_{xx}\right]d\tau,$$
$$v_2(x,t)=v_1(x,t)-\int_0^t\left[\frac{\partial v_1}{\partial\tau}-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{x}{3}v_1\right)_x+\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{4}{x}v_1^2\right)_x-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S(v_1^2)_{xx}\right]d\tau,$$
$$v_3(x,t)=v_2(x,t)-\int_0^t\left[\frac{\partial v_2}{\partial\tau}-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{x}{3}v_2\right)_x+\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S\left(\frac{4}{x}v_2^2\right)_x-\frac{\partial}{\partial\tau}S^{-1}\frac{u^{\alpha}}{s^{\alpha}}S(v_2^2)_{xx}\right]d\tau.$$

We give now iterations obtained in a practical form,

$$v_1(x,t)=x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right),$$
$$v_2(x,t)=x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right),$$
$$v_3(x,t)=x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right).$$

We continue in this manner to obtain the recursive relation to order n.

$$v_n(x,t)=x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right),$$

with $0<\alpha\le 1$. Hence,

$$v_n(x,t)=x^2\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

We deduce the solution of problem (4.11)-(4.12),

$$v(x,t)=\lim_{n\to+\infty}x^2\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}=x^2E_{\alpha}(t^{\alpha}),$$

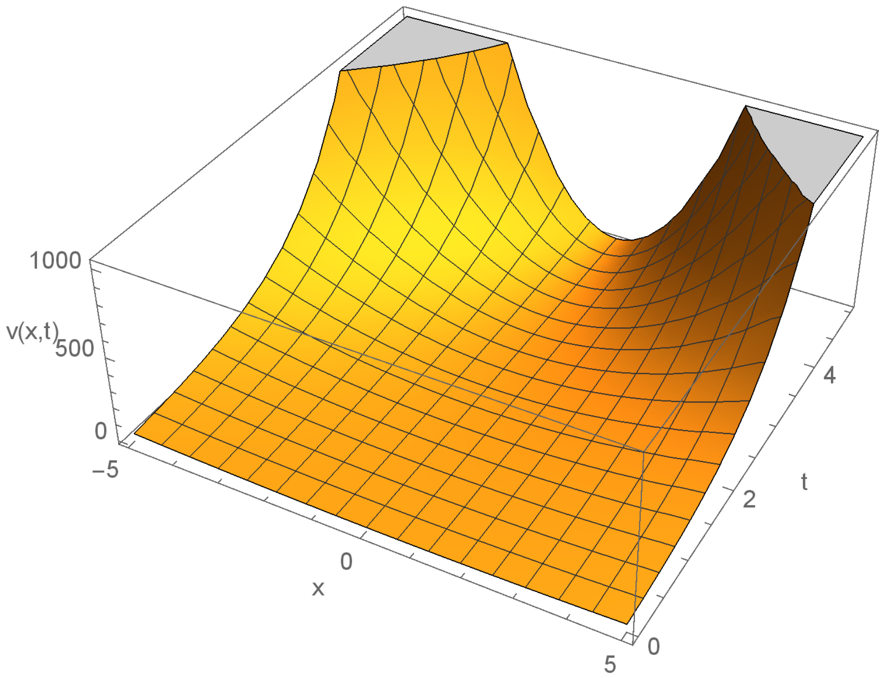
where E_{α} is the Mittag-Leffler function to one parameter define in this case by,

$$E_{\alpha}(t^{\alpha})=\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

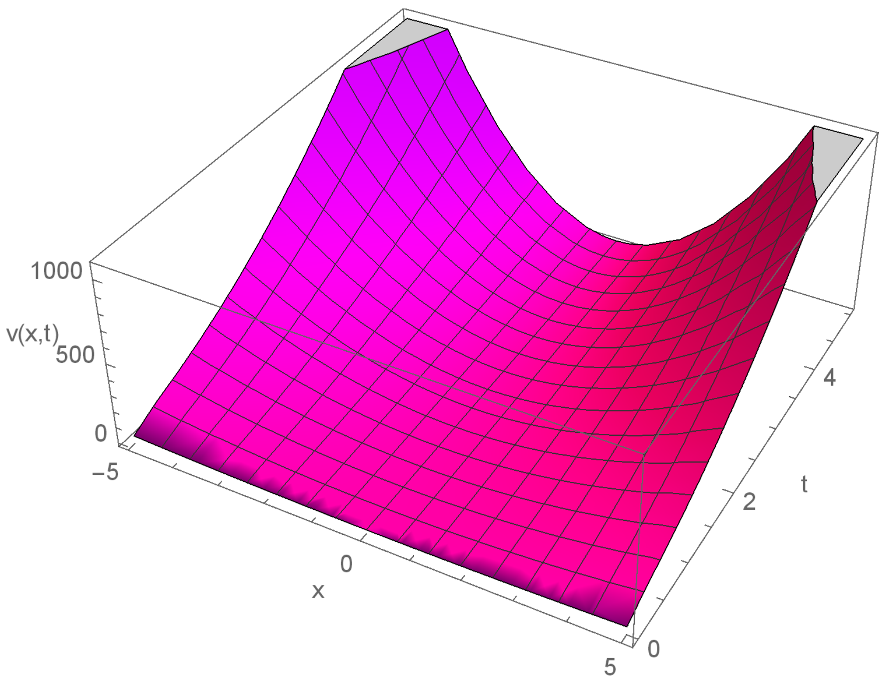
In special case, where $\alpha=1$, we obtain,

$$v(x,t)=x^2e^t.$$

In what follows, we present the graphs of the approximate solutions of the problem (4.11)-(4.12) for four remarkable values of the order alpha of differentiation.

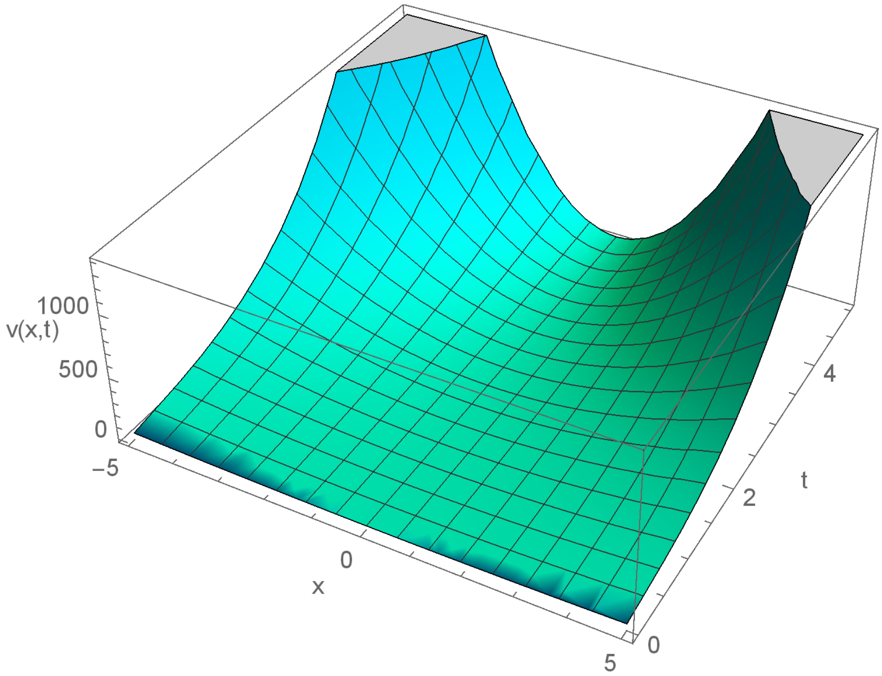


(a) $\alpha = 1$

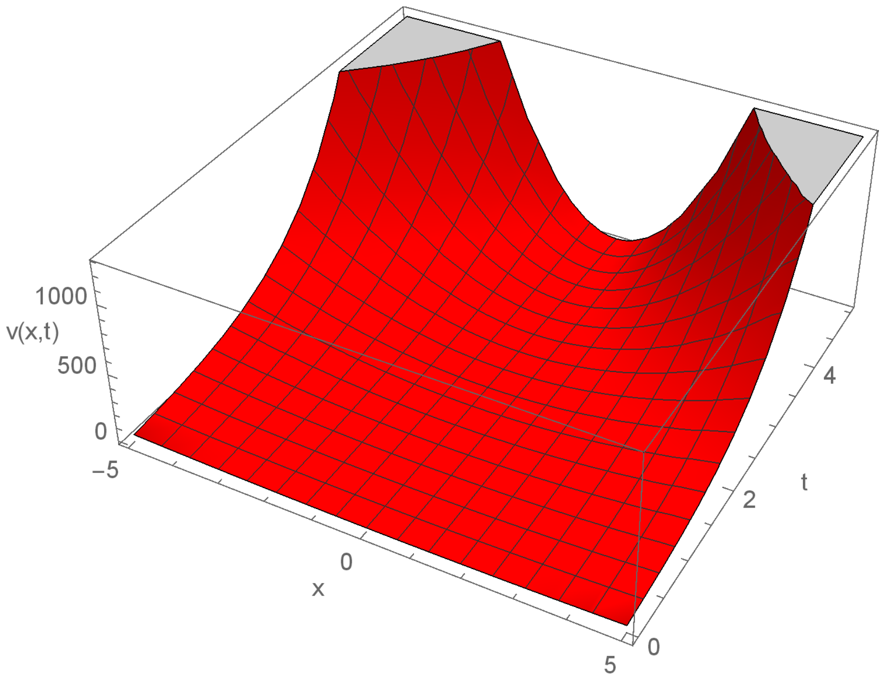


(b) $\alpha = 0.25$

Figure 1. The surface shows the solution $v(x,t)$ for equation (4.11) with initial condition (4.12) $v_0(x,t) = x^2$: FVIM results are, respectively, (a) $\alpha = 1$ and (b) $\alpha = 0.25$



(a) $\alpha = 0.5$



(b) $\alpha = 0.75$

Figure 2. The surface shows the solution $v(x,t)$ for equation (4.11) with initial condition (4.12) $v_0(x,t) = x^2$: FVIM results are, respectively, (a) $\alpha = 0.50$ and (b) $\alpha = 0.75$

5. Conclusion

The combined method of Shehu's integral transformation with the iterative variational method gives flexible and precise results. We have plotted the graphs of the approximate solutions corresponding to particular and remarkable values of the alpha scalar. This allowed us to analyse and compare the approximate solutions by their geometric aspects.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] G. E. Andrews, R. Askey and R. Roy, Special functions, in: *Encyclopedia of Mathematics and its Applications* **71**, Cambridge University Press, (1999), DOI: 10.1017/CBO9781107325937.

[2] R. Belgacem, D. Baleanu and A. Bokhari, Shehu transform and applications to Caputo fractional differential equations, *International Journal of Analysis and Applications* **17**(6) (2019), 917 – 927, URL: http://etamaths.com/index.php/ijaa/article/view/1958.

[3] K. Diethelm, *The Analysis Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag, Berlin — Heidelberg (2010), DOI: 10.1007/978-3-642-14574-2.

[4] T. M. Elzaki, The new integral transform "Elzaki transform", *Global Journal of Pure and Applied Mathematics* **7**(1) (2011), 57 – 64, URL: https://www.kau.edu.sa/Files/0057821/Researches/60337_31161.pdf.

[5] J.-H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Computer Methods in Applied Mechanics and Engineering* **167** (1998), 57 – 68, DOI: 10.1016/S0045-7825(98)00108-X.

[6] J.-H. He, Variational iteration method – a kind of non-linear analytical technique: some examples, *International Journal of Non-Linear Mechanics* **34** (1999), 699 – 708, DOI: 10.1016/S0020-7462(98)00048-1.

[7] M. Inokuti, H. Sekine and T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in: *Variational Methods in the Mechanics of Solids*, S. Nemat-Nasser (ed.), pp. 156 – 162, Pergamon Press, Oxford, UK (1980), DOI: 10.1016/B978-0-08-024728-1.50027-6.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. **204**, Elsevier Science B.V, Amsterdam (2006), URL: https://shop.elsevier.com/books/theory-and-applications-of-fractional-differential-equations/kilbas/978-0-444-51832-3#full-description.

[9] S. Maitama and W. Zhao, New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations, *International Journal of Analysis and Applications* **17**(2) (2019), 167 – 190, URL: https://www.etamaths.com/index.php/ijaa/article/view/1771.

[10] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Songs, Inc., New York (1993), URL: https://www.gbv.de/dms/ilmenau/toc/122837029.PDF.

[11] S. T. Mohyud-Din, W. Sikander, U. Khan and N. Ahmed, Optimal variational iteration method for nonlinear problems, *Journal of the Association of Arab Universities for Basic and Applied Sciences* **24** (2017), 191 – 197, DOI: 10.1016/j.jaubas.2016.09.004.

[12] M. A. Noor and S. T. Mohyud-Din, Variational iteration method for solving initial and boundary value problems of Bratu-type, *Applications and Applied Mathematics: An International Journal* **3**(1) (2008), 89 – 99, URL: https://digitalcommons.pvamu.edu/aam/vol3/iss1/8.

[13] M. D. Ortigueira and J. A. T. Machado, What is a fractional derivative?, *Journal of Computational Physics* **293** (2015), 4 – 13, DOI: 10.1016/j.jcp.2014.07.019.

[14] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, 1st edition, Academic Press, San Diego — California, USA (1998), URL: https://www.elsevier.com/books/fractional-differential-equations/podlubny/978-0-12-558840-9.

[15] M. Safari, D. D. Ganji and M. Moslemi, Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV–Burgers–Kuramoto equation, *Computers & Mathematics with Applications* **58** (2009), 2091 – 2097, DOI: 10.1016/j.camwa.2009.03.043.

[16] M. Safari, Application of He's variational iteration method for the analytical solution of space fractional diffusion equation, *Applied Mathematics* **2** (2011), 1091 – 1095, DOI: 10.4236/am.2011.29150.

[17] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives – Theory and Applications*, Gordon and Breach, Amsterdam, (1993).

[18] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *International Journal of Mathematical Education in Science and Technology* **24**(1) (1993), 35 – 43, DOI: 10.1080/0020739930240105.

[19] G.-C. Wu, Challenge in the variational iteration method – A new approach to identification of the Lagrange multipliers, *Journal of King Saud University - Sciences* **25** (2013), 175 – 178, DOI: 10.1016/j.jksus.2012.12.002.