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Research Article

Some Results in Nearly Gorenstein Numerical Semigroup of 6 Generators With Type 5

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Abstract. In this paper, some results are found in the nearly Gorenstein numerical semigroups of 6 generators with type 5. All nearly Gorenstein numerical semigroups with e generators and type p are given.

Keywords. Type of a numerical semigroup, Nearly Gorenstein, NG-vector

Mathematics Subject Classification (2020). 13H10, 20M14, 20M25

1. Introduction

Barucci and Fröberg [1] introduced the almost Gorenstein rings as a larger class of Cohen Macaulay rings [3] that are next to the Gorenstein [1]. This was generalized by Nari [4] and Goto *et al*. [5]. In semigroups, there is no correlation between the number and type of the generators. So far, results have been found that the type will not exceed the generative set in numerical semigroups and nearly Gorenstein numerical semigroup. Moscariello [6] proved the type of a 4-generated almost symmetric numerical semigroup is at most 3. Moscariello and Strazzanti[1], proved that the type of a 4-generated Nearly Gorenstein numerical semigroups is at most 3. The computation appearing in this paper were performed by using the GAP system [9] and in particular the Numerical Sgps package [2].

In this paper, some results are found in the nearly Gorenstein numerical semigroups of 6 generators with type 5.

[1]A. Moscariello and F. Strazzanti, Nearly Gorenstein vs almost Gorenstein affine monomial curves, arXiv preprint (2020), https://arxiv.org/abs/2003.05391.

2. Basic Definitions

Most of the definitions can also be found in [8], [1] or [7].

Definition 2.1. Let $\mathbb{N}$ is the set of non-negative integers and $S \subset \mathbb{N}$. If S is closed under the addition in $\mathbb{N}$ and $0 \in S$ and $\mathbb{N}\backslash S$ is finite then S is called a numerical semigroup. For all $n_1, n_2, \dots, n_e \in S$ it is denoted by

$$S = \langle n_1, n_2, \dots, n_e \rangle = \left\{ \sum_{i=1}^{e} a_i n_i : a_i \in \mathbb{N} \right\}$$

and the following is correct.

$$(n_{1,} n_2, \dots, n_e) = 1 \Leftrightarrow \mathbb{N}\backslash S \text{ is finite.}$$

Definition 2.2. Let the numerical semigroup S is given by

$$S = \langle n_1, n_2, \dots, n_e \rangle$$

then,

(i) the number $m(S) = n_1$ is called the multiplicity of S,

(ii) the number $e(S) = e$ is called the embedding dimension of S.

Definition 2.3. Let S is a numerical semigroup the largest integer that is not in S is called the Frobenius number of S and denoted by $F(S)$, i.e.,

$$F(S) = \max(\mathbb{N}\backslash S), \text{ or}$$

$$F(S) = \max\{x \in \mathbb{Z} : x \notin S\}.$$

Definition 2.4. The positive elements that is not in S and is denoted by $G(S)$. The elements of gaps is called genus of S and $g(S) = |G(S)|$.

Definition 2.5. $PF(S)$ is the set of pseudo-Frobenius number of S,

$$\begin{aligned} PF(S) &= \{x \notin S \mid x + s \in S, \text{ for every } s \in S\backslash\{0\}\} \\ &= \{x \notin S \mid x + n_i \in S, \text{ for every } i = 1, 2, \dots, e\}. \end{aligned}$$

Definition 2.6. The number of elements of the set of pseudo-Frobenius is called the type of S and is denoted by $t(S)$. That is, $\#PF(S) = t(S)$.

Definition 2.7. Let $S = \langle n_1, n_2, \dots, n_e \rangle$ be a numerical semigroup. Each $s \in S$ can be written as $s = \left\{\sum_{i=1}^{e} \alpha_i n_i \alpha_i \in \mathbb{N} \text{ for every } i\right\}$. Such a presentation of s is called a factorization of s.

Note 2.8. Let $(\leq_S)$ be the relation defined by $b \leq_S a$ if $a - b \in S$. It is easy to see that $(\mathbb{Z}, \leq_S)$ is a partially ordered set and that the pseudo-Frobenius numbers of S are maximal elements of the poset $(\mathbb{Z}/S, \leq_S)$.

Example 2.9.

$$S = \langle 10, 11, 12, 13, 14, 15 \rangle = \{0, 10, 11, 12, 13, 14, 15, 20, \rightarrow\},$$

$$G(S) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 18, 19\}, \text{ the set of gaps,}$$

$$F(S) = 19, \text{ the largest number that is not in } S,$$

$$m(S) = 10, \text{ multiplicity,}$$

$e(S) = 6$, embedding dimension,

$g(S) = 13$, the number of elements of gaps,

$PF(S) = \{16, 17, 18, 19\}$, pseudo-Frobenius of S,

$t(S) = 4$, the type of S.

Definition 2.10. A numerical semigroup S is almost symmetric. If for every $f \in \mathbb{Z} \setminus S$, we have $\{f, F(S) - f\} \subseteq PF(S)$ such that $F(S) - f \notin S$.

Example 2.11. Pseudo-Frobenius numbers of $S = \langle 10, 11, 13, 18, 19, 27 \rangle$ are $PF(S) = \{8, 9, 16, 17, 25\}$. So, S is almost symmetric semigroup.

Proposition 2.12. *The following statements hold:*

(i) *S is almost symmetric if and only if $n + F(S) - f \in S$ for all $f \in PF(S)$ and all $n \in G(S)$;*

(ii) *S is nearly Gorenstein if and only if for every $n_i \in G(S)$ there exists $f_i \in PF(S)$ such that $n_i + f_i - f \in S$ for all $f \in PF(S)$.*

Proof. See [1], [6]. □

Definition 2.13. Let $S = \langle n_1, n_2, \dots, n_e \rangle$ where $n_1 < n_2 < \cdots < n_e$ are minimal generators. We call a vector $f = \{f_1, \dots, f_e\} \in PF(S)$ nearly Gorenstein vector for S, briefly NG-vector, if $n_i + f_i - f \in S$ for every $f \in PF(S)$ and every $i = 1, 2, \dots, e$.

Proposition 2.14. *Let $f = \{f_1, \dots, f_e\} \in PF(S)$ be an NG-vector S. The following hold:*

(i) *$f_1 = F(S)$,*

(ii) *if i is the minimum index for which $f_i \neq F(S)$, then $f_i = F(S) - n_i + n_m$ for some $m < i$.*

Proof. See [1]. □

By Proposition 2.12, the existence of an NG-vector is equivalent to the nearly Gorensteinness of S, whereas S is almost symmetric if and only if it admits the NG-vector $(F(S), F(S), \dots, F(S))$.

Example 2.15.

$S = \langle 10, 11, 12, 13, 14, 17 \rangle = \{0, 10, 11, 12, 13, 14, 17, 20, \rightarrow\}$,

$G(S) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 16, 18, 19\}$,

$F(S) = 19$,

$PF(S) = \{15, 16, 18, 19\}$,

$n_1 = 10,\ n_2 = 11,\ n_3 = 12,\ n_4 = 13,\ n_5 = 14,\ n_6 = 17$.

By Proposition 2.14,

$f_1 = F(S) = 19$,

$11 + 18 - f \in S$ for every $f \in PF(S)$, $f_2 = 18$,

$12 + 15 - 18 = 9 \notin S$,

$12 + 16 - 19 = 9 \notin S$,

$12 + 18 - 15 = 15 \notin S$,

$12 + 19 - 16 = 15 \notin S$.

Since the numerical semigroup S has no f_3 value, there is no NG vector, so it is not nearly Gorenstein.

Example 2.16.

$S = \langle 10, 11, 12, 13, 14, 29\rangle = \{0, 10, 11, 12, 13, 14, 20, \rightarrow\}$

$G(S) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 16, 17, 18, 19\}$,

$F(S) = 19$,

$PF(S) = \{15, 16, 17, 18, 19\}$,

$n_1 = 10,\ n_2 = 11,\ n_3 = 12,\ n_4 = 13,\ n_5 = 14,\ n_6 = 29.$

By Proposition 2.14, $f_1 = F(S) = 19$,

$11 + 18 - f \in S$ for every $f \in PF(S)$ $f_2 = 18$,

$12 + 17 - f \in S$ for every $f \in PF(S)$ $f_3 = 17$,

$13 + 16 - f \in S$ for every $f \in PF(S)$ $f_4 = 16$,

$14 + 15 - f \in S$ for every $f \in PF(S)$ $f_5 = 15$,

$29 + 15 - f \in S$ for every $f \in PF(S)$ $f_6 = 15$,

$29 + 16 - f \in S$ for every $f \in PF(S)$ $f_6 = 16$,

$29 + 17 - f \in S$ for every $f \in PF(S)$ $f_6 = 17$,

$29 + 18 - f \in S$ for every $f \in PF(S)$ $f_6 = 18$,

$29 + 19 - f \in S$ for every $f \in PF(S)$ $f_6 = 19$.

Since S is the NG-vector of the numerical semigroup, S is nearly Gorenstein. Then, the NG-vector of S is $(19, 18, 17, 16, 15, 15)$, $(19, 18, 17, 16, 15, 16)$, $(19, 18, 17, 16, 15, 17)$, $(19, 18, 17, 16, 15, 18)$ and $(19, 18, 17, 16, 15, 19)$.

We remark that f could be unique or not. For instance, pseudo-Frobenius numbers of $S_1 = (51, 52, 75, 76, 83, 84)$ are $PF(S_1) = \{157, 174, 175, 176, 349, 350\}$ and it is easy to see that $(350, 349, 350, 349, 350, 349)$ is the NG-vector of S_1. Pseudo-Frobenius numbers of $S_2 = (10, 11, 12, 13, 14, 29)$ are $PF(S_2) = \{15, 16, 17, 18, 19\}$ and $(19, 18, 18, 16, 15, 15)$, $(19, 18, 18, 16, 15, 16)$, $(19, 18, 18, 16, 15, 17)$, $(19, 18, 18, 16, 15, 18)$ and $(19, 18, 18, 16, 15, 19)$ are the NG-vectors of S_2. The NG-vectors of S_2 above will be represented as; $(19, 18, 18, 16, 15, (15, 16, 17, 18, 19))$. In this paper, this notation will be used if the number of NG-vectors is too large.

3. Some Results in Nearly Gorenstein Numerical Semigroup of 6 Generators With Type 5

The nearly Gorenstein semigroup taken in this paper is not almost numerical semigroup.

Definition 3.1. The set of Nearly Gorenstein semigroups with all e-generators is called the nearly Gorenstein universal set and is denote by $\mathbb{S}^e$.
$S_1 = (10, 11, 12, 13, 28)$ is nearly Gorenstein. Moreover, $(29, 29, 27, 27, 27)$ and $(29, 29, 27, 27, 29)$ NG-vectors of are S_1. Therefore, $S_1 \in \mathbb{S}^5$. Although $S_2 = \langle 10, 11, 12, 13, 14, 17\rangle$ is a numerical semigroup with 6 generators, it is not nearly Gorenstein (see Example 2.15). So, $S_2 \notin \mathbb{S}^6$.

Definition 3.2. Let's denote the set of all nearly Gorenstein numerical semigroups of the type p ($p \geq 1$, $p \in \mathbb{N}$) with e generators $\mathcal{S}^e t(p)$. $H \in \mathcal{S}^5 t(3)$ since the pseudo-Frobenius numbers of $H = (10,11,12,13,27)$ are $PF(H) = \{14,28,29\}$.

Since the type of nearly Gorenstein numerical semigroup 4 generator is at most 3, $\mathcal{S}^4 t(4) = \emptyset$.

Here, $\mathcal{S}^e t(< p)$ ($\mathcal{S}^e t(> p)$) denotes the set of all nearly Gorenstein numerical semigroups of e generators with at most p (at least p) type.

The meaning of $S \in \mathcal{S}^6 t(< 5)$ is that S the nearly Gorenstein numerical semigroups of the type at most 5 with 6 generators. For instance, as $S = \{10,11,12,18,19,27\}$ is nearly Gorenstein $PF(S) = \{9,17,25,26\}$ and hence it has type 4, $S \in \mathcal{S}^6 t(< 5)$. But $H \notin \mathcal{S}^6 t(< 5)$ because $H = \{51,58,72,86,93,95\}$ is nearly Gorenstein $PF(H) = \{65,79,579,593,614,658\}$ and therefore it has type 6.

Now we will give some results related to the set of nearly Gorenstein numerical semigroups of 6 generators with the type 5.

Proposition 3.3. *Let $S \in \mathcal{S}^6 t(5)$ be a nearly Gorenstein numerical semigroup.*
Let $F(S) = f_6 = f_i \neq f_j \neq f_k \neq f_l$ with $\{i,j,k,l\} = \{2,3,4,5\}$.
If there is $f \in PF(S) \backslash \{f_l, f_k, f_j, F(S)\}$, then $PF(S) = \{F(S)/2, f_l, f_k, f_j, F(S)\}$, $PF(S) = \{f, F(S) - f, f_k, f_j, F(S)\}$, $PF(S) = \{f, f_l, F(S) - f, f_j, F(S)\}$.

Proof. *Case* 1. $i = 2$

Suppose that there exists $f \in PF(S) \backslash \{f_l, f_k, f_j, F(S)\}$. Since $F(S) = f_1$, there is a factorization

$$f_6 - f_3 = a_1 n_1 + a_2 n_2 + a_3 n_3 + a_4 n_4 + a_5 n_5 - n_6,$$

$$f_3 - f = d_1 n_1 + d_2 n_2 - n_3 + d_4 n_4 + d_5 n_5 + d_6 n_6,$$

$$f_6 - f_4 = b_1 n_1 + b_2 n_2 + b_3 n_3 + b_4 n_4 + b_5 n_5 - n_6,$$

$$f_4 - f = e_1 n_1 + e_2 n_2 + e_3 n_3 - n_4 + e_5 n_5 + e_6 n_6,$$

$$f_6 - f_5 = c_1 n_1 + c_2 n_2 + c_3 n_3 + c_4 n_4 + c_5 n_5 - n_6,$$

$$f_5 - f = p_1 n_1 + p_2 n_2 + p_3 n_3 + p_4 n_4 - n_5 + p_6 n_6 \text{ with } a_i, b_i, c_i, d_j, e_k, p_l \geq 0,$$

$i = \{1,2,3,4,5\}$, $j = \{1,2,4,5,6\}$, $k = \{1,2,3,5,6\}$, $l = \{1,2,3,4,6\}$ at least one of positive.
We assume that $d_6, e_6, p_6, a_1, b_1, b_2, c_1, c_2 > 0$.

$$\begin{aligned} f_6 - f &= (f_6 - f_3) + (f_3 - f) \\ &= (a_1 + d_1) n_1 + (a_2 + d_2) n_2 + (a_3 - 1) n_3 + (a_4 + d_4) n_4 + (a_5 + d_5) n_5 + (d_6 - 1) n_6 \end{aligned}$$

and since $f_6 - f \notin S$ and $d_6 > 0$, it follows that $a_3 = 0$ and $f_6 - f + n_3 \in S$ similarly, $e_6 > 0$, it follows that $b_4 = 0$ and $f_6 - f + n_4 \in S$. $p_6 > 0$, it follows that $c_5 = 0$ and $f_6 - f + n_5 \in S$. Moreover, $f_6 - f + n_i \in S$ for $i = 1,2,6$ and $f_6 = f_1$ by hypothesis, thus $f_1 - f \in PF(S)$.

Case 2. Similarly $f_1 - f \in PF(S)$ can be found for $i = 3,4,5$.
Since $f_1 - f < f_1$ and $S \in \mathcal{S}^6 t(5)$ imply that $f_1 - f = f$, i.e. $f = F(S)/2$, $f_1 - f = f_l$, $f_1 - f = f_k$. □

Example 3.4. (i) The semigroup $S = \langle 11,15,16,20,24,28 \rangle$ is nearly Gorenstein has the NG-vector $(34,34,29,25,21,34)$ and $PF(S) = \{17,21,25,29,34\}$. We also note that $(34,34,29,(25,29),(21,25),(17,21,34))$ are NG-vectors for S.

(ii) The semigroup $S = \langle 10,13,14,17,19,22\rangle$ is nearly Gorenstein has the NG-vector $(25,25,21,\ 18,16,25)$ and $PF(S) = \{9,16,18,21,25\}$. $(25,25,21,(18,21),(16,25),(16,25))$ are NG-vectors for S.

(iii) The semigroup $S = \langle 10,11,12,19,25,28\rangle$ is nearly Gorenstein has the NG-vector $(27,27,26,18,13,27)$ and $PF(S) = \{9,13,18,26,27\}$. $(27,(26,27),26,(18,27),(13,26),(9,18,27))$ are NG-vectors for S.

(iv) The semigroup $S = \langle 11,16,17,21,25,31\rangle$ is nearly Gorenstein has the NG-vector $(40,35,40,30,26,40)$ and $PF(S) = \{20,26,30,35,40\}$. $(40,35,40,30,26,(20,26,40))$ are NG-vectors for S.

(v) The semigroup $S = \langle 10,13,14,16,17,21\rangle$ is nearly Gorenstein has the NG-vector $(25,22,\ 25,19,1825)$ and $PF(S) = \{7,18,19,22,25\}$. $(25,22,25,19,(18,22,25),(18,25))$ are NG-vectors for S.

(vi) The semigroup $S = \langle 10,11,15,16,34,39\rangle$ is nearly Gorenstein has the NG-vector $(29,28,29,23,5,29)$ and $PF(S) = \{5,23,24,28,29\}$. $(29,28,(24,29),(23,28),(5,28,29),(5,23,24,28,29))$ are NG-vectors for S.

(vii) The semigroup $S = \langle 10,15,24,31,33,36\rangle$ is nearly Gorenstein has the NG-vector $(52,47,38,52,29,52)$ and $PF(S) = \{26,29,38,47,52\}$. $(52,47,38,52,29,(26,47,52))$ are NG-vectors for S.

(viii) The semigroup $S = \langle 10,14,15,16,19,21\rangle$ is nearly Gorenstein has the NG-vector $(27,23,22,\ 27,18,27)$ and $PF(S) = \{5,18,22,23,27\}$. $(27,23,(22,27),27,(18,23),(22,27))$ are NG-vectors for S.

(ix) The semigroup $S = \langle 10,12,17,25,28,43\rangle$ is nearly Gorenstein has the NG-vector $(33,31,26,33,15,33)$ and $PF(S) = \{15,18,26,31,33\}$. $(33,31,26,(18,33),(15,33),(15,18,26,31,33))$ are NG-vectors for S.

(x) The semigroup $S = \langle 10,15,24,27,28,33\rangle$ is nearly Gorenstein has the NG-vector $(46,41,32,29,46,46)$ and $PF(S) = \{23,29,32,41,46\}$. $(46,41,32,29,46,(23,41,46))$ are NG-vectors for S.

(xi) The semigroup $S = \langle 10,12,15,23,28,41\rangle$ is nearly Gorenstein has the NG-vector $(31,29,26,18,31,31)$ and $PF(S) = \{13,18,26,29,31\}$. $(31,29,26,(18,31),(13,26,31),(13,18,26,29,31))$ are NG-vectors for S.

(xii) The semigroup $S = \langle 10,11,25,26,34,49\rangle$ is nearly Gorenstein has the NG-vector $(39,38,24,23,39,39)$ and $PF(S) = \{15,23,24,38,39\}$. $(39,38,(24,39),(23,38),(15,38,39),(15,23,24,28,39))$ are NG-vectors for S.

Proposition 3.5. *Let $S \in \mathcal{S}^6 t(5)$ be a nearly Gorenstein numerical semigroup. Let $F(S) \neq f_2 \neq f_3 \neq f_4 = f_5 = f_6$. Then, $PF(S) = \{f_4/2, f_4, f_3, f_2, F(S)\}$.*

Proof. Suppose that there exists $f \in PF(S)\backslash\{f_4, f_3, f_2, F(S)\}$. Since $F(S) = f_1$, there is a factorization

$$f_5 - f_1 = a_1n_1 + a_2n_2 + a_3n_3 + a_4n_4 - n_5 + a_6n_6,$$

$$f_1 - f = -n_1 + d_2n_2 + d_3n_3 + d_4n_4 + d_5n_5 + d_6n_6,$$

$$f_5 - f_2 = b_1n_1 + b_2n_2 + b_3n_3 + b_4n_4 - n_5 + b_6n_6,$$
$$f_2 - f = e_1n_1 - n_2 + e_3n_3 + e_4n_4 + e_5n_5 + e_6n_6,$$
$$f_5 - f_3 = c_1n_1 + c_2n_2 + c_3n_3 + c_4n_4 - n_5 + c_6n_6,$$
$$f_3 - f = p_1n_1 + p_2n_2 - n_3 + p_4n_4 + p_5n_5 + p_6n_6 \text{ with } a_i, b_i, c_i, d_j, e_k, p_l \geq 0,$$

$i = \{1,2,3,4,6\}$, $j = \{2,3,4,5,6\}$, $k = \{1,3,4,5,6\}$, $l = \{1,2,4,5,6\}$ at least one of positive. We assume that $d_5, e_5, p_5, a_2, b_4, c_1, c_2 > 0$.

$$\begin{aligned} f_5 - f &= (f_5 - f_1) + (f_1 - f) \\ &= (a_1 - 1)n_1 + (a_2 + d_2)n_2 + (a_3 + d_3)n_3 + (a_4 + d_4)n_4 + (d_5 - 1)n_5 + (a_6 + d_6)n_6, \end{aligned}$$

and since $f_5 - f \notin S$ and $d_5 > 0$, then $a_1 = 0$ and $f_5 - f + n_1 \in S$ similarly, $e_5 > 0$, it follows that $b_2 = 0$ and $f_5 - f + n_2 \in S$. $p_5 > 0$, so $c_3 = 0$ and $f_5 - f + n_3 \in S$. Moreover, $f_5 - f + n_i \in S$ for $i = 4,5,6$ and $f_5 = f_4$ by hypothesis, thus $f_4 - f \in PF(S)$. Since $f_4 - f < f_4 < f_3 < f_2 < F(S)$ and $S \in \mathbb{S}^6t(5)$ implies that $f_4 - f = f$, i.e. $f = f_4/2$. □

Example 3.6. The semigroup $S = \langle 10,11,19,23,35,47 \rangle$ is nearly Gorenstein has the NG-vector $(37,36,28,24,24,24)$ and $PF(S) = \{12,24,28,36,37\}$.
We also note that $(37,36,28,(24,36),(12,24,36),(12,24,28,36,37))$ are NG-vectors for S.

Proposition 3.7. *Let $S \in \mathbb{S}^6t(5)$ be a nearly Gorenstein numerical semigroup. Let $F(S) \neq f_3 \neq f_4 \neq f_2 = f_5 = f_6$. If there is $f \in PF(S) \setminus \{f_4, f_3, f_2, F(S)\}$, then $PF(S) = \{f_2/2, f_4, f_3, f_2, F(S)\}$, $PF(S) = \{f, f_2 - f, f_3, f_2, F(S)\}$, $PF(S) = \{f, f_4, f_2 - f, f_2, F(S)\}$.*

Proof. In a similar way, $f_5 - f + n_1 \in S$, $f_5 - f + n_3 \in S$ and $f_5 - f + n_4 \in S$. Moreover, $f_5 - f + n_i \in S$ for $i = 2,5,6$ and $f_5 = f_2$ by hypothesis, therefore $f_2 - f \in PF(S)$. Since $f_2 - f < f_2 < F(S)$ and $S \in \mathbb{S}^6t(5)$ imply that $f_2 - f = f$, i.e. $f = f_2/2$ $PF(S) = \{f_2/2, f_4, f_3, f_2, F(S)\}$, $f_2 - f = f_4$, $F(S) = \{f, f_2 - f, f_3, f_2, F(S)\}$, $f_2 - f = f_3$, $PF(S) = \{f, f_4, f_2 - f, f_2, F(S)\}$. □

Example 3.8. (i) The semigroup $S = \langle 10,11,16,18,23,35 \rangle$ is nearly Gorenstein has the NG-vector $(25,24,19,17,24,24)$ and $PF(S) = \{12,17,19,24,25\}$. $(25,24,19,17,(12,24),(12,17,19,24,25))$ are NG-vectors for S.

(ii) The semigroup $S = \langle 10,11,19,23,35,47 \rangle$ is nearly Gorenstein has the NG-vector $(37,36,28,24,36,36)$ and $PF(S) = \{12,24,28,36,37\}$. $(37,36,28,(24,36),(12,24,36),(12,24,28,36,37))$ are NG-vectors for S.

(iii) The semigroup $S = \langle 10,11,23,25,37,49 \rangle$ is nearly Gorenstein has the NG-vector $(39,38,26,24,38,38)$ and $PF(S) = \{12,24,26,38,39\}$. $(39,38,(26,38),24,(12,24,38),(12,24,26,38,39))$ are NG-vectors for S.

Proposition 3.9. *Let $S \in \mathbb{S}^6t(5)$ be a nearly Gorenstein numerical semigroup. Let $F(S) \neq f_2 \neq f_4 \neq f_3 = f_5 = f_6$. If there is $f \in PF(S) \setminus \{f_4, f_3, f_2, F(S)\}$, then either $f = f_3/2$ or $PF(S) = \{f, f_3 - f, f_3, f_2, F(S)\}$.*

Proof. In a similar way, $f_5 - f + n_1 \in S$, $f_5 - f + n_2 \in S$ and $f_5 - f + n_4 \in S$. Moreover, $f_5 - f + n_i \in S$ for $i = 3,5,6$ and $f_5 = f_3$ by hypothesis, thus $f_3 - f \in PF(S)$. Since $f_3 - f < f_3 < f_2 < F(S)$ and $S \in \mathbb{S}^6t(5)$ either $f_3 - f = f$, i.e., $f = f_3/2$ or $f_3 - f = f_4$. $PF(S) = \{f, f_3 - f, f_3, f_2, F(S)\}$. □

Example 3.10. (i) The semigroup $S = \langle 10,12,15,26,29,43\rangle$ is nearly Gorenstein has the NG-vector $(33,31,28,17,28,28)$ and $PF(S) = \{14,17,28,31,33\}$. In addition, $(33,31,28,(17,31),(14,28,31),(14,17,28,31,33))$ are NG-vectors for S.

(ii) The semigroup $S = \langle 10,15,16,21,22,27\rangle$ is nearly Gorenstein has the NG-vector $(39,34,33,28,33,33)$ and $PF(S) = \{5,28,33,34,39\}$. Moreover, $(33,(34,39),(33,39),(28,33,34,39),33,(28,33))$ are NG-vectors for S.

Proposition 3.11. *Let $S \in \mathcal{S}^6t(5)$ be a nearly Gorenstein numerical semigroup. Let $F(S) \neq f_3 \neq f_2 = f_4 = f_5 = f_6$. Then, $PF(S) = \{f_2 - f, f, f_3, f_2, F(S)\}$.*

Proof. Suppose that there exists $f \in PF(S)\setminus\{f_3, f_2, F(S)\}$.
Since $F(S) = f_1$, there is a factorization,

$$f_4 - f_1 = a_1n_1 + a_2n_2 + a_3n_3 - n_4 + a_5n_5 + a_6n_6,$$
$$f_1 - f = -n_1 + d_2n_2 + d_3n_3 + d_4n_4 + d_5n_5 + d_6n_6,$$
$$f_4 - f_3 = b_1n_1 + b_2n_2 + b_3n_3 - n_4 + b_5n_5 + b_6n_6,$$
$$f_3 - f = e_1n_1 + e_2n_2 - n_3 + e_4n_4 + e_5n_5 + e_6n_6 \text{ with } a_i, b_i, d_j, e_k \geq 0,$$

$i = \{1,2,3,5,6\}$, $j = \{2,3,4,5,6\}$, $k = \{1,2,4,5,6\}$ at least one of positive. We assume that $d_4, e_4, a_3, b_1 > 0$.

$$\begin{aligned} f_4 - f &= (f_4 - f_1) + (f_1 - f) \\ &= (a_1 - 1)n_1 + (a_2 + d_2)n_2 + (a_3 + d_3)n_3 + (d_4 - 1)n_4 + (a_4 + d_5)n_5 + (a_6 + d_6)n_6 \end{aligned}$$

and since $f_4 - f \notin S$ and $d_4 > 0$, thus $a_1 = 0$ and $f_4 - f + n_1 \in S$, similarly, $e_4 > 0$, it follows that $b_3 = 0$ and $f_4 - f + n_3 \in S$. Moreover, $f_4 - f + n_i \in S$ for $i = 2,4,5,6$ and $f_4 = f_2$ by hypothesis, thus $f_2 - f \in PF(S)$. Since $S \in \mathcal{S}^6t(5)$, $PF(S) = \{f_2 - f, f, f_3, f_2, F(S)\}$. □

Example 3.12. The semigroup $S = \langle 10,13,15,18,21,37\rangle$ is nearly Gorenstein has the NG-vector $(32,29,27,29,29,29)$ and $PF(S) = \{5,24,27,29,32\}$. We also note that $(32,29,(27,32),(24,29,32),29,(5,29,32))$ are NG-vectors for S.

Proposition 3.13. *Let $S \in \mathcal{S}^6t(5)$ be a nearly Gorenstein numerical semigroup. Let $F(S) = f_4 = f_5 = f_6 \neq f_3 \neq f_2$. Then, $PF(S) = \{F(S) - f, f, f_3, f_2, F(S)\}$.*

Proof. Suppose that there exists $f \in PF(S)\setminus\{f_3, f_2, F(S)\}$.
Since $F(S) = f_1$, there is a factorization

$$f_4 - f_2 = a_1n_1 + a_2n_2 + a_3n_3 - n_4 + a_5n_5 + a_6n_6,$$
$$f_2 - f = d_1n_1 - n_2 + d_3n_3 + d_4n_4 + d_5n_5 + d_6n_6,$$
$$f_4 - f_3 = b_1n_1 + b_2n_2 + b_3n_3 - n_4 + b_5n_5 + b_6n_6,$$
$$f_3 - f = e_1n_1 + e_2n_2 - n_3 + e_4n_4 + e_5n_5 + e_6n_6 \text{ with } a_i, b_i, d_j, e_k \geq 0,$$

$i = \{1,2,3,5,6\}$, $j = \{1,3,4,5,6\}$, $k = \{1,2,4,5,6\}$ at least one of positive. We assume that $d_4, e_4, a_1, b_5 > 0$.

$$f_4 - f = (f_4 - f_2) + (f_2 - f)$$

$$= (a_1+d_1)n_1+(d_2-1)n_2+(a_3+d_3)n_3+(d_4-1)n_4+(a_4+d_5)n_5+(a_6+d_6)n_6.$$

and since $f_4-f\notin S$ and $d_4>0$, it follows that $a_2=0$ and $f_4-f+n_2\in S$ similarly, $e_4>0$, it follows that $b_3=0$ and $f_4-f+n_3\in S$. Moreover, $f_4-f+n_i\in S$ for $i=1,4,5,6$ and $f_4=f_1$ by hypothesis, thus $f_1-f\in PF(S)$. Since $S\in\mathbb{S}^6 t(5)$ give that $PF(S)=\{F(S)-f,f,f_3,f_2,F(S)\}$. □

Example 3.14. The semigroup $S=\langle 10,14,15,26,31,47\rangle$ is nearly Gorenstein has the NG-vector $(37,33,32,37,37,37)$ and $PF(S)=\{16,21,32,33,37\}$. Furthermore, $(37,33,32,(21,37),(16,32,37),(16,21,32,33,37))$ are NG-vectors for S.

4. Conclusion

The type of nearly Gorenstein numerical semigroup with 6 generator can be larger in the generative set. In order to understand it better, some results are found in the nearly Gorenstein numerical semigroup of 6 generators with type 5.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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