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Research Article

Hamacher Operations on Picture Fuzzy Matrices

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Abstract. In this article, Hamacher operations of *Picture Fuzzy Matrices* are proposed, which are direct extensions of Hamacher operations of *Intuitionistic Fuzzy Matrices*. By using these operations scalar multiplication and exponentiation operations are constructed for a picture fuzzy matrix and certain new relations are established.

Keywords. Fuzzy matrix, Intuitionistic fuzzy matrix, Picture fuzzy matrix, Hamacher sum and Hamacher product

Mathematics Subject Classification (2020). 03E72, 08A72, 15B15, 94D05

1. Introduction

Picture Fuzzy Sets (PFSs) are direct extensions of the fuzzy sets and *Intuitionistic Fuzzy Sets* (IFSs) [1, 2]. Dogra and Pal [5] define *Picture Fuzzy Matrix* (PFM) generalizing the concept of *Intuitionistic Fuzzy Matrix* (IFM). A many works on the IFMs had been done by Murugadas *et al*. [8], Muthuraji *et al*. [9], Ramakrishnan and Sriram [10], and Silambarasan and Sriram [11]. Cuong and Kreinovich [4] proposed PFSs, Cuong [3] presented some properties of PFSs, Dutta and Ganju [6] discussed decomposition theorem for PFSs, Thomason [12] introduced the notion of fuzzy matrices, IFMs as a generalization of fuzzy matrices developed by Mondal and Pal [7] is used to represent the intuitionistic fuzzy relation. Motivated by the existing operations, we present Hamacher operations on PFMs and construct scalar multiplication and power operations with respect to these operations.

2. Preliminaries

Definition 2.1 ([1,2])**.** An IFS E in X is given by $E = \{\langle x, \mu_E(x), \nu_E(x)/x \in X\rangle\}$, where $\mu_E(x): X \to [0,1]$ and $\nu_E(x): X \to [0,1]$, with the condition $0 \le \mu_E(x) + \nu_E(x) \le 1$.
The numbers $\mu_E(x)$ and $\nu_E(x)$ represent, the membership degree and non-membership degree of the element to the set E.

Definition 2.2 ([4])**.** A picture fuzzy set E on a universe X is an object in the form of $E = \{(x, \mu_E(x), \eta_E(x), \nu_E(x))/x \in X\}$ where $\mu_E(x) \in [0,1]$ is called the degree of positive membership of x in E, $\eta_E(x) \in [0,1]$ is called the degree of neutral membership of x in E and $\nu_E(x) \in [0,1]$ is called the degree of negative membership of x in E and where $\mu_E(x)$, $\eta_E(x)$ and $\nu_E(x)$ satisfy the following condition: $(\forall\ x \in X)\ \mu_E(x) + \eta_E(x) + \nu_E(x) \le 1$.

Now $(1 - (\mu_E(x) + \eta_E(x) + \nu_E(x)))$ could be called the degree of refusal membership of x in E. Let PFS(X) denote the set of all the picture fuzzy sets on a universe X.

Definition 2.3 ([5])**.** A picture fuzzy matrices of size $m \times n$ is defined as $E = (\langle e_{lm\mu}, e_{lm\eta}, e_{lm\nu}\rangle)$, where $e_{lm\mu} \in [0,1]$, $e_{lm\eta} \in [0,1]$, $e_{lm\nu} \in [0,1]$ are respectively, the measure of positive, neutral and negative membership of e_{lm} for $l = 1,2,3,\dots n$ and $m = 1,2,3,\dots n$ satisfying $0 \le e_{lm\mu} + e_{lm\eta} + e_{lm\nu} \le 1$.

Definition 2.4 ([5])**.** Let $E = (\langle e_{lm\mu}, e_{lm\eta}, e_{lm\nu}\rangle)$ and $F = t(\langle f_{lm\mu}, f_{lm\eta}, f_{lm\nu}\rangle)$ be two Picture fuzzy matrices of order m. Then $E \le F$. If $e_{lm\mu} \le f_{lm\mu}$, $e_{lm\eta} \le f_{lm\eta}$, $e_{lm\nu} \ge f_{lm\nu}$ for $l, m = 1,2,3,\dots n$.

3. Hamacher Operations on Picture Fuzzy Matrices

In this section, we define Hamacher operations on Picture fuzzy matrices and construct the expressions nE and E^n for a Picture fuzzy matrix E.

Definition 3.1. Let $E = (\langle e_{lm\mu}, e_{lm\eta}, e_{lm\nu}\rangle)$ and $F = (\langle f_{lm\mu}, f_{lm\eta}, f_{lm\nu}\rangle)$ be two Picture fuzzy matrices of same size. Then,

(i) $$E \oplus_S F = \left[\frac{e_{lm\mu} + f_{lm\mu} - 2e_{lm\mu}f_{lm\mu}}{1 - e_{lm\mu}f_{lm\mu}}, \frac{e_{lm\eta}f_{lm\eta}}{e_{lm\eta} + f_{lm\eta} - e_{lm\eta}f_{lm\eta}}, \frac{e_{lm\nu}f_{lm\nu}}{e_{lm\nu} + f_{lm\nu} - e_{lm\nu}f_{lm\nu}}\right]$$

is called the Hamacher sum of E and F.

(ii) $$E \odot_M F = \left[\frac{e_{lm\mu}f_{lm\mu}}{e_{lm\mu} + f_{lm\mu} - e_{lm\mu}f_{lm\mu}}, \frac{e_{lm\eta} + f_{lm\eta} - 2e_{lm\eta}f_{lm\eta}}{1 - e_{lm\eta}f_{lm\eta}}, \frac{e_{lm\nu} + f_{lm\nu} - 2e_{lm\nu}f_{lm\nu}}{1 - e_{lm\nu}f_{lm\nu}}\right]$$

is called the Hamacher Product of E and F.

(iii) $E \vee F = [\langle \max(e_{lm\mu}, f_{lm\mu}), \min(e_{lm\eta}, f_{lm\eta}), \min(e_{lm\nu}, f_{lm\nu})\rangle]$.

(iv) $E \wedge F = [\langle \min(e_{lm\mu}, f_{lm\mu}), \min(e_{lm\eta}, f_{lm\eta}), \max(e_{lm\nu}, f_{lm\nu})\rangle]$.

(v) $E^t = [\langle e_{ml\mu}, e_{ml\eta}, e_{ml\nu}\rangle]$

(vi) $E^C = [\langle e_{lm\nu}, e_{lm\eta}, e_{lm\mu}\rangle]$ (the complement of E).

(vii) $$E @_K F = \left[\left(\frac{e_{lm\mu} + f_{lm\mu}}{2}\right), \left(\frac{e_{lm\eta} + f_{lm\eta}}{2}\right), \left(\frac{e_{lm\nu} + f_{lm\nu}}{2}\right)\right].$$

Theorem 3.1. *Let* $E = (\langle e_{lm\mu}, e_{lm\eta}, e_{lmv}\rangle)$ *and* $F = (\langle f_{lm\mu}, f_{lm\eta}, f_{lmv}\rangle)$ *be two PFMs of same size, then,* $0 \le (E \oplus_S F) \le 1$ *is also PFM.*

Proof. Since, $0 \le (\langle e_{lm\mu}, e_{lm\eta}, e_{lmv}\rangle) \le 1$ and $0 \le (f_{lm\mu}, f_{lm\eta}, f_{lmv}) \le 1$ respectively, $0 \le (e_{lm\mu} + e_{lm\eta} + e_{lmv}) \le 1$, then, $0 \le e_{lm\mu} \le (1 - e_{lmv})$ and $e_{lmv} \le (1 - e_{lm\mu})$ and also, $e_{lm\eta} \le (1 - e_{lm\mu})$, $0 \le (f_{lm\mu} + f_{lm\eta} + f_{lmv}) \le 1$, then, $0 \le f_{lm\mu} \le (1 - f_{lmv})$ and $f_{lmv} \le (1 - f_{lm\mu})$ and also, $f_{lm\eta} \le (1 - f_{lm\mu})$. Then, we have

$$\frac{e_{lm\eta} f_{lm\eta}}{e_{lm\eta} + f_{lm\eta} - e_{lm\eta} f_{lm\eta}} \le \frac{(1 - e_{lm\mu})(1 - f_{lm\mu})}{1 + (1 - e_{lm\mu})(1 - f_{lm\mu})}$$

and

$$\frac{e_{lmv} f_{lmv}}{e_{lmv} + f_{lmv} - e_{lmv} f_{lmv}} \le \frac{(1 - e_{lm\mu})(1 - f_{lm\mu})}{1 + (1 - e_{lm\mu})(1 - f_{lm\mu})}.$$

Now,

$$\frac{e_{lm\mu} + f_{lm\mu} - 2e_{lm\mu} f_{lm\mu}}{1 - e_{lm\mu} f_{lm\mu}} + \frac{e_{lm\eta} f_{lm\eta}}{e_{lm\eta} + f_{lm\eta} - e_{lm\eta} f_{lm\eta}} + \frac{e_{lmv} f_{lmv}}{e_{lmv} + f_{lmv} - e_{lmv} f_{lmv}} \le 1$$

iff

$$E = (1,0,0) \quad \text{and} \quad F = (1,0,0).$$

Furthermore, we have

$$\frac{e_{lm\mu} + f_{lm\mu} - 2e_{lm\mu} f_{lm\mu}}{1 - e_{lm\mu} f_{lm\mu}} + \frac{e_{lm\eta} f_{lm\eta}}{e_{lm\eta} + f_{lm\eta} - e_{lm\eta} f_{lm\eta}} + \frac{e_{lmv} f_{lmv}}{e_{lmv} + f_{lmv} - e_{lmv} f_{lmv}} = 0$$

iff

$$E = (0,0,0) \quad \text{and} \quad F = (0,0,0).$$

Thus, the solution of

$$0 \le \left(\frac{e_{lm\mu} + f_{lm\mu} - 2e_{lm\mu} f_{lm\mu}}{1 - e_{lm\mu} f_{lm\mu}} + \frac{e_{lm\eta} f_{lm\eta}}{e_{lm\eta} + f_{lm\eta} - e_{lm\eta} f_{lm\eta}} + \frac{e_{lmv} f_{lmv}}{e_{lmv} + f_{lmv} - e_{lmv} f_{lmv}}\right) \le 1.$$ □

Similarly, we can prove the following theorems:

Theorem 3.2. *Let* $E = (\langle e_{lm\mu}, e_{lm\eta}, e_{lmv}\rangle)$ *and* $F = (\langle f_{lm\mu}, f_{lm\eta}, f_{lmv}\rangle)$ *be two PFMs of same size, then,* $0 \le (E \odot_M F) \le 1$ *is a PFM.*

Theorem 3.3. *If* n *is any positive integer and* E *is a PFM, then the scalar multiplication operation is defined by*

$$nE = \underbrace{E \oplus_S E \oplus_S E \oplus_S \cdots \oplus_S E}_{n} = \left(\frac{n e_{lm\mu}}{1 + (n-1) e_{lm\mu}}, \frac{e_{lm\eta}}{n - (n-1) e_{lm\eta}}, \frac{e_{lmv}}{n - (n-1) e_{lmv}}\right).$$

Proof. Mathematical induction can be used to prove that the above equation holds. For all positive integer n. The above equation is called $P(n)$.

(i): The above equation $P(n)$ true for $n=1$. Since,

$$nE=\left(\frac{(1+e_{lm\mu})^n-(1-e_{lm\mu})^n}{(1+e_{lm\mu})^n+(1-e_{lm\mu})^n},\frac{2(e_{lm\eta})^n}{(2-e_{lm\eta})^n+(e_{lm\eta})^n},\frac{2(e_{lm\nu})^n}{(2-e_{lm\nu})^n+(e_{lm\nu})^n}\right)$$
$$=(\langle e_{lm\mu},e_{lm\eta},e_{lm\nu}\rangle)=E.$$

Then $P(n)$ is true for $n=1$, i.e, $P(1)$ holds.

(ii): When $n=2$, we have

$$E\oplus_S E=2E=\left(\frac{2e_{lm\mu}(1-e_{lm\mu})}{(1-e_{lm\mu}e_{lm\mu})},\frac{e_{lm\eta}e_{lm\eta}}{e_{lm\eta}(2-e_{lm\eta})},\frac{e_{lm\nu}e_{lm\nu}}{e_{lm\nu}(2-e_{lm\nu})}\right)$$
$$=\left(\frac{2e_{lm\mu}}{1+(2-1)e_{lm\mu}},\frac{e_{lm\eta}}{2-(2-1)e_{lm\eta}},\frac{e_{lm\nu}}{2-(2-1)e_{lm\nu}}\right).$$

Similarly,

$$3E=\left(\frac{3e_{lm\mu}}{1+(3-1)e_{lm\mu}},\frac{e_{lm\eta}}{3-(3-1)e_{lm\eta}},\frac{e_{lm\nu}}{3-(3-1)e_{lm\nu}}\right).$$

In general,

$$nE=\left(\frac{ne_{lm\mu}}{1+(n-1)e_{lm\mu}},\frac{e_{lm\eta}}{n-(n-1)e_{lm\eta}},\frac{e_{lm\nu}}{n-(n-1)e_{lm\nu}}\right)$$

holds when $n=2$,

(iii): When $n=m$, we have

$$mE=\underbrace{E\oplus_S E\oplus_S E\oplus_S\cdots\oplus_S E}_{m}=\left(\frac{me_{lm\mu}}{1+(m-1)e_{lm\mu}},\frac{e_{lm\eta}}{m-(m-1)e_{lm\eta}},\frac{e_{lm\nu}}{m-(m-1)e_{lm\nu}}\right),$$

$$(m+1)E=mE\oplus_S E$$
$$=\left(\frac{(m+1-m\ e_{lm\mu}-e_{lm\mu})e_{lm\mu}}{(1+m\ e_{lm\mu}-e_{lm\mu}-m\ e_{lm\mu}e_{lm\mu})},\frac{e_{lm\eta}e_{lm\eta}}{(1+m-m\ e_{lm\eta})u_{lm\eta}},\frac{e_{lm\nu}e_{lm\nu}}{(1+m-m\ e_{lm\nu})e_{lm\nu}}\right)$$
$$=\left(\frac{(m+1)(1-e_{lm\mu})e_{lm\mu}}{(1+m\ e_{lm\mu})(1-e_{lm\mu})},\frac{e_{lm\eta}}{(1+m-m\ e_{lm\eta})},\frac{e_{lm\nu}}{(1+m-m\ e_{lm\nu})}\right)$$
$$=\left(\frac{(m+1)\ e_{lm\mu}}{(1+m\ e_{lm\mu})},\frac{e_{lm\eta}}{(m+1-m\ e_{lm\eta})},\frac{e_{lm\nu}}{(m+1-m\ e_{lm\nu})}\right)$$
$$=\left(\frac{(m+1)e_{lm\mu}}{(1+(m+1-1)e_{lm\mu})},\frac{e_{lm\eta}}{(m+1-(m+1-1)e_{lm\eta})},\frac{e_{lm\nu}}{(m+1-(m+1-1)e_{lm\nu})}\right).$$

Thus, when $n=m+1$,

$$nE=\underbrace{E\oplus_S E\oplus_S E\oplus_S\cdots\oplus_S E}_{n}=\left(\frac{n\ e_{lm\mu}}{1+(n-1)e_{lm\mu}},\frac{e_{lm\eta}}{n-(n-1)e_{lm\eta}},\frac{e_{lm\nu}}{n-(n-1)e_{lm\nu}}\right)$$

also holds. Using the induction hypothesis that $P(n)$ holds for any positive integer n. □

Similarly, we can prove the following theorem:

Theorem 3.4. *If n is any positive integer and E is an PFM, then the exponentiation operation is defined by*

$$E^n=\underbrace{E\odot_M E\odot_M E\odot_M\ldots\odot_M E}_{n}=\left(\frac{e_{lm\mu}}{n-(n-1)e_{lm\mu}},\frac{n\ e_{lm\eta}}{1+(n-1)e_{lm\eta}},\frac{n\ e_{lm\nu}}{1+(n-1)e_{lm\nu}}\right).$$

We prove the result of E^n is also a PFM.

4. Results of Some Algebraic Properties for Hamacher Operations of Picture Fuzzy Matrices

In this chapter, we discuss some algebraic properties with some special operations for Hamacher operations of picture fuzzy matrices.

Theorem 4.1. *Let $E=(\langle e_{lm\mu},e_{lm\eta},e_{lm\nu}\rangle)$ and $F=(\langle f_{lm\mu},f_{lm\eta},f_{lm\nu}\rangle)$ be two PFMs of same size, then*

(i) $E\oplus_S F=F\oplus_S E$,

(ii) $E\odot_M F=E\odot_M F$.

Proof. (i):
$$\begin{aligned}E\oplus_S F&=\left(\frac{e_{lm\mu}+f_{lm\mu}-2e_{lm\mu}f_{lm\mu}}{1-e_{lm\mu}f_{lm\mu}},\frac{e_{lm\eta}f_{lm\eta}}{e_{lm\eta}+f_{lm\eta}-e_{lm\eta}f_{lm\eta}},\frac{e_{lm\nu}f_{lm\nu}}{e_{lm\nu}+f_{lm\nu}-e_{lm\nu}f_{lm\nu}}\right)\\&=\left(\frac{f_{lm\mu}+e_{lm\mu}-2f_{lm\mu}e_{lm\mu}}{1-f_{lm\mu}e_{lm\mu}},\frac{f_{lm\eta}e_{lm\eta}}{f_{lm\eta}+e_{lm\eta}-f_{lm\eta}e_{lm\eta}},\frac{f_{lm\nu}e_{lm\nu}}{f_{lm\nu}+e_{lm\nu}-f_{lm\nu}e_{lm\nu}}\right)\\&=F\oplus_S E.\end{aligned}$$

Hence, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.2. *Let E and F be two PFMs of same size, then*

(i) $nE\oplus_S nF=n(E\oplus_S F)$,

(ii) $n(E\oplus_S E)=n(2E)=2nE=nE\oplus_S nE$.

Proof.

$$\begin{aligned}\text{(i)}:\quad nE\oplus_S nF&=\left(\frac{n\ e_{lm\mu}(1+(n-1)\ f_{lm\mu})+n\ f_{lm\mu}(1+(n-1)e_{lm\mu})-2\ n\ n\ e_{lm\mu}\ f_{lm\mu}}{(1+(n-1)f_{lm\mu})(1+(n-1)e_{lm\mu})-n\ n\ e_{lm\mu}\ f_{lm\mu}},\right.\\&\quad\cdot\frac{e_{lm\eta}f_{lm\eta}}{e_{lm\eta}(n-(n-1)f_{lm\eta})+f_{lm\eta}(n-(n-1)e_{lm\eta})-e_{lm\eta}f_{lm\eta}},\\&\quad\left.\cdot\frac{e_{lm\nu}\ f_{lm\nu}}{e_{lm\nu}(n-(n-1)f_{lm\nu})+f_{lm\nu}(n-(n-1)e_{lm\nu})-e_{lm\nu}f_{lm\nu}}\right)\\&=\left(\frac{n(e_{lm\mu}+f_{lm\mu}-2e_{lm\mu}f_{lm\mu})}{(1-e_{lm\mu}f_{lm\mu})+(n-1)(e_{lm\mu}+f_{lm\mu}-2e_{lm\mu}f_{lm\mu})},\right.\\&\quad\cdot\frac{e_{lm\eta}f_{lm\eta}}{n\ e_{lm\eta}+nf_{lm\eta}-2ne_{lm\eta}f_{lm\eta}+e_{lm\eta}f_{lm\eta}},\\&\quad\left.\cdot\frac{e_{lm\nu}f_{lm\nu}}{ne_{lm\nu}+n\ f_{lm\nu}-2n\ e_{lm\nu}f_{lm\nu}+e_{lm\nu}f_{lm\nu}}\right)\\&=\left(\frac{n\left(\frac{e_{lm\mu}+f_{lm\mu}-2e_{lm\mu}f_{lm\mu}}{1-e_{lm\mu}f_{lm\mu}}\right)}{1+(n-1)\left(\frac{e_{lm\mu}+f_{lm\mu}-2e_{lm\mu}f_{lm\mu}}{1-e_{lm\mu}f_{lm\mu}}\right)},\frac{\left(\frac{1-e_{lm\eta}f_{lm\eta}}{e_{lm\eta}+f_{lm\eta}-e_{lm\eta}f_{lm\eta}}\right)}{n-(n-1)\left(\frac{1-e_{lm\eta}f_{lm\eta}}{e_{lm\eta}+f_{lm\eta}-e_{lm\eta}f_{lm\eta}}\right)},\right.\\&\quad\left.\cdot\frac{\left(\frac{1-e_{lm\nu}f_{lm\nu}}{e_{lm\nu}+f_{lm\nu}-e_{lm\nu}f_{lm\nu}}\right)}{n-(n-1)\left(\frac{1-e_{lm\nu}f_{lm\nu}}{e_{lm\nu}+f_{lm\nu}-e_{lm\nu}f_{lm\nu}}\right)}\right)=n(E\oplus_S F).\end{aligned}$$

Hence, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.3. *Let E be a PFM, then*

(i) $mE \oplus_S nE = (m+n)E$,

(ii) $m(nE) = (mn)E$, *where* $m,n > 0$.

Proof.

$$\text{(i)}: \quad mE \oplus_S nE = \left(\frac{m\, e_{lm\mu}(1+(m-1)\, e_{lm\mu}) + n\, e_{lm\mu}(1+(n-1)e_{lm\mu}) - 2mn\, e_{lm\mu}\, e_{lm\mu}}{(1+(m-1)e_{lm\mu})(1+(n-1)e_{lm\mu}) - mn\, e_{lm\mu}e_{lm\mu}}, \right.$$
$$\cdot \frac{e_{lm\eta}e_{lm\eta}}{e_{lm\eta}(m-(m-1)\, e_{lm\eta}) + e_{lm\eta}(n-(n-1)\, e_{lm\eta}) - e_{lm\eta}e_{lm\eta}},$$
$$\left. \cdot \frac{e_{lmv}e_{lmv}}{e_{lmv}(m-(m-1)e_{lmv}) + e_{lmv}(n-(n-1)e_{lmv}) - e_{lmv}e_{lmv}} \right)$$
$$= \left(\frac{(1-e_{lm\mu})(m\, e_{lm\mu} + n\, e_{lm\mu})}{(1-e_{lm\mu})(1-e_{lm\mu}) + (m\, e_{lm\mu} + n\, e_{lm\mu})(1-e_{lm\mu})}, \frac{e_{lm\eta}}{(m+n)-(m+n-1)e_{lm\eta}}, \right.$$
$$\left. \cdot \frac{e_{lmv}}{(m+n)-(m+n-1)e_{lmv}} \right)$$
$$= \left(\frac{(m+n)\, e_{lm\mu}}{(1-e_{lm\mu}) + (m+n)e_{lm\mu}}, \frac{e_{lm\eta}}{(m+n)-(m+n-1)\, e_{lm\eta}}, \frac{e_{lmv}}{(m+n)-(m+n-1)e_{lmv}} \right)$$
$$= (m+n)E\,.$$

Hence, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.4. *Let E and F be two PFMs of same size, then*

(i) $E^m \odot_M E^n = E^{m+n}$, *where* $m,n > 0$.

(ii) $E^n \odot_M F^n = (E \odot_M F)^n$, *where* $m,n > 0$.

Proof.

$$\text{(i)}: \quad E^m \odot_M E^n = \left(\frac{e_{lm\mu}e_{lm\mu}}{e_{lm\mu}(n-(n-1)e_{lm\mu}) + e_{lm\mu}(m-(m-1)e_{lm\mu}) - e_{lm\mu}e_{lm\mu}}, \right.$$
$$\cdot \frac{m\, e_{lm\eta}(1+(n-1)e_{lm\eta}) + n\, e_{lm\eta}(1+(m-1)\, e_{lm\eta}) - 2mn\, e_{lm\eta}e_{lm\eta}}{(1+(m-1)e_{lm\eta})(1+(n-1)e_{lm\eta}) - mn\, e_{lm\eta}e_{lm\eta}},$$
$$\left. \cdot \frac{m\, e_{lmv}(1+(n-1)e_{lmv}) + n\, e_{lmv}(1+(m-1)\, e_{lmv}) - 2mn\, e_{lmv}e_{lmv}}{(1+(m-1)e_{lmv})(1+(n-1)e_{lmv}) - m\, n\, e_{lmv}e_{lmv}} \right)$$
$$= \left(\frac{e_{lm\mu}e_{lm\mu}}{e_{lm\mu}(m+n-m\, e_{lm\mu} - n\, e_{lm\mu} + e_{lm\mu})}, \right.$$
$$\cdot \frac{(1-e_{lm\eta})(m\, e_{lm\eta} + n\, e_{lm\eta})}{(1-e_{lm\eta})(1-e_{lm\eta}) + (m\, e_{lm\eta} + n\, e_{lm\eta})(1-e_{lm\eta})},$$
$$\left. \cdot \frac{(1-e_{lmv})(m\, e_{lmv} + n\, e_{lmv})}{(1-e_{lmv})(1-e_{lmv}) + (m\, e_{lmv} + n\, e_{lmv})(1-e_{lmv})} \right)$$
$$= \left(\frac{e_{lm\mu}}{(m+n)-(m+n-1)e_{lm\mu}}, \frac{(m\, e_{lm\eta} + n\, e_{lm\eta})}{(1-e_{lm\eta}) + (m\, e_{lm\eta} + n\, e_{lm\eta})}, \right.$$

$$\cdot \frac{(m\ e_{lm\nu}+n\ e_{lm\nu})}{(1-e_{lm\nu})+(m\ e_{lm\nu}+n\ e_{lm\nu})}\Bigg)$$

$$=\left(\frac{e_{lm\mu}}{(m+n)-((m+n)-1)\ e_{lm\mu}},\frac{(m+n)\ e_{lm\eta}}{1+((m+n)-1)\ e_{lm\eta}},\frac{(m+n)e_{lm\nu}}{1+((m+n)-1)\ e_{lm\nu}}\right)$$

$$=E^{m+n}.$$

Hence, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.5. *Let E,F and G be three PFMs of same size, then*

(i) $(E\oplus_S F)\oplus_S G=E\oplus_S(F\oplus_S G)$,

(ii) $(E\odot_M F)\odot_M G=E\odot_M(F\odot_M G)$.

Proof.

(i): $(E\oplus_S F)\oplus_S G$

$$=\left(\frac{e_{lm\mu}+f_{lm\mu}-2e_{lm\mu}f_{lm\mu}}{1-e_{lm\mu}f_{lm\mu}},\frac{e_{lm\eta}f_{lm\eta}}{e_{lm\eta}+f_{lm\eta}-e_{lm\eta}f_{lm\eta}},\frac{e_{lm\nu}f_{lm\nu}}{e_{lm\nu}+f_{lm\nu}-e_{lm\nu}f_{lm\nu}}\right)$$

$$\oplus_S(g_{lm\mu},g_{lm\eta},g_{lm\nu})$$

$$=\Bigg(\frac{e_{lm\mu}+f_{lm\mu}+g_{lm\mu}-2e_{lm\mu}f_{lm\mu}-2e_{lm\mu}g_{lm\mu}-2f_{lm\mu}g_{lm\mu}+3e_{lm\mu}f_{lm\mu}g_{lm\mu}}{1-e_{lm\mu}f_{lm\mu}-e_{lm\mu}g_{lm\mu}-f_{lm\mu}g_{lm\mu}+2e_{lm\mu}f_{lm\mu}g_{lm\mu}},$$

$$\cdot\frac{e_{lm\eta}f_{lm\eta}g_{lm\eta}}{e_{lm\eta}f_{lm\eta}+e_{lm\eta}g_{lm\eta}+f_{lm\eta}g_{lm\eta}-2e_{lm\eta}f_{lm\eta}g_{lm\eta}},$$

$$\cdot\frac{e_{lm\nu}f_{lm\nu}g_{lm\nu}}{e_{lm\nu}f_{lm\nu}+e_{lm\nu}g_{lm\nu}+f_{lm\nu}g_{lm\nu}-2e_{lm\nu}f_{lm\nu}g_{lm\nu}}\Bigg)$$

$$=\Bigg(\frac{e_{lm\mu}(1-f_{lm\mu}g_{lm\mu})+f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu}-2e_{lm\mu}(f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu})}{1-f_{lm\mu}g_{lm\mu}-e_{lm\mu}(f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu})},$$

$$\cdot\frac{e_{lm\eta}f_{lm\eta}g_{lm\eta}}{e_{lm\eta}(f_{lm\eta}+g_{lm\eta}-f_{lm\eta}g_{lm\eta})+f_{lm\eta}g_{lm\eta}-e_{lm\eta}f_{lm\eta}g_{lm\eta}},$$

$$\cdot\frac{e_{lm\nu}f_{lm\nu}g_{lm\nu}}{e_{lm\nu}(f_{lm\nu}+g_{lm\nu}-f_{lm\nu}g_{lm\nu})+f_{lm\nu}g_{lm\nu}-e_{lm\nu}f_{lm\nu}g_{lm\nu}}\Bigg)$$

$$=\Bigg(\frac{e_{lm\mu}+\left(\frac{f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu}}{1-f_{lm\mu}g_{lm\mu}}\right)-2\ e_{lm\mu}\left(\frac{f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu}}{1-f_{lm\mu}g_{lm\mu}}\right)}{1-e_{lm\mu}\left(\frac{f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu}}{1-f_{lm\mu}g_{lm\mu}}\right)},$$

$$\cdot\frac{e_{lm\eta}\left(\frac{f_{lm\eta}g_{lm\eta}}{f_{lm\eta}+g_{lm\eta}-f_{lm\eta}g_{lm\eta}}\right)}{e_{lm\eta}+\left(\frac{f_{lm\eta}g_{lm\eta}}{f_{lm\eta}+g_{lm\eta}-f_{lm\eta}g_{lm\eta}}\right)-e_{lm\eta}\left(\frac{f_{lm\eta}g_{lm\eta}}{f_{lm\eta}+g_{lm\eta}-f_{lm\eta}g_{lm\eta}}\right)},$$

$$\cdot\frac{e_{lm\nu}\left(\frac{f_{lm\nu}g_{lm\nu}}{f_{lm\nu}+g_{lm\nu}-f_{lm\nu}g_{lm\nu}}\right)}{e_{lm\nu}+\left(\frac{f_{lm\nu}g_{lm\nu}}{f_{lm\nu}+g_{lm\nu}-f_{lm\nu}g_{lm\nu}}\right)-e_{lm\nu}\left(\frac{f_{lm\nu}g_{lm\nu}}{f_{lm\nu}+g_{lm\nu}-f_{lm\nu}g_{lm\nu}}\right)}\Bigg)$$

$$=\Bigg((e_{lm\mu},e_{lm\eta},e_{lm\nu})\oplus_S\Bigg(\frac{f_{lm\mu}+g_{lm\mu}-2f_{lm\mu}g_{lm\mu}}{1-f_{lm\mu}g_{lm\mu}},\frac{f_{lm\eta}g_{lm\eta}}{f_{lm\eta}+g_{lm\eta}-f_{lm\eta}g_{lm\eta}},$$

$$\cdot \frac{f_{lmv} w_{lmv}}{f_{lmv} + w_{lmv} - f_{lmv} w_{lmv}}\Big)\Big) = E \oplus_S (F \oplus_S G).$$

Hence, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.6. *Let E be a PFM. Then*

(i) $E \oplus_S J = J \oplus_S E = J$,

(ii) $E \odot_M J = J \odot_M E = E$.

Proof. Let $J = (\langle 1, 0, 0 \rangle)$

$$E \oplus_S J = (e_{lm\mu}, e_{lm\eta}, e_{lmv}) \oplus_S J = J = J \oplus_S E,$$

$$E \odot_M J = (e_{lm\mu}, e_{lm\eta}, e_{lmv}) \odot_M J = E = J \odot_M E.$$ □

Theorem 4.7. *Let E be a PFM. Then*

(i) $J \oplus_S O = O \oplus_S J = J$,

(ii) $J \odot_M O = O \odot_M J = O$.

Theorem 4.8. *Let E,F and G be three PFMs of same size, then*

(i) $(E \wedge F) \oplus_S G = (E \oplus_S G) \wedge (F \oplus_S G)$,

(ii) $(E \vee F) \oplus_S G = (E \oplus_S G) \vee (F \oplus_S G)$.

Proof.

$$\begin{aligned}
\text{(i)}: \quad (E \wedge F) \oplus_S G &= \Bigg(\frac{\min(e_{lm\mu}, f_{lm\mu}) + g_{lm\mu} - 2\min(e_{lm\mu}, f_{lm\mu})\, g_{lm\mu}}{1 - \min(e_{lm\mu}, f_{lm\mu})\, g_{lm\mu}}, \\
&\quad \cdot \frac{\min(e_{lm\eta}, f_{lm\eta})\, g_{lm\eta}}{\min(e_{lm\eta}, f_{lm\eta}) + g_{lm\eta} - \min(e_{lm\eta}, f_{lm\eta})\, g_{lm\eta}}, \\
&\quad \cdot \frac{\max(e_{lmv}, f_{lmv})\, g_{lmv}}{\max(e_{lmv}, f_{lmv}) + g_{lmv} - \max(e_{lmv}, f_{lmv})\, g_{lmv}} \Bigg) \\
&= \Bigg(\min\left(\frac{e_{lm\mu} + g_{lm\mu} - 2 e_{lm\mu} g_{lm\mu}}{1 - e_{lm\mu} g_{lm\mu}}, \frac{f_{lm\mu} + g_{lm\mu} - 2 f_{lm\mu} g_{lm\mu}}{1 - f_{lm\mu} g_{lm\mu}} \right), \\
&\quad \cdot \min\left(\frac{e_{lm\eta} g_{lm\eta}}{e_{lm\eta} + g_{lm\eta} - e_{lm\eta} g_{lm\eta}}, \frac{f_{lm\eta} g_{lm\eta}}{f_{lm\eta} + g_{lm\eta} - f_{lm\eta} g_{lm\eta}} \right), \\
&\quad \cdot \max\left(\frac{e_{lmv} g_{lmv}}{e_{lmv} + g_{lmv} - e_{lmv} g_{lmv}}, \frac{f_{lmv} g_{lmv}}{f_{lmv} + g_{lmv} - f_{lmv} g_{lmv}} \right) \Bigg) \\
&= \Bigg(\left(\frac{e_{lm\mu} + g_{lm\mu} - 2 e_{lm\mu} g_{lm\mu}}{1 - e_{lm\mu} g_{lm\mu}}, \frac{e_{lm\eta} g_{lm\eta}}{e_{lm\eta} + g_{lm\eta} - e_{lm\eta} g_{lm\eta}}, \frac{e_{lmv} g_{lmv}}{e_{lmv} + g_{lmv} - e_{lmv} g_{lmv}} \right) \\
&\quad \wedge \Bigg(\frac{f_{lm\mu} + g_{lm\mu} - 2 f_{lm\mu} g_{lm\mu}}{1 - f_{lm\mu} g_{lm\mu}}, \frac{f_{lm\eta} g_{lm\eta}}{f_{lm\eta} + g_{lm\eta} - f_{lm\eta} g_{lm\eta}}, \\
&\quad \frac{f_{lmv} g_{lmv}}{f_{lmv} + g_{lmv} - f_{lmv} g_{lmv}} \Bigg) \Bigg) \\
&= (E \oplus_S G) \wedge (F \oplus_S G).
\end{aligned}$$

Thus, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.9. *Let E,F and G be three PFMs of same size, then*

(i) $(E\wedge F)\odot_M G=(E\odot_M G)\wedge(F\odot_M G)$,

(ii) $(E\vee F)\odot_M G=(E\odot_M G)\vee(F\odot_M G)$.

Proof.

(i):

$$
\begin{aligned}
(E\wedge F)\odot_M G&=\Bigg(\frac{\min(e_{lm\mu},\,f_{lm\mu})\,g_{lm\mu}}{\min(e_{lm\mu},\,f_{lm\mu})\;+g_{lm\mu}-\min(e_{lm\mu},\,f_{lm\mu})\,g_{lm\mu}},\\
&\quad\cdot\frac{\min(e_{lm\eta},\,f_{lm\eta})\;+g_{lm\eta}-2\min(e_{lm\eta},\,f_{lm\eta})\,g_{lm\eta}}{1-\min(e_{lm\eta},\,f_{lm\eta})\,g_{lm\eta}},\\
&\quad\cdot\frac{\max(e_{lm\nu},f_{lm\nu})\;+g_{lm\nu}-2\max(e_{lm\nu},f_{lm\nu})\,g_{lm\nu}}{1-\max(e_{lm\nu},f_{lm\nu})\,g_{lm\nu}}\Bigg)\\
&=\Bigg(\min\left(\frac{e_{lm\mu}g_{lm\mu}}{e_{lm\mu}+g_{lm\mu}-e_{lm\mu}g_{lm\mu}},\frac{f_{lm\mu}g_{lm\mu}}{f_{lm\mu}+g_{lm\mu}-f_{lm\mu}g_{lm\mu}}\right)\\
&\quad\cdot\min\left(\frac{e_{lm\eta}+g_{lm\eta}-2e_{lm\eta}g_{lm\eta}}{1-e_{lm\eta}g_{lm\eta}},\frac{f_{lm\eta}+g_{lm\eta}-2f_{lm\eta}g_{lm\eta}}{1-f_{lm\eta}g_{lm\eta}}\right)\\
&\quad\cdot\max\left(\frac{e_{lm\nu}+g_{lm\nu}-2e_{lm\nu}g_{lm\nu}}{1-e_{lm\nu}g_{lm\nu}},\frac{f_{lm\nu}+g_{lm\nu}-2f_{lm\nu}g_{lm\nu}}{1-f_{lm\nu}g_{lm\nu}}\right)\Bigg)\\
&=\Bigg(\left(\frac{e_{lm\mu}g_{lm\mu}}{e_{lm\mu}+g_{lm\mu}-e_{lm\mu}g_{lm\mu}},\frac{e_{lm\eta}+g_{lm\eta}-2e_{lm\eta}g_{lm\eta}}{1-e_{lm\eta}g_{lm\eta}},\frac{e_{lm\nu}+g_{lm\nu}-2e_{lm\nu}g_{lm\nu}}{1-e_{lm\nu}g_{lm\nu}}\right)\\
&\quad\wedge\left(\frac{f_{lm\mu}g_{lm\mu}}{f_{lm\mu}+g_{lm\mu}-f_{lm\mu}g_{lm\mu}},\frac{f_{lm\eta}+g_{lm\eta}-2f_{lm\eta}g_{lm\eta}}{1-f_{lm\eta}g_{lm\eta}},\frac{f_{lm\nu}+g_{lm\nu}-2f_{lm\nu}g_{lm\nu}}{1-f_{lm\nu}g_{lm\nu}}\right)\Bigg)\\
&=(E\odot_M G)\wedge(F\odot_M G).
\end{aligned}
$$

Thus, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.10. *Let E,F and G be three PFMs of same size, then*

(i) $(E\wedge F)@_K G=(E@_K G)\wedge(F@_K G)$,

(ii) $(E\vee F)@_K G=(E@_K G)\vee(F@_K G)$.

Proof.

(i): $(E\wedge F)@_K G$

$$
\begin{aligned}
&=\left(\left(\frac{\min(e_{lm\mu},\,f_{lm\mu})\;+g_{lm\mu}}{2}\right),\left(\frac{\min(e_{lm\eta},\,f_{lm\eta})\;+g_{lm\eta}}{2}\right),\left(\frac{\max(e_{lm\nu},f_{lm\nu})\;+g_{lm\nu}}{2}\right)\right)\\
&=\Bigg(\min\left(\frac{e_{lm\mu}+g_{lm\mu}}{2},\frac{f_{lm\mu}+g_{lm\mu}}{2}\right),\min\left(\frac{e_{lm\eta}+g_{lm\eta}}{2},\frac{f_{lm\eta}+g_{lm\eta}}{2}\right),\\
&\quad\max\left(\frac{e_{lm\nu}+g_{lm\nu}}{2},\frac{f_{lm\nu}+g_{lm\nu}}{2}\right)\Bigg)\\
&=\left(\left(\frac{e_{lm\mu}+g_{lm\mu}}{2},\frac{e_{lm\eta}+g_{lm\eta}}{2},\frac{e_{lm\nu}+g_{lm\nu}}{2}\right)\wedge\left(\frac{f_{lm\mu}+g_{lm\mu}}{2},\frac{f_{lm\eta}+g_{lm\eta}}{2},\frac{f_{lm\nu}+g_{lm\nu}}{2}\right)\right)\\
&=(E@_K G)\wedge(F@_K G).
\end{aligned}
$$

Thus, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.11. *Let E and F be an two PFMs of same size, then*

(i) $E@_K E = E$,

(ii) $E@_K F = F@_K E$,

(iii) $(E@_K F)^C = E^C @_K F^C$,

(iv) $(E^C @_K F^C)^C = E@_K F$.

Proof.

$$\text{(i)}: \quad E@_K E = \left(\frac{e_{lm\mu}+e_{lm\mu}}{2}\right), \left(\frac{e_{lm\eta}+e_{lm\eta}}{2}\right), \left(\frac{e_{lm\nu}+e_{lm\nu}}{2}\right) = (e_{lm\mu}, e_{lm\eta}, e_{lm\nu}) = E,$$

$$\text{(ii)}: \quad E@_K F = \left(\left(\frac{e_{lm\mu}+f_{lm\mu}}{2}\right), \left(\frac{e_{lm\eta}+f_{lm\eta}}{2}\right), \left(\frac{e_{lm\nu}+f_{lm\nu}}{2}\right)\right) = \left(\left(\frac{f_{lm\mu}+e_{lm\mu}}{2}\right), \left(\frac{f_{lm\eta}+e_{lm\eta}}{2}\right), \left(\frac{f_{lm\nu}+e_{lm\nu}}{2}\right)\right) = F@_K E,$$

$$\text{(iii)}: \quad (E@_K F)^C = \left(\left(\frac{e_{lm\mu}+f_{lm\mu}}{2}\right), \left(\frac{e_{lm\eta}+f_{lm\eta}}{2}\right), \left(\frac{e_{lm\nu}+f_{lm\nu}}{2}\right)\right)^C = \left(\left(\frac{e_{lm\nu}+f_{lm\nu}}{2}\right), \left(\frac{e_{lm\eta}+f_{lm\eta}}{2}\right), \left(\frac{e_{lm\mu}+f_{lm\mu}}{2}\right)\right) = (e_{lm\nu}, e_{lm\eta}, e_{lm\mu})@_K(f_{lm\nu}, f_{lm\eta}, f_{lm\mu}) = E^C @_K F^C,$$

$$\text{(iv)}: \quad (E^C @_K F^C)^C = ((e_{lm\nu}, e_{lm\eta}, e_{lm\mu})@_K(f_{lm\nu}, f_{lm\eta}, f_{lm\mu}))^C = \left(\left(\frac{e_{lm\mu}+f_{lm\mu}}{2}\right), \left(\frac{e_{lm\eta}+f_{lm\eta}}{2}\right), \left(\frac{e_{lm\nu}+f_{lm\nu}}{2}\right)\right) = E@_K F.$$

□

Theorem 4.12. *Let E and F be an two PFMs of same size, then*

(i) $(E^C \oplus_S F^C)^C \neq (E \odot_M F)$,

(ii) $(E^C \odot_M F^C)^C \neq (E \oplus_S F)$.

Proof.

$$\text{(i)}: \quad (E^C \oplus_S F^C)^C = ((e_{lm\nu}, e_{lm\eta}, e_{lm\mu}) \oplus_S (f_{lm\nu}, f_{lm\eta}, f_{lm\mu}))^C = \left(\frac{e_{lm\nu}+f_{lm\nu}-2e_{lm\nu}f_{lm\nu}}{1-e_{lm\nu}f_{lm\nu}}, \frac{e_{lm\eta}f_{lm\eta}}{e_{lm\eta}+f_{lm\eta}-e_{lm\eta}f_{lm\eta}}, \frac{e_{lm\mu}f_{lm\mu}}{e_{lm\mu}+f_{lm\mu}-e_{lm\mu}v_{lm\mu}}\right)^C = \left(\frac{e_{lm\mu}f_{lm\mu}}{e_{lm\mu}+f_{lm\mu}-e_{lm\mu}v_{lm\mu}}, \frac{e_{lm\eta}f_{lm\eta}}{e_{lm\eta}+f_{lm\eta}-e_{lm\eta}f_{lm\eta}}, \frac{e_{lm\nu}+f_{lm\nu}-2e_{lm\nu}f_{lm\nu}}{1-e_{lm\nu}f_{lm\nu}}\right) \neq (E \odot_M F).$$

Thus, (i) holds.

(ii): It can be proved similarly. □

Theorem 4.13. *Let E and F be an two PFMs of same size, then*

(i) $(E\vee F)^C=((E^C)\wedge(F^C))$,

(ii) $(E\wedge F)^C=((E^C)\vee(F^C))$,

(iii) $((E^C)\vee(F^C))^C=(E\wedge F)$,

(iv) $((E^C)\wedge(F^C))^C=(E\vee F)$,

(v) $((E\vee F)@_K(E\wedge F))=(E@_KF)$,

(vi) $((E\wedge F)@_K(E\vee F))=(F@_KE)$.

Proof.

$$\begin{aligned}
\text{(i)}: \quad (E\vee F)^C &= (\max(e_{lm\mu},\ f_{lm\mu}),\min(e_{lm\eta},\ f_{lm\eta}),\min(e_{lmv},f_{lmv}))^C\\
&= (\min(e_{lmv},f_{lmv}),\min(e_{lm\eta},\ f_{lm\eta}),\max(e_{lm\mu},\ f_{lm\mu}))\\
&= ((e_{lmv},e_{lm\eta},e_{lm\mu})\wedge(f_{lmv},f_{lm\eta},f_{lm\mu}))\\
&= ((E^C)\wedge(F^C)),
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}: \quad (E\wedge F)^C &= (\min(e_{lm\mu},\ f_{lm\mu}),\min(e_{lm\eta},\ f_{lm\eta}),\max(e_{lmv},f_{lmv}))^C\\
&= (\max(e_{lmv},f_{lmv}),\min(e_{lm\eta},\ f_{lm\eta}),\min(e_{lm\mu},\ f_{lm\mu}))\\
&= ((e_{lmv},e_{lm\eta},e_{lm\mu})\vee(f_{lmv},f_{lm\eta},f_{lm\mu}))\\
&= ((E^C)\vee(F^C)),
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}: \quad ((E^C)\vee(F^C))^C &= (\min(e_{lm\mu},\ f_{lm\mu}),\min(e_{lm\eta},\ f_{lm\eta}),\max(e_{lmv},f_{lmv}))\\
&= (E\wedge F),
\end{aligned}$$

$$\begin{aligned}
\text{(v)}: \quad ((E\vee F)@_K(E\wedge F)) &= \left(\left(\frac{\max(e_{lm\mu},\ f_{lm\mu})+\min(e_{lm\mu},\ f_{lm\mu})}{2}\right),\right.\\
&\quad \cdot\left(\frac{\min(e_{lm\eta},\ f_{lm\eta})+\min(e_{lm\eta},\ f_{lm\eta})}{2}\right),\\
&\quad \cdot\left.\left(\frac{\min(e_{lmv},f_{lmv})+\max(e_{lmv},f_{lmv})}{2}\right)\right)\\
&= \left(\left(\frac{e_{lm\mu}+\ f_{lm\mu}}{2}\right),\left(\frac{e_{lm\eta}+f_{lm\eta}}{2}\right),\left(\frac{e_{lmv}+f_{lmv}}{2}\right)\right)=(E@_KF\ ).
\end{aligned}$$

Thus, (i), (ii), (iii), (v) holds.

(iv) and (vi): It can be proved similarly. □

5. Conclusion

We have developed Hamacher operations of *Picture Fuzzy Matrices* including scalar multiplication and power operation, which provide a good complement to the existing operations on Hamacher operations of picture fuzzy matrices. The properties of these operations are investigated.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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