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Research Article

On Fractional Calculus Operators and the Basic Analogue of Generalized Mittag-Leffler Function

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Abstract. In the present paper, we have derived some unified image formulas of the generalized q-Mittag-Leffler function under fractional calculus operators. We have derived the integral and derivative formulas of Saigo's for the generalized q-Mittag-Leffler function in terms of basic hypergeometric series ${}_2\Phi_1[a,b;c\,|\,q,z]$ and with the help of main results we have obtained the known formulas of the generalized q-Mittag-Leffler function such as Riemann-Liouville fractional integral & derivatives. The Kober and Weyl integrals of the generalized q-Mittag-Leffler function are also obtained as special cases.

Keywords. Saigo's fractional q-calculus operator, Generalized q-Mittag-Leffler function, q-gamma function, q-shifted factorial and basic hypergeometric series

Mathematics Subject Classification (2020). 33E12, 33D05, 26A33

1. Introduction

The basic analogue of generalized Mittag-Leffler function is defined as [3] for $\alpha,\beta,\gamma,\delta\in\mathbb{C}$

$$E^{\gamma,\delta,m}_{\alpha,\beta,p}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_{mn}}{(q^{\delta};q)_{pn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)},\quad (m,p>0) \tag{1.1}$$

where $\min\{Re(\alpha),Re(\beta),Re(\gamma),Re(\delta)\}>0$ and $Re(\alpha)+p\ge m$.

The series (1.1) is convergent for $|z|<(1-q)^{-\alpha}$ when $0<|q|<1$ and for $q\to 1$ it is absolutely convergent for all value of z provided $Re(\alpha)+p>m$.

Particular $p = \delta = 1$ and replacing z by $z(1-q)$ the basic analogue of generalized Mittag-Leffler function (1.1) reduce to q-Mittag-Leffler function given by Chanchlani and Garg [7]

$$E_{\alpha,\beta,1}^{\gamma,1,m}(z(1-q);q) = E_{\alpha,\beta}^{\gamma,m}(z;q) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{\Gamma_q(\alpha n+\beta)} \frac{z^n}{[n]_q!}. \tag{1.2}$$

Particular $m = p = 1$ and $\delta = 1$, the basic analogue of generalized Mittag-Leffler function (1.1) reduce to generalized small q-Mittag-Leffler function given by Purohit and Kalla [11]

$$E_{\alpha,\beta,1}^{\gamma,1,1}(z;q) = e_{\alpha,\beta}^{\gamma}(z;q) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n}{\Gamma_q(\alpha n+\beta)} \frac{z^n}{(q;q)_n}, \quad |z| < (1-q)^{-\alpha}. \tag{1.3}$$

For $q \to 1$ the basic analogue of generalized Mittag-Leffler function (1.1) reduce to Mittag-Leffler function defined by Salim and Faraj [13]

$$E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q \to 1) = E_{\alpha,\beta,p}^{\gamma,\delta,m}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{mn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_{pn}}, \quad (m,p>0). \tag{1.4}$$

2. Preliminaries and Definitions

To establish our main results, we use the following definitions:

Definition 2.1. For $\alpha \in \mathbb{C}$ and $0 < |q| < 1$, the q-shifted factorial is defined as

$$(\alpha;q)_m = \prod_{\kappa=0}^{n-1}(1-\alpha q^\kappa) = \frac{(\alpha;q)_\infty}{(\alpha q^m;q)_\infty}, \quad m \in \mathbb{N} \tag{2.1}$$

and in terms of q-gamma function

$$(\alpha;q)_m = \frac{\Gamma_q(\alpha+m)}{\Gamma_q(\alpha)}(1-q)^m \quad \text{and} \quad (\alpha;q)_0 = 1. \tag{2.2}$$

Definition 2.2. The q-binomial series is defined [8] as

$${}_1\Phi_0\left[\begin{matrix}\alpha\\ -\end{matrix}\;;\;q,x\right] = \sum_{\kappa=0}^{\infty} \frac{(\alpha;q)_\kappa}{(q;q)_\kappa} x^\kappa = \frac{(\alpha x;q)_\infty}{(x;q)_\infty}. \tag{2.3}$$

Definition 2.3. The q-analogue of exponential function is given by [8]

$$e_q^z = {}_1\Phi_0\left[\begin{matrix}0\\ -\end{matrix}\;;\;q,x\right] = \sum_{\kappa=0}^{\infty} \frac{x^\kappa}{(q;q)_\kappa}, \quad |z|<1 \tag{2.4}$$

and

$$E_q^z = {}_0\Phi_0\left[\begin{matrix}-\\ -\end{matrix}\;;\;q,-x\right] = \sum_{\kappa=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_\kappa} x^\kappa = (-x;q)_\infty. \tag{2.5}$$

Definition 2.4. The q-analogue of the power function is defined and denoted as

$$(x-y)_n = x^n \left(\frac{y}{x};q\right)_n = x^n \prod_{\kappa=0}^{\infty} \left[\frac{1-(y/x)q^\kappa}{1-(y/x)q^{\kappa+n}}\right], \quad (x \neq 0). \tag{2.6}$$

Definition 2.5. The q-beta function is defined [8] as

$$B_q(\alpha_1,\alpha_2) = \frac{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)}{\Gamma_q(\alpha_1+\alpha_2)} = \int_0^1 t^{\alpha_1-1}(1-tq)_{\alpha_2-1} d_q t, \tag{2.7}$$

where q-gamma function is defined as

$$\Gamma_q(\alpha)=\frac{(q;q)_\infty\,(1-q)^{1-\alpha}}{(q^\alpha;q)_\infty}=\frac{(q;q)_{\alpha-1}}{(1-q)^{\alpha-1}},\quad \alpha\neq 0,-1,-2,\dots. \tag{2.8}$$

Definition 2.6. Heine's q-transformation formula for Gauss summation theorem is given by Gasper and Rahman [8] is defined as

$$ {}_2\Phi_1\left[\begin{array}{c} q^u,q^v \\ q^w \end{array};\,q,z\right]=\sum_{\kappa=0}^{\infty}\frac{(q^u;q)_\kappa\,(q^v;q)_\kappa}{(q^w;q)_\kappa\,(q;q)_\kappa}z^\kappa,\quad |z|<1 \tag{2.9}$$

and

$$ {}_2\Phi_1\left[\begin{array}{c} q^u,q^v \\ q^w \end{array};\,q,q^{w-u-v}\right]=\frac{\Gamma_q(w)\,\Gamma_q(w-u-v)}{\Gamma_q(w-u)\,\Gamma_q(w-v)},\quad |q^{w-u-v}|<1. \tag{2.10}$$

Definition 2.7. For $Re(u)>0$, $v,w\in\mathbb{C}$ Saigo's fractional q-integral operator has been defined by Garg and Chanchlani [5] in the following form:

$$\text{(I)}\quad I_q^{u,v,w}f(x)=\frac{x^{-v-1}}{\Gamma_q(u)}\int_0^x\left(\frac{tq}{x};q\right)_{u-1}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa\,(q^{-w};q)_\kappa}{(q^u;q)_\kappa\,(q;q)_\kappa}q^{(w-v)\kappa}$$
$$\times(-1)^\kappa q^{-\binom{\kappa}{2}}\left(\frac{t}{x}-1\right)_\kappa f(t)d_qt, \tag{2.11}$$

$$\text{(II)}\quad K_q^{u,v,w}f(x)=\frac{q^{-u(u+1)/2-v}}{\Gamma_q(u)}\int_x^\infty\left(\frac{x}{t};q\right)_{u-1}t^{-v-1}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa\,(q^{-w};q)_\kappa}{(q^u;q)_\kappa\,(q;q)_\kappa}q^{(w-v)\kappa}$$
$$\times(-1)^\kappa q^{-\binom{\kappa}{2}}\left(\frac{x}{qt}-1\right)_\kappa f(tq^{1-u})d_qt. \tag{2.12}$$

Definition 2.8. For $l-1<Re(u)\le l$, $v,w\in\mathbb{C}$ Saigo's fractional q-derivative are defined [5] as follows

$$\text{(I)}\quad D_q^{u,v,w}f(x)=D_q^l\{I_q^{-u+l,-v-l,u+w-l}f(x)\},\quad l\in\mathrm{N}, \tag{2.13}$$

$$\text{(II)}\quad P_q^{u,v,w}f(x)=q^{u(u+v)}(-q^{-(u+v)}D_q)^l\{K_q^{-u+l,-v-l,u+w}f(x)\}. \tag{2.14}$$

3. Main Results

Theorem 3.1. *If eq.* (1.1) *satisfied then for* $Re(u)>0$ *and* $v,w\in\mathbb{C}$, *the Saigo's first kind fractional* q*-integral of basic analogue of generalized Mittag-Leffler function is given by the following formula:*

$$I_q^{u,v,w}\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\}=\frac{z^{-v}}{\Gamma_q(u)}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{B_q(u,n+1)z^n}{\Gamma_q(\alpha n+\beta)}\,{}_2\phi_1\left[\begin{array}{c} q^{u+v},q^{-w} \\ q^{u+n+1} \end{array};q,q^{w-v+n+1}\right]. \tag{3.1}$$

Proof. After using q-integral $\int_0^x f(z)d_qz=x(1-q)\sum_{\kappa=0}^{\infty}q^\kappa f(xq^\kappa)$ (Gasper and Rahman [8]), the first integral of Saigo's (2.11) reduces into the following summation form

$$I_q^{u,v,w}f(z)=\frac{(1-q)^u}{z^v}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}q^{(w-v+1)\kappa}\sum_{s=0}^{\infty}q^s\frac{(q^{u+\kappa};q)_s}{(q;q)_s}f(zq^{s+\kappa}). \tag{3.2}$$

Now by replacing $f(z)$ with $E^{\gamma,\delta,m}_{\alpha,\beta,p}(z;q)$ and using eq. (1.1), we get the following

$$\Rightarrow \quad I_q^{u,v,w}\{E^{\gamma,\delta,m}_{\alpha,\beta,p}(z;q)\} = \frac{(1-q)^u}{z^v}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}q^{(w-v+1)\kappa} \times \sum_{s=0}^{\infty}q^s\frac{(q^{u+\kappa};q)_s}{(q;q)_s}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n q^{sn+\kappa n}}{\Gamma_q(\alpha n+\beta)}. \tag{3.3}$$

Interchanging the order of second and third summation

$$= \frac{(1-q)^u}{z^v}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}q^{(w-v+1)\kappa}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n q^{\kappa n}}{\Gamma_q(\alpha n+\beta)}\sum_{s=0}^{\infty}\frac{(q^{u+\kappa};q)_s}{(q;q)_s}q^{(n+1)s}. \tag{3.4}$$

Again interchanging the order of first and second summation and using q-binomial series

$$= \frac{(1-q)^u}{z^v}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}\frac{(q^{u+n+\kappa+1};q)_\infty}{(q^{n+1};q)_\infty}q^{(w-v+n+1)\kappa} \tag{3.5}$$

$$= \frac{(1-q)^u}{z^v}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa(q^{n+1};q)_{u+\kappa}}q^{(w-v+n+1)\kappa} \tag{3.6}$$

$$= z^{-v}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n}{\Gamma_q(\alpha n+\beta)}\frac{(1-q)^u}{(q^{n+1};q)_u}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa(q^{u+n+1};q)_\kappa}q^{(w-v+n+1)\kappa}. \tag{3.7}$$

Now according the Heine's q-transformation formula for Gauss summation theorem, we get

$$= \frac{z^{-v}}{\Gamma_q(u)}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{B_q(u,n+1)z^n}{\Gamma_q(\alpha n+\beta)}\,{}_2\phi_1\left[\begin{matrix}q^{u+v},q^{-w}\\ q^{u+n+1}\end{matrix};q,q^{w-v+n+1}\right]. \tag{3.8}$$

□

Theorem 3.2. *If eq.* (1.1) *satisfied then for* $Re(u)>0$ *and* $v,w\in\mathbb{C}$*, the Saigo's second kind fractional* q*-integral of basic analogue of generalized Mittag-Leffler function is given by the following formula:*

$$K_q^{u,v,w}\{E^{\gamma,\delta,m}_{\alpha,\beta,p}(z;q)\} = \frac{z^{-v}}{q^{u(u+1)/2}\Gamma_q(u)}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{B_q(u,v-n)z^n}{\Gamma_q(\alpha n+\beta)q^{un}}\,{}_2\phi_1\left[\begin{matrix}q^{u+v},q^{-w}\\ q^{u+v-n}\end{matrix};q,q^{w-n}\right]. \tag{3.9}$$

Proof. After using the q-integral $\int_x^\infty f(z)d_qz = x(1-q)\sum_{\kappa=1}^{\infty}q^{-\kappa}f(xq^{-\kappa})$ (Gasper and Rahman [8]), the second integral of Saigo's (2.12) reduces into the following summation form

$$K_q^{u,v,w}f(z) = \frac{(1-q)^u}{z^v q^{u(u+1)/2}}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}q^{w\kappa}\sum_{s=0}^{\infty}q^{vs}\frac{(q^{u+\kappa};q)_s}{(q;q)_s}f(zq^{-u-s-\kappa}). \tag{3.10}$$

Now by replacing $f(z)$ with $E^{\gamma,\delta,m}_{\alpha,\beta,p}(z;q)$ and using equation (1.1), we get

$$\Rightarrow \quad K_q^{u,v,w}\{E^{\gamma,\delta,m}_{\alpha,\beta,p}(z;q)\} = \frac{(1-q)^u}{z^v q^{u(u+1)/2}}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}q^{w\kappa} \times \sum_{s=0}^{\infty}q^{vs}\frac{(q^{u+\kappa};q)_s}{(q;q)_s}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n q^{-(u+s+\kappa)n}}{\Gamma_q(\alpha n+\beta)}. \tag{3.11}$$

Interchanging the order of second and third summation

$$= \frac{(1-q)^u}{z^v q^{u(u+1)/2}}\sum_{\kappa=0}^{\infty}\frac{(q^{u+v};q)_\kappa(q^{-w};q)_\kappa}{(q;q)_\kappa}q^{w\kappa}\sum_{n=0}^{\infty}\frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}}\frac{z^n q^{-(u+\kappa)n}}{\Gamma_q(\alpha n+\beta)}\sum_{s=0}^{\infty}\frac{(q^{u+\kappa};q)_s}{(q;q)_s}q^{(v-n)s}. \tag{3.12}$$

Again interchanging the order of first and second summation and using q-binomial series

$$= \frac{(1-q)^u}{z^v q^{u(u+1)/2}} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{z^n q^{-un}}{\Gamma_q(\alpha n+\beta)} \sum_{\kappa=0}^{\infty} \frac{(q^{u+v};q)_\kappa (q^{-w};q)_\kappa}{(q;q)_\kappa} \frac{(q^{u+v-n+\kappa};q)_\infty}{(q^{v-n};q)_\infty} q^{(w-n)\kappa} \tag{3.13}$$

$$= \frac{(1-q)^u}{z^v q^{u(u+1)/2}} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{z^n q^{-un}}{\Gamma_q(\alpha n+\beta)} \sum_{\kappa=0}^{\infty} \frac{(q^{u+v};q)_\kappa (q^{-w};q)_\kappa}{(q;q)_\kappa (q^{v-n};q)_{u+\kappa}} q^{(w-n)\kappa} \tag{3.14}$$

$$= \frac{z^{-v}}{q^{u(u+1)/2}} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{z^n q^{-un}}{\Gamma_q(\alpha n+\beta)} \frac{(1-q)^u}{(q^{v-n};q)_u} \sum_{\kappa=0}^{\infty} \frac{(q^{u+v};q)_\kappa (q^{-w};q)_\kappa}{(q;q)_\kappa (q^{u+v-n};q)_\kappa} q^{(w-n)\kappa}. \tag{3.15}$$

Now according the Heine's q-transformation formula for Gauss summation theorem, we get

$$= \frac{z^{-v}}{q^{u(u+1)/2}\Gamma_q(u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(u,v-n) z^n}{\Gamma_q(\alpha n+\beta) q^{un}} \, {}_2\phi_1 \left[\begin{matrix} q^{u+v}, q^{-w} \\ q^{u+v-n} \end{matrix} ; q, q^{w-n} \right]. \tag{3.16}$$

□

Theorem 3.3. *If eq.* (1.1) *satisfied then for* $Re(u) \in (\lambda-1,\lambda]$, $\lambda \in \mathrm{N}$ *and* $v,w \in \mathbb{C}$, *the Saigo's first kind fractional* q*-derivative of basic analogue of generalized Mittag-Leffler function is given by the following formula:*

$$D_q^{u,v,w}\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \frac{z^v}{\Gamma_q(-u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(-u,n+1) z^n}{\Gamma_q(\alpha n+\beta)} \times {}_2\phi_1 \left[\begin{matrix} q^{-u-v}, q^{-u-w} \\ q^{-u+n+1} \end{matrix} ; q, q^{u+v+w+n+1} \right]. \tag{3.17}$$

Proof. On using the definition (eq. (1.1)) of basic analogue of generalized Mittag-Leffler function, we have

$$\Rightarrow \quad D_q^{u,v,w}\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{1}{\Gamma_q(\alpha n+\beta)} D_q^{u,v,w}\{z^n\}. \tag{3.18}$$

Now using the result, $D_q^{u,v,w}\{z^n\} = \frac{\Gamma_q(n+1)\Gamma_q(u+v+w+n+1)}{\Gamma_q(v+n+1)\Gamma_q(w+n+1)} z^{n+v}$ due to Garg and Chanchlani ([5, eq. (3.13), p. 177]), we get

$$= \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{z^{n+v}}{\Gamma_q(\alpha n+\beta)} \frac{\Gamma_q(n+1)\Gamma_q(u+v+w+n+1)}{\Gamma_q(v+n+1)\Gamma_q(w+n+1)} \tag{3.19}$$

$$= \frac{z^v}{\Gamma_q(-u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(-u,n+1) z^n}{\Gamma_q(\alpha n+\beta)} \, {}_2\phi_1 \left[\begin{matrix} q^{-u-v}, q^{-u-w} \\ q^{-u+n+1} \end{matrix} ; q, q^{u+v+w+n+1} \right]. \tag{3.20}$$

□

Theorem 3.4. *If eq.* (1.1) *satisfied then for* $Re(u) \in (\lambda-1,\lambda]$, $\lambda \in \mathrm{N}$ *and* $v,w \in \mathbb{C}$, *the Saigo's second kind fractional* q*-derivative of basic analogue of generalized Mittag-Leffler function is given by the following formula:*

$$P_q^{u,v,w}\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \frac{z^v}{q^{-u(u-1)/2}\Gamma_q(-u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(-u,-v-n) z^n q^{u(v+n)}}{\Gamma_q(\alpha n+\beta)} \times {}_2\phi_1 \left[\begin{matrix} q^{-u-v}, q^{-u-w} \\ q^{-u-v-n} \end{matrix} ; q, q^{u+w-n} \right]. \tag{3.21}$$

Proof. The result can be proved similarly Theorem 3.3, after using the result, $P_q^{u,v,w}\{z^n\} = \frac{\Gamma_q(-v-n)\Gamma_q(u+w-n)}{\Gamma_q(-n)\Gamma_q(-v+w-n)} q^{u(v+n)} q^{u(u-1)/2} z^{n+v}$ due to Garg and Chanchlani ([5, eq. (3.14), p. 177]). □

Remark. For particular $v = -u$, Theorem 3.3 and Theorem 3.4 reduces to Riemann-Liouville and Weyl fractional derivatives of the basic analogue of Mittag-Leffler function (1.1), respectively.

4. Special Cases

Corollary 4.1. *On taking $v = -u$ in Theorem 3.1 and using the result $I_q^{u,-u,w} f(z) = I_q^u f(z)$,* [5, eq. (2.7)]*, we get the Riemann-Liouville q-integral of the basic analogue of Mittag-Leffler function, which also obtained by Bhadana and Meena* ([3, eq. (3.7), p. 176])

$$I_q^u\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \frac{z^u}{\Gamma_q(u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(u,n+1)}{\Gamma_q(\alpha n+\beta)} z^n. \tag{4.1}$$

Corollary 4.2. *On taking $v = 0$ in Theorem 3.1 and using the result $I_q^{u,0,w} f(z) = I_q^{w,u} f(z)$,* [5, eq. (2.9)]*, we obtain the Kober q-integral of the basic analogue of Mittag-Leffler function:*

$$I_q^{w,u}\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \frac{1}{\Gamma_q(u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(u,w+n+1)}{\Gamma_q(\alpha n+\beta)} z^n. \tag{4.2}$$

Corollary 4.3. *On taking $v = -u$ in Theorem 3.2 and using the result $K_q^{u,-u,w} f(z) = K_q^u f(z)$,* [5, eq. (2.8)]*, then we obtain the Riemann-Liouville q-integral of the basic analogue of Mittag-Leffler function:*

$$K_q^u\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \frac{z^u\,\Gamma_q(-u)}{q^{u(u+1)/2}} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{z^n q^{-un}}{\Gamma_q(\alpha n+\beta) B_q(-u,-n)}. \tag{4.3}$$

Corollary 4.4. *On taking $v = 0$ in Theorem 3.2 and using the result $K_q^{u,0,w} f(z) = q^{-u(u+1)/2} K_q^{w,u} f(z)$,* [5, eq. (2.10)]*, then we obtain the Weyl q-integral of the basic analogue of Mittag-Leffler function:*

$$K_q^{w,u}\{E_{\alpha,\beta,p}^{\gamma,\delta,m}(z;q)\} = \frac{1}{\Gamma_q(u)} \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_{mn}}{(q^\delta;q)_{pn}} \frac{B_q(u,w-n)q^{-un}}{\Gamma_q(\alpha n+\beta)} z^n. \tag{4.4}$$

Corollary 4.5. *If we take $\alpha = \beta = 1$, $\gamma = \delta$, $m = p$ and replacing z by $\frac{z}{(1-q)}$ then Theorem 3.1 and Theorem 3.3 reduce to Saigo's fractional integral and derivative of q-exponential function and hence we obtain following known results given by Garg et al.* ([6, eqs. (3.25) and (3.26), p. 151])

$$I_q^{u,v,w}\{E_{1,1,p}^{\delta,\delta,p}(z;q)\} = \frac{z^{-v}}{(1-q)^{-v}} \sum_{n=0}^{\infty} \frac{z^n(1-q)^{-n}\Gamma_q(w-v+n+1)}{\Gamma_q(-v+n+1)\Gamma_q(u+w+n+1)}$$

$$\Rightarrow \quad I_q^{u,v,w}\{e_q^z\} = \frac{\Gamma_q(-v+w+1)}{\Gamma_q(-v+1)\Gamma_q(u+w+1)} z^{-v}\, {}_3\phi_2\left[\begin{matrix} 0, q, q^{-v+w+1} \\ q^{-v+1}, q^{u+w+1} \end{matrix}\; ; q, z\right], \tag{4.5}$$

where $Re(-v+w+1) > 0$ and $|z| < 1$.

$$D_q^{u,v,w}\{E_{1,1,p}^{\delta,\delta,p}(z;q)\} = \frac{z^v}{(1-q)^v} \sum_{n=0}^{\infty} \frac{z^n(1-q)^{-n}\Gamma_q(u+v+w+n+1)}{\Gamma_q(v+n+1)\Gamma_q(w+n+1)}$$

$$\Rightarrow \quad D_q^{u,v,w}\{e_q^z\} = \frac{\Gamma_q(u+v+w+1)}{\Gamma_q(v+1)\Gamma_q(w+1)} z^v\, {}_3\phi_2\left[\begin{matrix} 0, q, q^{u+v+w+1} \\ q^{v+1}, q^{w+1} \end{matrix}\; ; q, z\right], \tag{4.6}$$

where $Re(u+v+w+1) > 0$ and $|z| < 1$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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