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Research Article

On the Measure of Quantum Correlations

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Abstract. In this paper, we present novel qualities of the measure of noncommutative (so quantum) correlations for general quantum systems. In other words, the fundamental difference between classical and non-commutative probability will be studied. In particular, we introduce the notion of coefficient of quantum correlations $d(\omega, A)$. The main theorem says that there are quantum correlations if and only if $d(\omega, A) > 0$. Our presentation is done within C^*-algebraic description of Quantum Theory.

Keywords. Quantum correlations, C^*-algebra, Decomposition theory

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1. Introduction

Deviations from classical probability when distant quantum systems become correlated are interesting both fundamentally and operationally. In particular, this question is very important topic in discussions of foundations of *Quantum Probability Theory* as well as it has applications in the emerging technologies of quantum computing, quantum cryptography.

In the light of the above remarks it is clear that a general definition of a measure of quantum correlations is of great importance, and this is the goal of this paper. Thus, we will be concerned with a definition and a characterization of a measure of quantum correlations for general quantum systems. As we must distinguish classical and quantum features of probability theory it is crucial to have a rigorous approach. Therefore, to this end, a general C^*-algebraic approach to *Quantum Theory* will be employed. In particular, by quantum probability space we mean any pair $(\mathfrak{A},\varphi)$ with $\mathfrak{A}$ a C^*-algebra of operators on some Hilbert space $\mathcal{H}$ and φ a continuous positive normalized functional on $\mathfrak{A}$. For a comprehensive account on quantum correlations and the complete bibliography we refer the reader to our lectures[1]. But, for a discussion along noncommutative integration lines we refer the reader to [8].

The paper is organized as follows. In Section 2 we set up notation and we will provide preliminaries. Section 3 contains the main result — the definition of measure of quantum correlations (so entanglement) and the theorem saying that there are quantum correlations if and only if the measure of correlations is not equal to 0.

2. Preliminaries

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras with unit $\mathbb{1}$. The (projective) tensor product $\mathfrak{A}=\mathfrak{A}_1\otimes\mathfrak{A}_2$ will describe a composite system while $\mathfrak{A}_1\equiv\mathfrak{A}_1\otimes\mathbb{1}$ and $\mathfrak{A}_2\equiv\mathbb{1}\otimes\mathfrak{A}_2$ stand for its subsystems.

By the way, we note that in most applications in physics, at least one subalgebra is nuclear one, so there is the unique tensor product. However, to keep the full generality we will consider the projective tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

We write $\mathfrak{S}_{\mathfrak{A}}$ ($\mathfrak{S}_{\mathfrak{A}_1}$, $\mathfrak{S}_{\mathfrak{A}_2}$) for the set of all states (so all linear, normalized, positive forms) on $\mathfrak{A}$ ($\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively).

In $\mathfrak{S}$ one can distinguish the subset of classically correlated states, i.e. states which can be written as a convex combinations of product states. To get some intuition about the problem we give an example which is extracted from the book of Kadison and Ringrose (see [6, Exercise 11.5.11]).

Example 2.1. Let $\mathfrak{A}_1=B(\mathcal{H})$ and $\mathfrak{A}_2=B(\mathcal{K})$ where $\mathcal{H}$ and $\mathcal{K}$ are 2-dimensional Hilbert spaces. Consider the vector state $\omega_x(\cdot)=(x,\cdot\, x)$ with $x=\frac{1}{\sqrt{2}}(e_1\otimes f_1+e_2\otimes f_2)$ where $\{e_1,e_2\}$ and $\{f_1,f_2\}$ are orthonormal bases in $\mathcal{H}$ and $\mathcal{K}$, respectively. Let ρ be any state in the norm closure of the convex hull of product states, i.e. $\rho\in\overline{\mathrm{conv}}(\mathfrak{S}_1\otimes\mathfrak{S}_2)$. Then, one can show that

$$\|\omega_x-\rho\|\geq\frac{1}{4}. \tag{2.1}$$

Remark 2.2. The reader should note that ω_x can always be approximated by a finite linear combination of simple tensors. However, here we wish to approximate ω_x by a convex combination of positive (normalized) functionals and this makes the difference.

Guided by Example 2.1 we define:

Definition 2.3. Let $\mathfrak{A}_i$, $i=1,2$ be a C^*-algebra, $\mathfrak{S}$ the set of all states on $\mathfrak{A}\equiv\mathfrak{A}_1\otimes\mathfrak{A}_1$. The subset $\overline{\mathrm{conv}}(\mathfrak{S}_{\mathfrak{A}_1}\otimes\mathfrak{S}_{\mathfrak{A}_2})$ in $\mathfrak{S}$ will be called the set of separable states and will be denoted by $\mathfrak{S}_{\mathrm{sep}}$. The closure is taken with respect to the norm of $\mathfrak{A}^*$. The subset $\mathfrak{S}\setminus\mathfrak{S}_{\mathrm{sep}}\equiv\mathfrak{S}_{\mathrm{ent}}\subset\mathfrak{S}$ is called the subset of entangled states.

[1] W. A. Majewski, *Quantum correlations; quantum probability approach*, arXiv:1407.4754v4 [quant-ph] https://arxiv.org/pdf/1407.4754.pdf.

Remark 2.4. As a separable state has the form of an arbitrary classical state, it is naturally to adopt the convention that $\mathfrak{S}_{\text{sep}}$ contains only classical correlations. We emphasize that a classically correlated state may well contain nontrivial correlations. On the other hand, the set of entangled states is the set where the quantum (so extra) correlations can occur. In other words, quantum mechanics allows correlations between values of measurements performed at spatially separated locations that can never occur according to laws of classical physics.

The crucial ingredient of a description of $\mathfrak{S}_{\text{sep}}$ and $\mathfrak{S}_{\text{ent}}$ is the geometrical characterization of the set of all states $\mathfrak{S}$. We remind the reader that in geometrical description of a convex compact set one can distinguish two types of convex closed sets: simplexes and non-simplexes. Let K be a convex compact set. A point $x \in K$, if K is simplex, admits the unique decomposition in the form of convex combination of extreme points of K. This is not the case for K being not simplex. The following proposition justifies our interest in non-simplexes, cf. [2, Example 4.2.6].

Proposition 2.5. *Let $\mathfrak{A}$ be a C^*-algebra. Then the following conditions are equivalent*

(i) *The state space $\mathfrak{S}_{\mathfrak{A}}$ is a simplex.*

(ii) *$\mathfrak{A}$ is abelian algebra.*

Therefore in quantum case, where $\mathfrak{A}$ is a noncommutative C^*-algebra with unit and $\mathfrak{S}$ is convex *-weakly compact, the set of states $\mathfrak{S}$ is not a simplex (contrary to the classical case). Consequently, in quantum case, all possible decompositions of a given state should be taken into account.

To this end, we need to apply the decomposition theory. Now, for the convenience of the reader, we repeat the relevant material from [2–5, 9, 10] without proofs thus making our exposition self-contained. As usually, let $\mathfrak{A}$ stand for a C^*-algebra with unit, and $\mathfrak{S}$ its state space. $M_1(\mathfrak{S})$ denotes the set of all probability Radon measures on $\mathfrak{S}$. We note that $M_1(\mathfrak{S})$ is a compact subset of the vector space of real, regular Borel measures on $\mathfrak{S}$. As the next step we recall the concept of the barycenter $b(\mu)$ of a measure $\mu \in M_1(\mathfrak{S})$. It is defined by

$$b(\mu) = \int d\mu(\varphi)\varphi, \tag{2.2}$$

where the integral is understood in the weak sense. We note that $b(\mu) \in \mathfrak{S}$. The set $M_\omega(\mathfrak{S})$ is defined as a subset of $M_1(\mathfrak{S})$ with the fixed barycenter ω, i.e.

$$M_\omega(\mathfrak{S}) = \{\mu \in M_1(\mathfrak{S}), b(\mu) = \omega\}. \tag{2.3}$$

$M_\omega(\mathfrak{S})$ is a convex closed subset of $M_1(\mathfrak{S})$, hence compact in the weak *-topology. Thus, it follows by the Krein-Milman theorem that there are "many" extreme points in $M_\omega(\mathfrak{S})$.

Finally, we briefly sketch an approximation property for a positive measure, see [1], [3, Vol. I], and [9].

For a Borel measure μ on a locally compact Hausdorff space E and a function $f \in C_{\mathfrak{K}}(E)$ we denote $\mu(f) = \int_E f\, d\mu$, where $C_{\mathfrak{K}}(E)$ stands for the set of continuous functions with compact support. Denote by $\mathfrak{M}(E)$ the collection of Radon measures on E. Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathfrak{M}(E)$. We say that the net $\{\mu_n\}$ is weakly convergent to μ if $\mu_n(f) \to \mu(f)$ for any function $f \in C_{\mathfrak{K}}(E)$. This topology of simple convergence is called the vague topology (and sometimes also called the weak*-topology).

Dirac's (point) measure δ_a, where $a \in E$ is determined by the condition:

$$\delta_a(f) = f(a). \tag{2.4}$$

We say that a measure μ has a finite support if it can be written as a linear (finite) combination of δ_a's. Now, we are in position to give (see [1, Chapter 3, Section 2, Corollary 3]):

Theorem 2.6. *Any positive finite measure μ on E is a limit point, in the vague topology, of a convex hull of positive measures having a finite support contained in the support of μ.*

Remark 2.7. (i) This result will be not valid in the non-commutative setting. It is taken from the (classical) measure theory.

(ii) A slightly stronger formulation can be find in [9]. Namely, every probability measure λ in $\mathfrak{M}(E)$ is a weak limit of discrete (with finite support) measures belonging to the collection of probability measures in $\mathfrak{M}(E)$ which have the same barycenter as λ.

3. Measure of Quantum Correlations

Now we are in a position to proceed with the study of coefficient of (quantum) correlations for a quantum composite system specified by $(\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_{\mathfrak{A}})$, where $\mathfrak{A}_i$ are C^*-algebras with unit. We begin with the definition of the restriction maps

$$(r_1\omega)(A) = \omega(A \otimes \mathbb{1}), \tag{3.1}$$

$$(r_2\omega)(B) = \omega(\mathbb{1} \otimes B), \tag{3.2}$$

where $\omega \in \mathfrak{S}_{\mathfrak{A}}$, $A \in \mathfrak{A}_1$, and $B \in \mathfrak{A}_2$. Clearly, $r_i : \mathfrak{S}_{\mathfrak{A}} \to \mathfrak{S}_{\mathfrak{A}_i}$ and the restriction map r_i is continuous (in weak-$*$ topology), $i = 1,2$ (see [2, Proposition 4.1.37]). We recall that the decomposition procedure is starting with a "good" measure on the state space $\mathfrak{S}$ (so from $M_\omega(\mathfrak{S})$). Hence, let us take a "good" measure μ on $\mathfrak{S}_{\mathfrak{A}}$. Define

$$\mu_i(F_i) = \mu(r_i^{-1}(F_i)) \tag{3.3}$$

for $i = 1,2$, where F_i is a Borel subset in $\mathfrak{S}_{\mathfrak{A}_i}$. It is easy to check that the formula (3.3) provides the well defined measures μ_i on $\mathfrak{S}_{\mathfrak{A}_i}$, $i = 1,2$. Having two measures μ_1, μ_2 on $\mathfrak{S}_{\mathfrak{A}_1}$, and $\mathfrak{S}_{\mathfrak{A}_2}$ respectively, we want to "produce" a new measure $\boxtimes\mu$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. To this end, firstly, let us consider the case of finitely supported probability measure μ:

$$\mu = \sum_{i=1}^{N} \lambda_i \delta_{\rho_i}, \tag{3.4}$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$, and δ_{ρ_i} denotes the Dirac's measure. We define

$$\mu_1 = \sum_{i=1}^{N} \lambda_i \delta_{r_1\rho_i} \tag{3.5}$$

and

$$\mu_2 = \sum_{i=1}^{N} \lambda_i \delta_{r_2 \rho_i}. \tag{3.6}$$

Then

$$\boxtimes\mu = \sum_{i=1}^{N} \lambda_i \delta_{r_1 \rho_i} \times \delta_{r_2 \rho_i} \tag{3.7}$$

provides a well defined measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. Here $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$ is understood as a measure space obtained as a product of two measure spaces $\mathfrak{S}_{\mathfrak{A}_1}$ and $\mathfrak{S}_{\mathfrak{A}_2}$. A measure structure on $\mathfrak{S}_{\mathfrak{A}_i}$ is defined as the Borel structure determined by the corresponding weak-$*$ topology on $\mathfrak{S}_{\mathfrak{A}_i}$, $i = 1,2$.

We note that an arbitrary fixed decomposition of a state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ corresponds to a measure μ such that $\omega = \int_{\mathfrak{S}} \nu d\mu(\nu)$. But, in general, there are many decompositions. Consequently, we will be interested in measures from the following set

$$M_\omega(\mathfrak{S}_{\mathfrak{A}}) \equiv M_\omega = \{\mu : \omega = \int_{\mathfrak{S}} \nu d\mu(\nu)\},$$

i.e. the set of all Radon probability measures on $\mathfrak{S}_{\mathfrak{A}}$ with the fixed barycenter ω.

Take an arbitrary measure μ from M_ω. By Theorem 2.6 (cf. also Remark 2.7) there exists a net of discrete measures (having a finite support) μ_k such that $\mu_k \to \mu$, and the convergence is understood in the weak-*topology on $\mathfrak{S}_{\mathfrak{A}}$.

Defining μ_1^k (μ_2^k) analogously as μ_1 (μ_2 respectively; cf. equations (3.5), (3.6)), one has $\mu_1^k \to \mu_1$ and $\mu_2^k \to \mu_2$, where again the convergence is taken in the weak-* topology on $\mathfrak{S}_{\mathfrak{A}_1}$ ($\mathfrak{S}_{\mathfrak{A}_2}$ respectively). To see this, note that $\mu_k \to \mu$ means that for any continuous function $f \in C(\mathfrak{S}_{\mathfrak{A}})$,

$$\mu_k(f) \to \mu(f). \tag{3.8}$$

But note, that $g \circ r_i$ is in $C(\mathfrak{S}_{\mathfrak{A}})$ for any $g \in C(\mathfrak{S}_{\mathfrak{A}_i})$. Thus, plugging $f = g \circ r_i$ in (3.8) one gets the convergence of μ_i^k.

Then define, for each k, $\boxtimes\mu^k$ as it was done in (3.7). We can verify that $\{\boxtimes\mu^k\}$ is convergent (in weak $*$-topology) to a measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. To see this, take a continuous function g on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. Observe that this two variable function gives rise to the following function $g(r_1 \cdot, r_2 \cdot) = \tilde{g}(\cdot)$, and obviously $\tilde{g}$ is a continuous function on $\mathfrak{S}_{\mathfrak{A}}$. Therefore,

$$\begin{aligned}
\boxtimes\mu_k(g) &= \left(\sum_{i=1}^{N_k} \lambda_{i,k} \delta_{r_1 \rho_{i,k}} \times \delta_{r_2 \rho_{i,k}} \right)(g) \\
&= \sum_{i=1}^{N_k} \lambda_{i,k} g(r_1 \rho_{i,k}, r_2 \rho_{i,k}) \\
&= \sum_{i=1}^{N_k} \lambda_{i,k} \tilde{g}(\rho_{i,k}) = \left(\sum_{i=1}^{N_k} \lambda_{i,k} \delta_{\rho_{i,k}} \right)(\tilde{g}),
\end{aligned}$$

and the last term is convergent, by definition, to $\mu(\tilde{g})$.

Consequently, taking the weak-* limit we arrive at the measure $\boxtimes\mu$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. It follows easily that $\boxtimes\mu$ does not depend on the chosen approximation procedure.

To grasp the idea which is behind the construction let us consider a very simple example:

Example 3.1. Let us fix a state ω and take a discrete, finite supported, measure μ_0 in $M_\omega(\mathfrak{S}_\mathfrak{A})$; i.e. μ_0 is of the form

$$\mu_0 = \sum_{i=1}^{N} \lambda_i \delta_{\rho_i},$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Note, that $\rho_i \in \operatorname{supp}\mu_0$ for any i. Define, as before, $\boxtimes\mu_0 = \sum_i \lambda_i \delta_{r_1\rho_i} \times \delta_{r_2\rho_i}$ and note that the measure $\boxtimes\mu_0$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$ defines the new state $\tilde{\omega}$ in the following way:

$$\tilde{\omega}(A_1 \otimes A_2) = \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \varphi(A_1 \otimes A_2)(d \boxtimes \mu_0)(\varphi), \tag{3.9}$$

where $\varphi \in \mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$, i.e. $\varphi = (\varphi_1, \varphi_2)$. We have defined $\varphi(A_1 \otimes A_2)$ as $\varphi_1(A_1)\varphi_2(A_2)$ which can be considered as $(\varphi_1 \otimes \varphi_2)(A_1 \otimes A_2) \equiv \varphi(A_1 \otimes A_2)$. Thus

$$\begin{aligned}
\tilde{\omega}(A_1 \otimes A_2) &= \sum_{i=1}^{N} \lambda_i \cdot (r_1\rho_i, r_2\rho_i)(A_1 \otimes A_2) \\
&= \sum_{i=1}^{N} \lambda_i \cdot (r_1\rho_i)(A_1)(r_2\rho_i)(A_2) \\
&= \sum_{i=1}^{N} \lambda_i \cdot (r_1\rho_i) \otimes (r_2\rho_i)(A_1 \otimes A_2).
\end{aligned} \tag{3.10}$$

Hence $\tilde{\omega}$ is a separable state which originates from the given state ω.

Now, again guided by Example 2.1, we are in position to give the definition of the coefficient of quantum correlations, $d(\omega, A_1, A_2) \equiv d(\omega, A)$, where $A_i \in \mathfrak{A}_i$ (see [7]):

Definition 3.2. Let a quantum composite system $(\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_\mathfrak{A})$ be given. Take a $\omega \in \mathfrak{S}_\mathfrak{A}$. We define the coefficient of quantum correlations as

$$d(\omega, A) = \inf_{\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})} \left| \int_{\mathfrak{S}_\mathfrak{A}} \xi(A) d\mu(\xi) - \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A)(d \boxtimes \mu)(\xi) \right| \tag{3.11}$$

The formula (3.11) can be considered as a "measure" of extra non classical type of correlations. Namely, following the strategy of Kadison-Ringrose example, see Example 2.1, an evaluation of a distance between the given state ω and the set of approximative separable states is done.

It is a simple matter to see that $d(\omega, A)$ is equal to 0 if the state ω is a separable one. To show this let ω be a separable state, i.e.

$$\omega = \lim_{N \to \infty} \sum_{i=1}^{N} \lambda_i^{(N)} \omega_i^{(N)} \tag{3.12}$$

where each $\omega_i^{(N)}$ is a product state such that $\omega_i^{(N)}(A \otimes B) = \omega_{i,1}^{(N)}(A)\omega_{i,2}^{(N)}(B)$, where $\omega_{i,k}^{(N)}(\cdot)$ is a state on $\mathfrak{A}_k$. Define the sequence of measures $\mu^{(N)}$ in the following way:

$$\mu^{(N)} = \sum_{i=1}^{N} \lambda_i^{(N)} \delta_{\omega_i^{(N)}} = \sum_{i=1}^{N} \lambda_i^{(N)} \delta_{\omega_{i,1}^{(N)}} \times \delta_{\omega_{i,2}^{(N)}} \tag{3.13}$$

where $\delta_{\omega_i^{(N)}}$ denotes Dirac's measure. If necessary, passing to a subsequence, we may suppose also that $\mu^{(N)}$ converges to $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$ (it is always possible as $\{\mu^{(N)}\} \subset M_\omega(\mathfrak{S}_\mathfrak{A})$, which a compact set). Taking a weak limit of $\{\mu^{(N)}\}$ and keeping in mind arguments given prior to Example 3.1 one gets a measure μ such that

$$\int \varphi d\mu(\varphi) = \omega, \tag{3.14}$$

and $d(\omega, A) = 0$. The converse statement is much less obvious. However, we are able to prove it.

Theorem 3.3. *Let $\mathfrak{A}$ be the tensor product of two C^*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$. Then state $\omega \in \mathfrak{S}_\mathfrak{A}$ is separable if and only if $d(\omega, A) = 0$ for all $A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$*

Proof. Clearly, we need to prove the "only if" part. The basic idea of the proof of the statement that $d(\omega, A) = 0$ implies separability of ω relies on the study of continuity properties of the function

$$M_\omega(\mathfrak{S}_\mathfrak{A}) \ni \mu \mapsto \int_{\mathfrak{S}_\mathfrak{A}} \xi(A) d\mu(\xi) - \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A)(d \boxtimes \mu)(\xi) \tag{3.15}$$

and the proof falls naturally into few steps.

(i) $M_\omega(\mathfrak{S}_\mathfrak{A})$ is a compact set. It was already stated in the previous section. However, for the convenience of the reader we give here a proof of this statement.

We begin by recalling that $\mathfrak{S}_\mathfrak{A}$ is a compact set ($\mathfrak{A}$ has the unit $\mathbb{1}$) and the set of positive Radon measures $M^+(\mathfrak{S}_\mathfrak{A})$ is also compact (all in weak-$*$ topologies!). Take $\{\mu_\alpha\} \subset M_\omega(\mathfrak{S}_\mathfrak{A})$ such that $\mu_\alpha \to \mu$ (weakly). But this implies

$$\int \hat{A}(\varphi) d\mu_\alpha(\varphi) = \omega(A) \equiv \hat{A}(\omega) \to \int \hat{A}(\varphi) d\mu(\varphi).$$

Thus $\int \hat{A}(\varphi) d\mu(\varphi) = \omega(A)$. Hence $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$. So $M_\omega(\mathfrak{S}_\mathfrak{A})$ being a closed subset of a compact set $M^+(\mathfrak{S}_\mathfrak{A})$ is a compact set.

(ii) The mapping $M_\omega(\mathfrak{S}_\mathfrak{A}) \ni \mu \mapsto \boxtimes\mu \in M^+(\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2})$ is weakly continuous.

To prove this, we should show that for any $\mu_0 \in M_\omega(\mathfrak{S}_\mathfrak{A})$ and any neighborhood $V(\boxtimes\mu_0; g_1, \dots, g_k)$ of $\boxtimes\mu_0$ there exists a neighborhood $U(\mu_0; f_1, \dots, f_k)$ of μ_0 such that $\boxtimes\big(U(\mu_0; f_1, \dots, f_k)\big) \subseteq V(\boxtimes\mu_0; g_1, \dots, g_k)$ where $V \equiv V(\boxtimes\mu_0; g_1, \dots, g_k) = \{\boxtimes\mu : |\boxtimes\mu_0(g_i) - \boxtimes\mu(g_i)| < \epsilon, i = 1, \dots, k\}$, $g_i \in C(\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2})$ while $U \equiv U(\mu_0; f_1, \dots, f_k) = \{\mu : |\mu_0(f_i) - \mu(f_i)| < \epsilon_1, i = 1, \dots, k\}$, $f_i \in C(\mathfrak{S}_\mathfrak{A})$.

The first step of the proof is to take μ_0 and μ in $M_\omega(\mathfrak{S}_\mathfrak{A})$ such that

$$|\mu_0(f) - \mu(f)| < \epsilon \quad \text{for } f \in C(\mathfrak{S}_\mathfrak{A}). \tag{3.16}$$

So, for simplicity, we put $k = 1$ in the definition of neighborhoods U and V. Let f be of the form

$$f(\rho) = g(r_1(\rho), r_2(\rho)) \quad \rho \in \mathfrak{S}_\mathfrak{A}, \tag{3.17}$$

where $g(\cdot,\cdot)$ is a continuous (two variables) function on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. We note that f is satisfying (3.16).

Let μ_0^n (μ^n) be a weak-* Riemann approximation for μ_0 (μ respectively). Then

$$|\mu_0^n(f)-\mu^n(f)| \le |\mu_0^n(f)-\mu_0(f)| + |\mu_0(f)-\mu(f)| + |\mu(f)-\mu^n(f)| < \epsilon,$$

for all f of the form (3.17).

As a next step, let us consider a sequence $\boxtimes\mu_0^n$ ($\boxtimes\mu^n$) defining $\boxtimes\mu_0$ ($\boxtimes\mu$ respectively). Note, that for any f of the form (3.17), one has

$$\begin{aligned}|\boxtimes\mu_0^n(f)-\mu_0^n(f)| &= \left|\sum_{i=1}^{N_n}\lambda_{i,n}\delta_{r_1\rho_{i,n}}\times\delta_{r_2\rho_{i,n}}(f) - \sum_{i=1}^{N_n}\lambda_{i,n}\delta_{\rho_{i,n}}(f)\right| \\ &= \left|\sum_{i=1}^{N_n}\lambda_{i,n}g(r_1(\rho_{i,n}),r_2(\rho_{i,n})) - \sum_{i=1}^{N_n}\lambda_{i,n}g(r_1(\rho_{i,n}),r_2(\rho_{i,n}))\right| \\ &= 0,\end{aligned}$$

where $N_n < \infty$, and analogously for the second sequence. Therefore for any f of the form (3.17) one has

$$\begin{aligned}|\boxtimes\mu_0(g)-\boxtimes\mu(g)| &\le |\boxtimes\mu_0(g)-\boxtimes\mu_0^n(g)| + |\boxtimes\mu_0^n(g)-\mu_0^n(f)| + |\mu_0^n(f)-\mu_0(f)| \\ &\quad + |\mu_0(f)-\mu(f)| + |\mu(f)-\mu^n(f)| + |\mu^n(f)-\boxtimes\mu^n(g)| \\ &\quad + |\boxtimes\mu^n(g)-\boxtimes\mu(g)| \\ &< 5\epsilon,\end{aligned}$$

for large enough n. Thus we have shown that for any $g \in C(\mathfrak{S}_{\mathfrak{A}_1}\times\mathfrak{S}_{\mathfrak{A}_2})$

$$|\boxtimes\mu_0(g)-\boxtimes\mu(g)| < 5\epsilon, \tag{3.18}$$

provided that $|\mu_0(f)-\mu(f)| < \epsilon$. Therefore, if $V = \{\boxtimes\mu; |\boxtimes\mu_0(g_i)-\boxtimes\mu(g_i)| < \epsilon, i=1,\dots,k\}$ with $g_i \in C(\mathfrak{S}_{\mathfrak{A}_1}\times\mathfrak{S}_{\mathfrak{A}_2})$ then there exists $U = \{\mu; |\mu_0(f_i)-\mu(f_i)| < \frac{1}{5}\epsilon, i=1,\dots,k\}$ with f_i of the form (3.17) such that $\boxtimes(U)\subseteq V$. But this means the continuity of the considered mapping.

(iii) The continuity proved in the second step implies that the function (3.15) is a real valued, continuous function defined on a compact space. Hence, by Weierstrass theorem, infimum is attainable. Therefore, the condition $d(\omega,A)=0$ means that

$$\omega(A) = \int_{\mathfrak{S}_{\mathfrak{A}}}\xi(A)d\mu_0(\xi) = \int_{\mathfrak{S}_{\mathfrak{A}_1}\times\mathfrak{S}_{\mathfrak{A}_2}}\xi(A)d\boxtimes\mu_0(\xi), \tag{3.19}$$

for all $A = A_1\otimes A_2$. But, this means the separability of ω.

□

Finally, we wish to end the paper with a brief remark. The presented study of geometrical properties of $\mathfrak{S}_{\mathfrak{A}}$ through decomposition theory (so the important part of the measure theory) allows a definition of a nice measure of quantum correlations. On the other hand, we note that in [7] properties of the coefficient $d(\omega,A)$ were studied within the theory of positive linear maps. Thus, in that analysis, a very different approach was used.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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