Communications in Mathematics and Applications
Vol. 14, No. 1, pp. 1–7, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i1.1828

http://www.rgnpublications.com

Research Article

Some Fixed Point Theorems for (α,β,z)-Contraction Mapping under Simulation Functions in Banach Space

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Received: September 18, 2022 **Accepted:** October 20, 2022

Abstract. In this paper, we prove some fixed point results in the setting of a Banach space via a cyclic (α,β,z)-admissible mapping imbedded in simulation function. Our results extend and generalize some well known results in the existing literature.

Keywords. Banach space, Fixed point, (α,β,z)-admissible mapping, Simulation functions

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

Banach contraction principle is widely and extensively used in fixed point theory. Many researches studied and generalized the Banach contraction principle in different direction and proved the existence of fixed and common fixed point theorems. Fixed point theorems have a great importance in the application of mathematical analysis, especially in differential and integral equations.

Recently, the concept of cyclic (α,β)-admissible mapping was introduced by Alizadeh *et al.* [1] and obtained a generalization of Banach contraction principle. Khojasteh *et al.* [7] introduced simulation function and the notion of z-contraction with respect to simulation function to generalize Banach contraction principle. The concept of Khojasteh *et al.* [7] is further modified by Argoubi *et al.* [4]. Karapınar [6] introduced the notion of α-admissible z-contraction and obtained corresponding fixed point results in metric spaces. For more results in different type contractions and z-contraction we refer to the papers in [2,3,5,8,9] and references therein.

In this paper, we use the concept of cyclic (α,β,z)-admissible mapping and prove some fixed point results in the setting of a Banach space. Now we will give some basic definition and results in Banach spaces before presenting our main results.

2. Preliminaries

Definition 2.1. A norm on a linear space S is a mapping $\|\cdot\|: S \to R^+$ which satisfies for each $x,y \in S$; $\lambda \in R$ such that

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda|\ \|x\|$,

(iii) $\|x+y\| \le \|x\| + \|y\|$.

A linear space with a norm is called a normed linear space.

Definition 2.2. A normed space $(X,\|\cdot\|)$ is said to be complete if every Cauchy sequence in X is convergent. A complete normed space is called a Banach space.

Definition 2.3 ([1]). Let X be a nonempty set, f be a self-mapping on X and $\alpha,\beta: X \to [0,+\infty)$ be two mappings. We say that f is a cyclic (α,β)-admissible mapping if $x \in X$ with $\alpha(x) \ge 1 \Rightarrow \beta(fx) \ge 1$ and $x \in X$ with $\beta(x) \ge 1 \Rightarrow \alpha(fx) \ge 1$.

Definition 2.4 ([7]). Let z be the family of all mappings $\zeta: [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t\ \forall\ s,t > 0$;

(ζ_3) for any sequence $\{t_n\},\{s_n\} \subset (0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0 \Rightarrow \lim_{n\to\infty} \sup \zeta(t_n,s_n) < 0.$$

We say that $\zeta_1 \in z$ is a simulation function.

Definition 2.5 ([4]). A simulation function is a function $\zeta: [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies the following conditions:

(i) $\zeta(t,s) < s - t$, for all $t,s > 0$;

(ii) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (l,\infty) > 0,$$

then

$$\lim_{n\to\infty} \sup \zeta(t_n,s_n) < 0.$$

It is clear that any simulation function is the same according to the Definitions 2.4 and 2.5.

We denote by Ψ the family of all nondecreasing functions $\psi: [0,\infty) \to [0,\infty)$ such that

(ψ_1) ψ is continuous;

(ψ_2) $\psi^{-1}(\{0\}) = 0$.

3. Main Result

Lemma 3.1 ([8]). *Let $T: X \to X$ be cyclic (α,β)-admissible mapping. Assume that there exist $x_0, x_1 \in X$ such that $\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) \geq 1$ and $\beta(x_0) \geq 1 \Rightarrow \alpha(x_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\alpha(x_n) \geq 1 \Rightarrow \beta(x_m) \geq 1$ and $\beta(x_n) \geq 1 \Rightarrow \alpha(x_m) \geq 1$ for all $m,n \in N$ with $n < m$.*

Definition 3.2. Let $(X, \|\_\|)$ be a Banach space, $T: X \to X$ be a mapping and $\alpha,\beta: R \to [0,\infty)$ be two functions. Then T is said to be a (α,β,z)-contraction mapping if T satisfies the following conditions:

(i) T is a cyclic (α,β)-admissible;

(ii) there exists a simulation function $\zeta \in z$ such that $\alpha(x)\beta(y) \geq 1$,

$$\zeta(\psi(\|Tx - Ty\|), \psi(m(x,y))) \geq 0 \text{ holds for all } x,y \in X,$$

where $m(x,y) = \max\left\{\|x-y\|, \frac{[1+\|x-Tx\|]\|y-Ty\|}{1+\|x-y\|}\right\}$.

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a Banach space, $T: X \to X$ be a mapping and $\alpha,\beta: X \to [0,\infty)$ be two functions. Suppose that the below conditions are followed:*

(i) *T is (α,β,z)-contraction mapping;*

(ii) *there exists element $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*

(iii) *T is continuous, then T has a fixed point $u \in X$ such that $Tu = u$.*

Proof. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$. The proof divided into the following steps:

Step 1. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in N \cup \{0\}$. If $x_n = x_{n+1}$ for all $n \in N \cup \{0\}$, then T has a fixed point and the proof is finished.
Hence, we assume that $x_n \neq x_{n+1}$for all $n \in N \cup \{0\}$. That is $\|x_n - x_{n+1}\| \neq 0$ for all $n \in N \cup \{0\}$. Since T is a cyclic (α,β)-admissible mapping, we have $\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) = \beta(Tx_0) \geq 1 \Rightarrow \alpha(x_2) = \alpha(Tx_1) \geq 1$ and $\beta(x_0) \geq 1 \Rightarrow \alpha(x_1) = \alpha(Tx_0) \geq 1 \Rightarrow \beta(x_2) = \beta(Tx_1) \geq 1$, then the above process is continued, we have $\alpha(x_n) \geq 1 \Rightarrow \beta(x_n) \geq 1$ for all $n \in N \cup \{0\}$. Thus $\alpha(x_n)\beta(x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$.
Therefore, we get $\zeta(\psi(\|Tx_n - Tx_{n+1}\|), \psi(m(x_n,x_{n+1}))) \geq 0$ for all $n \in N \cup \{0\}$, where

$$\begin{aligned} m(x,y) &= \max\left\{\|x_n - x_{n+1}\|, \frac{[1+\|x_n - Tx_n\|]\|x_{n+1} - Tx_{n+1}\|}{1+\|x_n - x_{n+1}\|}\right\} \\ &= \max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\}. \end{aligned}$$

It follows that

$$\zeta(\psi(\|x_{n+1} - x_{n+2}\|), \psi(\max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\})) \geq 0.$$

By condition (ζ_2), we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\|x_{n+1} - x_{n+2}\|), \psi(\max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\})) \\ &< \psi(\max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\}) - \psi(\|x_{n+1} - x_{n+2}\|). \end{aligned} \tag{3.1}$$

Consequently, we obtain that $\forall\ n = 1,2,3,\dots$

$$\psi(\|x_{n+1} - x_{n+2}\|) < \psi(\max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\}).$$

If $\max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\} = (\|x_{n+1} - x_{n+2}\|)$ for some n, then

$$\psi(\|x_{n+1} - x_{n+2}\|) < \psi(\|x_{n+1} - x_{n+2}\|)$$

which is a contradiction.
Hence $\max\{\|x_n - x_{n+1}\|, \|x_{n+1} - x_{n+2}\|\} = (\|x_n - x_{n+1}\|)$, for all $n = 1,2,3,\dots$ and hence from (3.1)

$$\begin{aligned} 0 &\le \zeta(\psi(\|x_{n+1} - x_{n+2}\|), \psi(\|x_n - x_{n+1}\|)) \\ &< \psi(\|x_n - x_{n+1}\|) - \psi(\|x_{n+1} - x_{n+2}\|) \end{aligned} \tag{3.2}$$

which implies

$$\psi(\|x_{n+1} - x_{n+2}\|) < \psi(\|x_n - x_{n+1}\|) \quad \text{for all } n = 1,2,3,\dots.$$

It follows that the sequence $\{\psi(\|x_n - x_{n+1}\|)\}$ is non-increasing sequence. Therefore, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \psi(\|x_n - x_{n+1}\|) = r.$$

We now show that

$$\lim_{n\to\infty} \psi(\|x_n - x_{n+1}\|) = 0.$$

Note that if $r \ne 0$ that is $r > 0$ then by condition (ζ_2). We have

$$0 \le \lim_{n\to\infty} \sup \zeta(\psi(\|x_n - x_{n+1}\|), \psi(\|x_{n+1} - x_{n+2}\|)) < 0$$

which is contradiction. Hence we have $r = 0$, since $\psi \in \Psi$. We have

$$\lim_{n\to\infty} (\|x_n - x_{n+1}\|) = 0. \tag{3.3}$$

Step 2. To prove $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that is $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of x_n with $m(k) > n(k) > k$ and $m(k)$ is the smallest index in N such that $\|x_{m(k)} - x_{n(k)}\| \ge \epsilon$. So

$$\|x_{n(k)} - x_{m(k)-1}\| < \epsilon.$$

Then we have

$$\begin{aligned} \epsilon &\le \|x_{m(k)} - x_{n(k)}\| \\ &\le \|x_{n(k)} - x_{m(k)-1}\| + \|x_{m(k)-1} - x_{m(k)}\| \\ &< \epsilon + \|x_{m(k)-1} - x_{m(k)}\|. \end{aligned}$$

Taking $k \to \infty$in the above inequality and using (3.3), we get

$$\lim_{k\to\infty} (\|x_{n(k)} - x_{m(k)}\|) = \epsilon. \tag{3.4}$$

Again, by triangular inequality

$$\begin{aligned} \|x_{n(k)-1} - x_{m(k)-1}\| &\le \|x_{n(k)-1} - x_{n(k)}\| + \|x_{n(k)} - x_{m(k)}\| + \|x_{m(k)} - x_{m(k)-1}\| \\ &\le 2\|x_{n(k)} - x_{n(k)-1}\| + \|x_{m(k)-1} - x_{n(k)-1}\| + 2\|x_{m(k)-1} - x_{m(k)}\|. \end{aligned}$$

Taking $k \to \infty$in the above inequality and using (3.3)and (3.4), we get

$$\lim_{k\to\infty} (\|x_{n(k)} - x_{m(k)}\|) = \lim_{k\to\infty} (\|x_{n(k)-1} - x_{m(k)-1}\|) = \epsilon. \tag{3.5}$$

Since $\alpha(x_0) > 1$ and $\beta(x_0) > 1$ and by Lemma 3.1, we conclude that $\alpha(x_{n(k)-1})\beta(x_{m(k)-1}) \ge 1$, for all $k = 1,2,3,\dots$.

Since T is a (α,β,z)-contraction, we have

$$\zeta(\psi(\|Tx_{n(k)-1}-Tx_{m(k)-1}\|),\psi(m(x_{n(k)-1},x_{m(k)-1})))\geq 0 \quad \text{for all } x,y\in X$$

where

$$\begin{aligned} m(x_{n(k)-1},x_{m(k)-1}) &= \max\left\{\|x_{n(k)-1}-x_{m(k)-1}\|,\frac{[1+\|x_{n(k)-1}-Tx_{n(k)-1}\|]\,\|x_{m(k)-1}-Tx_{m(k)-1}\|}{1+\|x_{n(k)-1}-x_{m(k)-1}\|}\right\} \\ &= \max\left\{\|x_{n(k)-1}-x_{m(k)-1}\|,\frac{[1+\|x_{n(k)-1}-x_{n(k)}\|]\,\|x_{m(k)-1}-x_{m(k)}\|}{1+\|x_{n(k)-1}-x_{m(k)-1}\|}\right\}. \end{aligned}$$

Let $S_k:=\psi(m(x_{n(k)-1},x_{m(k)-1}))$ and $t_k:=\psi(\|x_{n(k)}-x_{m(k)}\|)$.
Then it follows from (3.3), (3.4) and (3.5) that

$$\lim_{k\to\infty} S_k=\lim_{k\to\infty} t_k=\psi(\epsilon). \tag{3.6}$$

Since $\psi(\epsilon)>0$, it follows from condition (ζ_3) that

$$0\leq \lim_{n\to\infty}\sup\zeta(\psi(\|Tx_{n(k)-1}-Tx_{m(k)-1}\|),\psi(m(x_{n(k)-1},x_{m(k)-1})))<0$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

Step 3. Finally, to prove T has a fixed point. Since $\{x_n\}$ is a Cauchy sequence in the Banach space X, then there exists $u\in X$ such that $x_n\to u$. The continuity of T implies that $Tx_{2n}\to Tu$. Since $x_{2n+1}=Tx_{2n}$ and $x_{2n+1}\to u$ by uniqueness of limit, we get $Tu=u$. So u is a fixed point of T. □

Note that the continuity of the mapping T is Theorem 3.3 can be dropped if the condition (iii) is replaced by a suitable one as in the following result.

Theorem 3.4. *Let $(X,\|\cdot\|)$ be a Banach space, $T:X\to X$ be a mapping and $\alpha,\beta:X\to[0,\infty)$ be two functions. Suppose the below conditions are followed:*

(i) *T is (α,β,z)-contraction,*

(ii) *there exists an element $x_0\in X$ such that $\alpha(x_0)>1$ and $\beta(x_0)>1$,*

(iii) *if $\{x_n\}$ is a sequence in X converges to $x\in X$ with $\alpha(x_n)\geq 1$ (or $\beta(x_n)\geq 1$), for all $n\in N$ then $\alpha(x)\geq 1$ (or $\beta(x)\geq 1$) for $n\in N$.*

Then T has a fixed point.

Proof. Following the same steps as in the proof of Theorem 3.3 we construct a sequence $\{x_n\}$ in X by defining $x_{2n+1}=Tx_{2n}$ for all $n\in N$ such that $x_n\to u\in X$, $\alpha(x_n)\geq 1$, $\beta(x_n)\geq 1$ for all $n\in N$. By condition (iii), we have $\alpha(u)\geq 1$ and $\beta(u)\geq 1$ for all $n\in N$. So $\alpha(u)\beta(u)\geq 1$.
Claim $Tu=u$. We have

$$\begin{aligned} m(x_n,u) &= \max\left\{\|x_n-u\|,\frac{[1+\|x_n-Tx_n\|]\,\|u-Tu\|}{1+\|x_n-u\|}\right\} \\ &= \max\left\{\|x_n-u\|,\frac{[1+\|x_n-x_{n+1}\|]\,\|u-Tu\|}{1+\|x_n-u\|}\right\}. \end{aligned}$$

Let $S_n:=\psi(m(x_n,u))$ and $t_n:=\psi(\|x_{n+1}-Tu\|)$.
Then

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty} t_n=\psi(\|u-Tu\|).$$

Assume $\psi(\|u-Tu\|)>0$. Therefore

$$\psi(\|u-Tu\|)\neq 0 \text{ and } \lim_{n\to+\infty}\psi(\|x_{n+1}-Tu\|)\neq 0. \tag{3.7}$$

Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} t_n > 0$ and it follows from (ζ_3) that

$$0\leq \lim_{n\to\infty}\sup\zeta(\psi(\|x_{n+1}-Tu\|),\psi(m(x_n,u)))<0$$

which is a contradiction. Thus $\psi(\|u-Tu\|)=0$.

From (ψ_2) we have $\|u-Tu\|=0$. Hence u is a fixed point of T. □

Corollary 3.5. *Let $(X,\|\cdot\|)$ be a Banach space, $T:X\to X$ be a mapping and $\alpha:X\to[0,\infty)$ be a function. Suppose that below conditions are followed:*

(i) *There exists $\zeta\in z$ such that, if $x,y\in X$ with $\alpha(x)\beta(y)\geq 1$ then*

$$\zeta(\psi(\|Tx-Ty\|),\psi(m(x,y)))\geq 0,$$

where $m(x,y)=\max\left\{\|x-y\|,\frac{[1+\|x-Tx\|]\,\|y-Ty\|}{1+\|x-y\|}\right\}$.

(ii) *If $x\in X$ with $\alpha(x)\geq 1$ then $\alpha(Tx)\geq 1$.*

(iii) *There exists $x_0\in X$ such that $\alpha(x_0)\geq 1$.*

(iv) *T is continuous then T has a fixed point.*

Proof. It follows from Theorem 3.3 by taking the function $\beta:X\to[0,\infty)$ to be α. □

Corollary 3.6. *Let $(X,\|\cdot\|)$ be a Banach space, $T:X\to X$ be a mapping and $\alpha:X\to[0,\infty)$ be a function. Suppose that below conditions are followed:*

(i) *There exists $\zeta\in z$ such that, if $x,y\in X$ with $\alpha(x)\beta(y)\geq 1$ then*

$$\zeta(\psi(\|Tx-Ty\|),\psi(m(x,y)))\geq 0,$$

where $m(x,y)=\max\left\{\|x-y\|,\frac{[1+\|x-Tx\|]\,\|y-Ty\|}{1+\|x-y\|}\right\}$.

(ii) *If $x\in X$ with $\alpha(x)\geq 1$ then $\alpha(Tx)\geq 1$.*

(iii) *There exists $x_0\in X$ such that $\alpha(x_0)\geq 1$.*

(iv) *If $\{x_n\}$ is a sequence in X converges to $x\in X$ with $\alpha(x_n)\geq 1$ for all $n\in N$, then*

$$\alpha(x)\geq 1 \text{ for all } n\in N.$$

Then T has a fixed point.

Proof. It follows from Theorem 3.4 by taking the function $\beta:X\to[0,\infty)$ to be α. □

4. Conclusion

In this paper, we establish some fixed point results for cyclic (α,β,z)-admissible mapping imbedded in simulation function in the setting of a Banach space. Our results extend and generalized the results [5] and [8].

Acknowledgments

The authors are thankful to the learned referee for his/her deep observations and their suggestions, which greatly helped us to improve the paper significantly.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. A. Alizadeh, F. Moradlou and P. Salimi, Some fixed point results for (α,β)-(ψ,ϕ)-contractive mappings, *Filomat* **28**(3) (2014), 635 – 647, DOI: 10.2298/Fil1403635a.

[2] H. H. Alsulami, E. Karapınar, F. Khojasteh and A.-F. Roldán-López-de-Hierro, A proposal to the study of contractions in quasi-metric spaces, *Discrete Dynamics in Nature and Society* **2014** (2014), Article ID 269286, DOI: 10.1155/2014/269286.

[3] E. Ameer, M. Arshad, D. Y. Shin and S. Yun, Common fixed point theorems of generalized multivalued (ψ,ϕ)-contractions in complete metric spaces with application, *Mathematics* **7**(2) (2019), 194, DOI: 10.3390/math7020194.

[4] H. Argoubi, B. Samet and C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *Journal of Nonlinear Sciences and Applications* **8**(6) (2015), 1082 – 1094, DOI: 10.22436/jnsa.008.06.18.

[5] S.-H. Cho, Fixed point theorems for (α,β)-z-contractions in metric spaces, *International Journal of Mathematical Analysis* **13**(4) (2019), 161 – 174, DOI: 10.12988/ijma.2019.9318.

[6] E. Karapınar, Fixed point results via simulations functions, *Filomt* **30**(8) (2016), 2343 – 2350, DOI: 10.2298/FIL1608343k.

[7] F. Khojasteh, S. Shukla and S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat* **29**(6) (2015), 1189 – 1194, DOI: 10.2298/FIL1506189k.

[8] T. Mani, R. Krishnakumar and D. Dhamodharan, Fixed point of (α,β,Z)-Contraction mapping under simulation functions in Banach space, *Malaya Journal of Matematik* **9**(1) (2001), 745 – 750, DOI: 10.26637/MJM0901/0131.

[9] A.-F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro and J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, *Journal of Computational and Applied Mathematics* **275** (2015), 345 – 355, DOI: 10.1016/j.cam.2014.07.011.