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Research Article

# Analytical Study of a 3D-MHD System with Exponential Damping

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**Abstract.** In this paper, we investigate the magnetohydrodynamic system with exponential type damping $\alpha(e^{\beta|u|^2}-1)u$. We prove existence of a global in time weak solution and a global in time unique strong solutions, for any $\alpha,\beta\in(0,\infty)$. The proofs are based on energy methods and use compactness argument for the existence results, and Gronwall lemma for the uniqueness. Friedrich approximation and standard techniques are also used.

**Keywords.** Magnetohydrodynamic system, Exponential damping, Existence, Uniqueness, Weak solution, Strong solution, Global solution

**Mathematics Subject Classification (2020).** 35A01, 35A02, 35B45.



## 1. Introduction

In this paper, we study the following magnetohydrodynamic system with exponential damping:

$$(S)\quad \begin{cases} \partial_t u-\Delta u+u\cdot\nabla u+b\cdot\nabla b+\alpha(e^{\beta|u|^2}-1)u=-\nabla p, & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t b-\Delta b+b\cdot\nabla u-u\cdot\nabla b=0, & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ \operatorname{div} u=0,\ \operatorname{div} b=0, & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ u(0,x)=u^0(x),\ b(0,x)=b^0(x), & \text{in } \mathbb{R}^3, \end{cases}$$

where $u = u(t,x) = (u_1,u_2,u_3)$, $b = b(t,x) = (b_1,b_2,b_3)$, $p = p(t,x)$ denote respectively the unknown velocity, the magnetic field and the pressure of the fluid at the point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$. The viscosity of fluid is $\nu = 1$. Reels $\alpha > 0$ and $\beta > 0$ denote the parameters of the damping term and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is an initial given velocity. If $u^0$ is quite regular, the divergence free condition determines the pressure $p$. About magnetohydrodynamic equations without damping terms, the paper ([5]) is one of the most complete references in the literature. Here, our purpose is to study the well-posedness of the magnetohydrodynamic system with exponential damping $\alpha(e^{\beta|u|^2}-1)u$. To the best of our knowledge, this model has not yet a physical motivation. However, it is a challenging mathematical model that may be have applications in the future, as for many mathematical models that were considered first by mathematicians and that turned to be of importance for scientists and engineers later on. We will show that the Cauchy problem $(S)$ has a global in time weak solution and a global in time unique strong solutions, for any $\alpha, \beta \in (0,\infty)$. The proofs are based on energy methods and use compactness argument for the existence results, and Gronwall lemma for the uniqueness. First, we introduce the following functional spaces:

$$\mathcal{E}_\beta = \{f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} \text{ measurable; } (e^{\beta|f|^2}-1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$$

and

$$\mathcal{F}_\beta = \{f \in \mathcal{E}_\beta \,;\, (e^{\beta|f|^2}-1)|\nabla f|^2,\ e^{\beta|f|^2}|\nabla|f|^2|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}.$$

Our main results are the following two theorems.

**Theorem 1.1.** *Let $u^0, b^0 \in L^2(\mathbb{R}^3)$ be divergence free vectors fields. Then, there is a global solution of $(S)$; $(u,b) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3) \cap C(\mathbb{R}^+, H^{-1}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$. Moreover, $u \in \mathcal{E}_\beta$ and for all $t \geq 0$, we have*

$$\|(u,b)(t)\|_{L^2}^2 + 2\int_0^t \|\nabla(u,b)\|_{L^2}^2 + 2\alpha\int_0^t \|(e^{\beta|u|^2}-1)|u|^2\|_{L^1} \leq \|(u^0,b^0)\|_{L^2}^2. \tag{1.1}$$

**Theorem 1.2.** *Let $u^0, b^0 \in H^1(\mathbb{R}^3)$ be divergence free vectors fields, such that $\|(u^0,b^0)\|_{H^1} \leq c_0$ for some positif constant $c_0$. Then, there is a unique solution of the system $(S)$; $(u,b) \in L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3)$ $\cap C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^2(\mathbb{R}^3))$. Moreover, $u \in \mathcal{F}_\beta$ and for all $t \in \mathbb{R}^+$, we have*

$$\|(u,b)(t)\|_{L^2}^2 + 2\int_0^t \|\nabla(u,b)\|_{L^2}^2 + 2\alpha\int_0^t \|(e^{\beta|u|^2}-1)|u|^2\|_{L^1} \leq \|(u^0,b^0)\|_{L^2}^2 \tag{1.2}$$

*and*

$$\|(u,b)(t)\|_{H^1}^2 + \int_0^t \|\nabla(u,b)\|_{H^1}^2 + \alpha\int_0^t \|(e^{\beta|u|^2}-1)|\nabla u|^2\|_{L^1} + \alpha\beta\int_0^t \|e^{\beta|u|^2}|\nabla|u|^2|^2\|_{L^1} \leq \|(u^0,b^0)\|_{H^1}^2. \tag{1.3}$$

In the following three subsections, we will prove respectively the existence of a global in time weak solution, the existence of the global in time strong solution and finally the uniqueness of such strong solution.

# 2. Proof of Main Results

## 2.1 Existence of a Global Weak Solution

For $R > 0$, let $B_R = \{x \in \mathbb{R}^3;\ |x| < R\}$ and the Friedrichs operator ([1]), $J_R$ defined by

$$J_R(D)f = \mathcal{F}^{-1}(\mathbf{1}_{B_R}(\xi)\widehat{f}).$$

Let us consider the approximate system of nonlinear ordinary differential equations, where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$ and $n \in \mathbb{N}$:

$$(S_n)\quad \begin{cases} \partial_t u - \Delta J_n u_n + J_n(J_n u_n \cdot \nabla J_n u_n) + J_n(J_n b_n \cdot \nabla J_n b_n) + \alpha J_n[(e^{\beta|J_n u_n|^2} - 1)J_n u_n] = -\nabla p_n, \\ \partial_t b_n - \Delta J_n b_n + J_n(J_n b_n \cdot \nabla J_n u_n) - J_n(J_n u_n \cdot \nabla J_n b_n) = 0, \\ \operatorname{div} u_n = 0,\ \operatorname{div} b_n = 0, \\ u_n(0,x) = J_n u^0(x),\ b_n(0,x) = J_n b^0(x). \end{cases}$$

The unknown pressure is given in terms of the velocity and the magnetic field; this is a standard procedure based on applying the divergence operator and using the incompressibility condition, to obtain

$$p_n = (-\Delta)^{-1}\big(\operatorname{div} J_n(J_n u_n \cdot \nabla J_n u_n + J_n b_n \cdot \nabla J_n b_n) + \alpha \operatorname{div} J_n[(e^{\beta|J_n u_n|^2} - 1)J_n u_n]\big).$$

System $(S_n)$ has the following form:

$$\partial_t U_n = H_n(U_n),\quad U_n(0) = U_n^0 = (J_n u^0, J_n b^0),$$

where $H_n$ is locally Lipschitzienne. Then, by ordinary differential equation theory, we get a unique maximal solution of $(S_n)$; $U_n = (u_n, b_n) \in C^1([0, T_n^*), L^2(\mathbb{R}^3))$. By the divergence-free properties ($\operatorname{div} u_n = 0$, $\operatorname{div} b_n = 0$) and the facts $J_n u_n = u_n$, $J_n b_n = b_n$, while taking the inner product in $L^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|U_n\|_{L^2}^2 + \|\nabla U_n\|_{L^2}^2 + \alpha\|(e^{\beta|u_n|^2} - 1)|u_n|^2\|_{L^1} \le 0. \tag{2.1}$$

Integrating with respect to time, it holds that

$$\|U_n(t)\|_{L^2}^2 + 2\int_0^t \|\nabla U_n\|_{L^2}^2 d\tau + 2\alpha\int_0^t \|(e^{\beta|u_n|^2} - 1)|u_n|^2\|_{L^1} d\tau \le \|U^0\|_{L^2}^2.$$

Following ideas in ([3]), this energy estimate allows to perform a compactness argument based on Ascoli's theorem, the Cantor diagonal process. For $n$ tends to infinity, we obtain the solution subject of Theorem 1.1. This classical compactness argument is frequently used in ([1]). Continuity in time of the solution can be proved in a standard way as in ([4]), for example.

## 2.2 Existence of the Global Strong Solution

Now, we turn to the proof of Theorem 1.2. Taking the derivative of the system of equations $(S_n)$ with respect to the variable $x_j$ and taking the dot product with $\partial_j U_n$, then summing up with respect to index $j$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla U_n\|_{L^2}^2 + \|\Delta U_n\|_{L^2}^2 + \alpha\|(e^{\beta|u_n|^2} - 1)|\nabla u_n|^2\|_{L^1} + \alpha\beta\|e^{\beta|u_n|^2}|\nabla|u_n|^2|^2\|_{L^1} \le \sum_{j=1}^{4} I_{j,n},$$

where

$$I_{1,n} = \left|\sum_{j=1}^{3}\langle \partial_j(u_n \nabla u_n)/\partial_j u_n\rangle_{L^2}\right|,$$

$$I_{2,n} = \left| \sum_{j=1}^{3} \langle \partial_j (b_n \cdot \nabla b_n) / \partial_j u_n \rangle_{L^2} \right|,$$

$$I_{3,n} = \left| \sum_{j=1}^{3} \langle \partial_j (u_n \nabla b_n) / \partial_j b_n \rangle_{L^2} \right|$$

and

$$I_{4,n} = \left| \sum_{j=1}^{3} \langle \partial_j (b_n \cdot \nabla u_n) / \partial_j b_n \rangle_{L^2} \right|.$$

By definition, we have

$$I_{1,n} = |\langle u_n \nabla u_n / \Delta u_n \rangle_{L^2}|.$$

Using Cauchy-Schwarz inequality, we obtain

$$I_{1,n} \leq \|u_n \nabla u_n\|_{L^2} \|\Delta u_n\|_{L^2}.$$

Classical Sobolev product laws imply that

$$I_{1,n} \leq C \|u_n\|_{\dot{H}^1} \|\nabla u_n\|_{\dot{H}^{1/2}} \|\Delta u_n\|_{L^2}.$$

By definition of Sobolev norms, it comes that

$$I_{1,n} \leq C \|\nabla u_n\|_{L^2}^{3/2} \|\Delta u_n\|_{L^2}^{3/2}.$$

By interpolation inequality, we have

$$I_{1,n} \leq C \|u_n\|_{L^2}^{1/2} \|\nabla u_n\|_{L^2}^{1/2} \|\Delta u_n\|_{L^2}^{2}.$$

Using Sobolev embedding, we obtain

$$I_{1,n} \leq C_1 \|u_n\|_{H^1} \|\Delta u_n\|_{L^2}^{2}.$$

As $U_n = (u_n, b_n)$, we deduce that

$$I_{1,n} \leq C \|U_n\|_{H^1} \|\Delta U_n\|_{L^2}^{2}.$$

Proceeding as in the case of $I_{1,n}$, it holds that

$$\begin{aligned}
I_{2,n} &= |\langle b_n \cdot \nabla b_n / \Delta u_n \rangle_{L^2}| \\
&\leq \|b_n \cdot \nabla b_n\|_{L^2} \|\Delta u_n\|_{L^2} \\
&\leq C \|b_n\|_{\dot{H}^1} \|\nabla b_n\|_{\dot{H}^{1/2}} \|\Delta u_n\|_{L^2} \\
&\leq C \|U_n\|_{\dot{H}^1} \|\nabla U_n\|_{\dot{H}^{1/2}} \|\Delta U_n\|_{L^2} \\
&\leq C \|\nabla U_n\|_{L^2}^{3/2} \|\Delta U_n\|_{L^2}^{3/2} \\
&\leq C \|U_n\|_{L^2}^{1/2} \|\nabla U_n\|_{L^2}^{1/2} \|\Delta U_n\|_{L^2}^{2} \\
&\leq C \|U_n\|_{H^1} \|\Delta U_n\|_{L^2}^{2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
I_{3,n} &= \left| \sum_{j=1}^{3} \langle \partial_j (u_n \nabla b_n) / \partial_j b_n \rangle_{L^2} \right| \\
&= |\langle u_n \nabla b_n / \Delta b_n \rangle_{L^2}| \\
&\leq \|u_n \nabla b_n\|_{L^2} \|\Delta b_n\|_{L^2}
\end{aligned}$$

$$\leq C\|u_n\|_{\dot H^1}\|\nabla b_n\|_{\dot H^{1/2}}\|\Delta b_n\|_{L^2}$$
$$\leq C\|U_n\|_{\dot H^1}\|\nabla U_n\|_{\dot H^{1/2}}\|\Delta U_n\|_{L^2}$$
$$\leq C\|U_n\|_{H^1}\|\Delta U_n\|_{L^2}^2$$

and

$$I_{4,n} \leq C\|U_n\|_{H^1}\|\Delta U_n\|_{L^2}^2.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla U_n\|_{L^2}^2 + \|\Delta U_n\|_{L^2}^2 + \alpha\|(e^{\beta|u_n|^2}-1)|\nabla u_n|^2\|_{L^1} + \alpha\beta\|e^{\beta|u_n|^2}|\nabla|u_n|^2|^2\|_{L^1}$$
$$\leq 4C\|U_n\|_{H^1}\|\Delta U_n\|_{L^2}^2.$$

Summing up inequality above with the $L^2$ energy estimate (2.1), we get

$$\frac{1}{2}\frac{d}{dt}\|U_n\|_{H^1}^2 + \|\nabla U_n\|_{H^1}^2 + \alpha\|(e^{\beta|u_n|^2}-1)|\nabla u_n|^2\|_{L^1} + \alpha\beta\|e^{\beta|u_n|^2}|\nabla|u_n|^2|^2\|_{L^1} + \alpha\|(e^{\beta|u_n|^2}-1)|u_n|^2\|_{L^1}$$
$$\leq 4C\|U_n\|_{H^1}\|\Delta U_n\|_{L^2}^2.$$

We suppose that $c_0 \in (0, 1/8C)$. As

$$4C\|U_n^0\|_{H^1} \leq 4C\|U^0\|_{H^1} < 1 \iff \|U_n^0\|_{H^1} \leq \|U^0\|_{H^1} < \frac{1}{4C}$$

and by continuity of the function $(t \longmapsto \|U_n(t)\|_{H^1})$, we get

$$t_n = \sup\left\{t \geq 0/\ \|U_n\|_{L^\infty([0,t],H^1)} < \frac{1}{2}\left(\|U^0\|_{H^1} + \frac{1}{4C}\right)\right\} \in (0,\infty].$$

As, $\frac{1}{2}(\|U^0\|_{H^1} + \frac{1}{4C}) \in (\|U^0\|_{H^1}, \frac{1}{4C})$. Then, for $t \in [0, t_n)$, we have

$$\frac{1}{2}\frac{d}{dt}\|U_n\|_{H^1}^2 + \|\nabla U_n\|_{H^1}^2 + \alpha\|(e^{\beta|u_n|^2}-1)|\nabla u_n|^2\|_{L^1} + \alpha\beta\|e^{\beta|u_n|^2}|\nabla|u_n|^2|^2\|_{L^1} + \alpha\|(e^{\beta|u_n|^2}-1)|u_n|^2\|_{L^1}$$
$$\leq 4C\frac{1}{2}(\|U^0\|_{H^1} + \frac{1}{4C})\|\Delta U_n\|_{L^2}^2.$$

Then, for $t \in [0, t_n)$, we have

$$\|U_n(t)\|_{H^1}^2 + 2(1 - 4C\|U^0\|_{H^1})\int_0^t \|\nabla U_n\|_{H^1}^2 + 2\alpha\int_0^t \|(e^{\beta|u_n|^2}-1)|\nabla u_n|^2\|_{L^1}$$
$$+ 2\alpha\beta\int_0^t \|e^{\beta|u_n|^2}|\nabla|u_n|^2|^2\|_{L^1} + \alpha\int_0^t \|(e^{\beta|u_n|^2}-1)|u_n|^2\|_{L^1}$$
$$\leq \|U^0\|_{H^1}^2,$$

which gives $t_n = \infty$ and for $t \in [0,\infty)$, we have, under the condition $4C\|U^0\|_{H^1} < 1/2$,

$$\|U_n(t)\|_{H^1}^2 + \int_0^t \|\nabla U_n\|_{H^1}^2 + 2\alpha\int_0^t \|(e^{\beta|u_n|^2}-1)|\nabla u_n|^2\|_{L^1}$$
$$+ 2\alpha\beta\int_0^t \|e^{\beta|u_n|^2}|\nabla|u_n|^2|^2\|_{L^1} + \alpha\int_0^t \|(e^{\beta|u_n|^2}-1)|u_n|^2\|_{L^1} \leq \|U^0\|_{H^1}^2.$$

This implies that $(U_n)$ is bounded in

$$L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3) \cap C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot H^2(\mathbb{R}^3)) \cap \mathcal{E}_\beta \cap \mathcal{F}_\beta.$$

As for the week solution above, following ideas in ([3]), this energy estimate allows to perform a classical compactness argument ([1]) in order to finish the proof of the existence of the strong solution, in theorem 1.2, that satisfies the energy estimate:

$$\|U(t)\|_{H^1}^2+\int_0^t\|\nabla U\|_{H^1}^2+2\alpha\int_0^t\|(e^{\beta|u|^2}-1)|\nabla u|^2\|_{L^1}$$
$$+2\alpha\beta\int_0^t\|e^{\beta|u|^2}|\nabla|u|^2|^2\|_{L^1}+\alpha\int_0^t\|(e^{\beta|u|^2}-1)|u|^2\|_{L^1}\leq\|U^0\|_{H^1}^2.$$

Continuity in time of the solution can be proved in a standard way, see for example ([4]).

## 2.3 Uniqueness of the Global Strong Solution

Now, we will prove the uniqueness of the strong solution to the system $(S)$. To do so, let $U=(u,b)$ be a solution given by the precedent subsection, and let $V=(v,c)$ be an other solution of $(S)$ having the same initial data. We take the difference of the corresponding systems, we denote $w=u-v$, $d=b-c$, where $p$ is the pressure term corresponding to $U$ and $q$ is the one corresponding to $V$. Thus, we get for $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$,

$$(\delta S)\quad\begin{cases}\partial_t w-\Delta w+w\cdot\nabla u+v\cdot\nabla w+d\cdot\nabla b+c\cdot\nabla d+\alpha(e^{\beta|u|^2}-1)u-\alpha(e^{\beta|v|^2}-1)v=-\nabla(p-q)\\ \partial_t d-\Delta d+d\cdot\nabla u+c\cdot\nabla w-w\cdot\nabla b-v\cdot\nabla d=0\\ \operatorname{div}w=0,\ \operatorname{div}d=0\\ w(0,x)=0,\ d(0,x)=0,\end{cases}$$

Taking the $L^2$-scalar product of the first equation with $w$ and the $L^2$-scalar product of the second equation with $d$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2+\|d\|_{L^2}^2)+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^2}^2\alpha\langle(e^{\beta|u|^2}-1)u-\alpha(e^{\beta|v|^2}-1)v,w\rangle_{L^2}$$
$$+\langle w\cdot\nabla u,w\rangle_{L^2}-\langle d\cdot\nabla b,w\rangle_{L^2}+\langle d\cdot\nabla u,d\rangle_{L^2}-\langle w\cdot\nabla b,d\rangle_{L^2}=0.$$

In fact, by the divergence free condition, we have

$$\langle\nabla(p-q),w\rangle_{L^2}=0,$$
$$\langle v\cdot\nabla w,w\rangle_{L^2}=0,$$
$$\langle v\cdot\nabla d,d\rangle_{L^2}=0.$$

Also, since

$$\langle c\cdot\nabla d,w\rangle_{L^2}+\langle c\cdot\nabla w,d\rangle_{L^2}=\langle c\cdot\nabla(d+w),(d+w)\rangle_{L^2}-\langle c\cdot\nabla d,d\rangle_{L^2}-\langle c\cdot\nabla w,w\rangle_{L^2},$$

it vanishes thanks to the divergence free condition.

It holds that

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2+\|d\|_{L^2}^2)+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^2}^2\alpha\langle(e^{\beta|u|^2}-1)u-\alpha(e^{\beta|v|^2}-1)v,w\rangle_{L^2}$$
$$\leq|\langle w\cdot\nabla u,w\rangle_{L^2}|+|\langle d\cdot\nabla b,w\rangle_{L^2}|+|\langle d\cdot\nabla u,d\rangle_{L^2}|+|\langle w\cdot\nabla b,d\rangle_{L^2}|.$$

As for the exponential damping term, in the left hand side, we recall the following lemma from ([2]).

**Lemma 2.1.** *If $\beta>0$, then, for all $x,y\in\mathbb{R}^d$, we have*

$$\big((e^{\beta|x|^2}-1)x-(e^{\beta|y|^2}-1)y\big)\cdot(x-y)\geq\frac{1}{2}\big[(e^{\beta|x|^2}-1)+(e^{\beta|y|^2}-1)\big]|x-y|^2.$$

This lemma allows to control the difference of damping terms in the right hand side, by a positive lower bound. Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|w\|_{L^2}^2+\|d\|_{L^2}^2)+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^2}^2$$
$$\le |\langle w\cdot\nabla u,w\rangle_{L^2}|+|\langle d\cdot\nabla b,w\rangle_{L^2}|+|\langle d\cdot\nabla u,d\rangle_{L^2}|+|\langle w\cdot\nabla b,d\rangle_{L^2}|.$$

By the divergence free condition, $I_2:=|\langle d\cdot\nabla b,w\rangle_{L^2}|=|\langle d\otimes b,\nabla w\rangle_{L^2}|$. Using Cauchy-Schwarz inequality, it holds that $I_2\le(\int|d||b|dx)^{1/2}\|\nabla w\|_{L^2}$. By Hölder inequality, we have $I_2\le\|d\|_{L^3}\|b\|_{L^6}\|\nabla w\|_{L^2}$. Since $\|b\|_{L^6}\le C\|b\|_{\dot H^1}$, it comes that $\|b\|_{L^6}\le C_0$, where $C_0=C\|(u_0,b_0)\|_{H^1}$, according to the energy estimate in the existence theorem. That is $I_2\le C_0\|d\|_{L^3}\|\nabla w\|_{L^2}$. As, $\dot H^{1/2}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, it comes that $I_2\le C_0'\|d\|_{\dot H^{1/2}}\|\nabla w\|_{L^2}$. Interpolation inequality leads to

$$I_2\le C_0'\|d\|_{L^2}^{1/2}\|d\|_{\dot H^1}^{1/2}\|w\|_{\dot H^1}\le C_0'\|(w,d)\|_{L^2}^{1/2}\|(w,d)\|_{\dot H^1}^{3/2}.$$

By Young's product inequality $xy\le\frac{1}{4}x^4+\frac{3}{4}y^{4/3}$, we obtain

$$I_2\le C_0'\|(w,d)\|_{L^2}^2+\frac{1}{4}\|(w,d)\|_{\dot H^1}^2.$$

Similarly, we obtain

$$\begin{aligned}I_3:&=|\langle d\cdot\nabla u,d\rangle_{L^2}|\\&\le\|d\|_{L^3}\|u\|_{L^6}\|\nabla d\|_{L^2}\\&\le 2C_0'\|d\|_{L^2}^{1/2}\|d\|_{\dot H^1}^{3/2}\\&\le C_0'\|d\|_{L^2}^2+\frac{1}{4}\|d\|_{\dot H^1}^2\\&\le C_0'\|(w,d)\|_{L^2}^2+\frac{1}{4}\|(w,d)\|_{\dot H^1}^2,\\I_1:&=|\langle w\cdot\nabla u,w\rangle_{L^2}|\\&\le 2C_0'\|w\|_{L^2}^{1/2}\|w\|_{\dot H^1}^{3/2}\\&\le C_0'\|w\|_{L^2}^2+\frac{1}{4}\|w\|_{\dot H^1}^2\\&\le C_0'\|w\|_{L^2}^2+\frac{1}{4}\|(w,d)\|_{\dot H^1}^2\end{aligned}$$

and

$$\begin{aligned}I_4:&=|\langle w\cdot\nabla b,d\rangle_{L^2}|\\&\le\|w\|_{L^3}\|b\|_{L^6}\|\nabla d\|_{L^2}\\&\le C_0\|w\|_{L^2}^{1/2}\|w\|_{\dot H^1}^{1/2}\|d\|_{\dot H^1}\\&\le C_0\|(w,d)\|_{L^2}^{1/2}\|(w,d)\|_{\dot H^1}^{3/2}\\&\le C_0'\|(w,d)\|_{L^2}^2+\frac{1}{4}\|(w,d)\|_{\dot H^1}^2.\end{aligned}$$

Then,

$$\frac{d}{dt}\|(w,d)\|_{L^2}^2+\|(w,d)\|_{\dot H^1}^2\le 4C_0'\|(w,d)\|_{L^2}^2.$$

Since $\|(w,d)(0)\|_{L^2}^2=0$, then Gronwall lemma gives the uniqueness.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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