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Research Article

Measure of Rotatability for a Class of Balanced Ternary Design

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Abstract. In the present study we mainly focused on a measure of rotatability for a class of Balanced ternary designs. It is very much useful to assess the degree of a rotatability for a given response surface design.

Keywords. Response surface designs, Second order rotatable designs, Measure of rotatability

Mathematics Subject Classification (2020). Primary 62K05, Secondary 05B05

1. Introduction

Response surface methodology is a statistical technique which is very useful in design and analysis of scientific experiments. In many experimental situations the experimenter is concerned with explaining certain aspects of a functional relationship $Y = f(x_1, x_2, \dots, x_v) + e$, where Y is the response, $x_1, x_2, \dots, x_v$ are the levels of v-quantitative variables or factors and e is the random error. Response surface methods are useful where several independent variables influence a dependent variable. The independent variables are assumed to be continuous and controlled by the experimenter. The response is assumed to be a random variable. For example, if a chemical engineer wishes to find the temperature (x_1) and pressure (x_2) that maximizes the yield (response) of his process, the observed response Y may be written as a function of the levels of the temperature (x_1) and pressure (x_2) as $Y = f(x_1, x_2) + e$.

Initially, Response surface methodology for multi factor responses was coined by Box and Hunter [4]. Das and Narasimham [6] constructed *Balanced incomplete block designs* (BIBD). Billington [1] provided a list of *balanced ternary designs* (BTD) with some necessary conditions. Billington [2] suggested balanced n-ary designs. Donovan [7] introduced balanced ternary designs from 1-designs. Kunkle and Sarvate [11] focused on further development of balanced ternary designs. Draper and Guttman [8] studied an index of rotatability. Khuri [12] proposed a measure of rotatability for the given response surface designs. Draper and Pukelsheim [9] studied another look at rotatability. Park *et al*. [14] suggested a new measure of rotatability for second order models. Kaski and Ostergard [15] suggested enumeration of balanced ternary designs. Victorbabu and Surekha [18] studied measure of rotatability for second order response surface designs using BIBD. Kanna *et al*. [10] constructed a new class of SORD using balanced ternary designs. Varalakshmi and Rajyalakshmi [17] estimated the optimum responses for SORD using balanced ternary designs.

In this paper we follow the works of Box and Hunter [4], Das and Narasimham [6], Billington and Robinson [1], Kanna *et al*. [10], Victorbabu and Surekha [18] suggested a measure of rotatability for the given balanced ternary design.

2. Methodology

The second order response surface design be $D = ((X_{iu}))$

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i}^{v}\sum_{j}^{v} b_{ij} x_{iu} x_{ju} + \varepsilon_u, \tag{2.1}$$

where

x_{iu}: level of the ith factor ($i = 1,2,\ldots,v$) in the uth run ($u = 1,2,\ldots,N$) of the experiment, and e_u's: uncorrelated random errors with mean zero and variance σ^2. From the conditions of Box and Hunter [4], Das and Narasimham [6], and Das and Giri [5] the spherical variance function for estimation of responses in the second order response surface is achieved if the design points satisfy the conditions. They have suggested the variances and covariances along with the non-singularity condition and the rotatable condition.

SORD using BTD

Balanced Ternary Design (BTD)

Following the works of Billington [2], Donovan [7], Kaski and Ostergard [15], BTD is denote by $(V, B, \rho_1, \rho_2, R, K, \lambda)$ with V rows and B columns. Here (i, j)th entry represents the number of times the element i appears in block j.

Let $n = (n_{ij})$ be the incidents matrix of balanced ternary design, it should consists of the following conditions:

(i) Each treatment may appear or does not appear, if it appear it occur 0, 1 or 2 times in a block.

(ii) Each treatment have the same number of replications say R.

(iii) In BTD each element occurs singly in ρ_1 blocks and doubly in ρ_2 blocks (Billington [2]).

(iv) The inner product of any two distinct rows of an incidence matrices of BTD is λ.

The parametric relations of BTD are

(1) $VR = BK$.

(2) $\lambda(V-1) = \rho_1(k-1) + 2\rho_1(k-2)$.

(3) $R = \rho_1 + 2\rho_2$.

Conditions of a Measure of Rotatability

As per Box and Hunter [4], Das and Narasimham [6], and Park *et al*. [14], the necessary and sufficient conditions for a measure of rotatability for any general second order response surface designs is as follows:

$$\left.\begin{array}{l} V(b_i) \text{ are equal for } i,\ V(b_{ii}) \text{ are equal for } i,\ V(b_{ij}) \text{ are equal for } i,j, \text{ where } i \neq j, \\ \operatorname{cov}(b_i,b_{ii}) = \operatorname{cov}(b_i,b_{ij}) = \operatorname{cov}(b_{ii},b_{ij}) = \operatorname{cov}(b_{ij},b_{il}) = 0 \text{ for all } i \neq j \neq l,\ l \neq i, \\ V(b_i) \text{ are equal for } i, \\ V(b_{ii}) \text{ are equal for } i, \\ V(b_{ij}) \text{ are equal for } i,j, \text{ where } i \neq j, \end{array}\right\} \tag{2.2}$$

Park *et al*. [14] suggested measure ($P_v(D)$) can be used to assess the degree of rotatability for any general second order response surface design (cf. Park *et al*. [14, p. 661]).

$$P_v(D) = \frac{1}{1+R_v(D)}, \tag{2.3}$$

where

$$R_v(D) = \left[\frac{N}{\sigma^2}\right]^2 \frac{6v[v(\widehat{b}_{ij}) + 2\operatorname{cov}(\widehat{b}_{ii},\widehat{b}_{ij}) - 2V(\widehat{b}_{ii})]^2(v-1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8} \tag{2.4}$$

and g is the scaling factor. On simplification $R_v(D)$ becomes

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8}. \tag{2.5}$$

3. Measure of Rotatability for Second Order Design using BTD

Measure of rotatability for second order response surface designs using BTD is suggested.

Theorem 3.1. *The design points,* $[1-(V,B,\rho_1,\rho_2,R,K,\lambda)] \times 2^{t(k)} \cup (a,0,0,\dots,0)2^1 \cup n_0$ *will give a v-dimensional measure of second order response surface design using BTD in* $N = B \times 2^{t(k)} + 2V + n_0$ *design points, with* $a^4 = (3\lambda - R)2^{t(K)-1}$.

Proof. For the design points generated from a BTD, assume that the simple symmetric condition is true. Further conditions are as follows:

$$\sum x_{iu}^2 = \rho_1 2^{t(k)} + +2\rho_2 2^{t(k)} + 2a^2 = N\lambda_2, \tag{3.1}$$

$$\sum x_{iu}^4 = \rho_1 2^{t(k)} + +2\rho_2 2^{t(k)} + 2a^4 = cN\lambda_4, \tag{3.2}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\lambda_4. \tag{3.3}$$

From (3.1) and (3.2), we can get the value of 'c'. Measure of rotatability is given by

$$P_v(D) = \frac{1}{1+R_v(D)}, \tag{3.4}$$

where

$$R_v(D) = \left[\frac{N}{\sigma^2}\right]^2 \frac{6v[v(\widehat{b}_{ij}) + 2\operatorname{cov}(\widehat{b}_{ii}, \widehat{b}_{ij}) - 2V(\widehat{b}_{ii})]^2(v-1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8} \tag{3.5}$$

and g is the scaling factor

$$g = \begin{cases} \dfrac{1}{a}, & \text{if } a < \sqrt{(B-R)2^{t(k)-1}+v}, \\ \sqrt{(B-R)2^{t(k)-1}+v}, & \text{if } a > \sqrt{(B-R)2^{t(k)-1}+v}. \end{cases} \tag{3.6}$$

On simplification $R_v(D)$ becomes

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8}. \tag{3.7}$$

□

Example 3.1. We illustrate Theorem 3.1 using measure of rotatability for BTD with parameters $(3,3,1,1,3,3,2)$. For the above BTD the design points $[1-(3,3,1,1,3,3,2)]2^{t(3)} \cup (a,0,0,\dots,0)2^1 \cup (n_0 = 6)$ will give a measure of rotatability for second order response surface design in $N = 36$ design points for five factors.

From (3.1), (3.2) and (3.3) we have

$$\sum x_{iu}^2 = 24 + 2a^2 = N\lambda_2, \tag{3.8}$$

$$\sum x_{iu}^4 = 24 + 2a^4 = cN\lambda_4, \tag{3.9}$$

$$\sum x_{iu}^2 x_{ju}^2 = 16 = N\lambda_4. \tag{3.10}$$

From (3.9) and (3.10), we obtain rotatability value by taking $c = 3$. Hence, we get a SORD with $a = 1.861$ and $c = 3$. Instead of taking $a = 1.861$, suppose we take $a = 1.3$, we get $c = 1.857013$. The scaling factor $g = 0.7692308$, $R_v(D) = 0.1526335$ and $P_v(D) = 0.8675785$.

Hence, we get a nearly SORD using a BTD with $N = 36$, $a = 1.9$, $c = 3.1290$, scaling factor $g = 0.5773503$ and measure of rotatability $P_v(D) = 0.9968809$.

Table 1 gives the values of measure of rotatability for second order response surface designs using BTD. It can be verified that $P_v(D)$ is 1 if and only if the design is rotatable and it is smaller than one for a non-rotatable design or nearly rotatable design.

Table 1. Measure of rotatability for second order response surface designs using BTD

$(3,3,1,1,3,3,2)$, $a = 1.861$, $N = 36$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.625	1	0.05091429	0.9515524
1.3	1.857013	0.7692308	0.1526335	0.8675785
1.6	2.3192	0.625	0.1203295	0.8925946
1.861	3	0.5773503	0	1
1.9	3.129012	0.5773503	0.003128862	0.9968809
2.2	4.4282	0.5773503	0.1478853	0.8711672
2.5	6.382812	0.5773503	0.3365245	0.7482093
2.8	9.1832	0.5773503	0.4864739	0.672733
3.1	13.04401	0.5773503	0.592586	0.6279096
3.4	18.2042	0.5773503	0.6654838	0.6004261

$(3,4,2,1,4,3,3)$, $a = 2.115$, $N = 42$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.416667	1	0.09189091	0.9158424
1.3	1.571342	0.7692308	0.3245765	0.7549583
1.6	1.879467	0.625	0.4436863	0.6926713
1.9	2.419342	0.5773503	0.0862696	0.9205818
2.115	3.000812	0.5773503	0	1
2.2	3.285467	0.5773503	0.008041765	0.9920224
2.5	4.588542	0.5773503	0.101007	0.9082594
2.8	6.455467	0.5773503	0.206795	0.8286412
3.1	9.029342	0.5773503	0.29065	0.7748034
3.4	12.46947	0.5773503	0.3513621	0.7399941

$(3,6,4,1,6,3,5)$, $a = 2.449$, $N = 55$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.25	1	0.1925	0.8385744
1.3	1.342805	0.7692308	0.7489182	0.571782
1.6	1.52768	0.625	1.313583	0.4322299
1.9	1.851605	0.5773503	0.5786632	0.6334473
2.2	2.37128	0.5773503	0.0668932	0.9373009
2.449	2.998561	0.5773503	1.649963e-07	1
2.5	3.153125	0.5773503	0.001609435	0.9983932
2.8	4.27328	0.5773503	0.04815054	0.9540614
3.1	5.817605	0.5773503	0.1088473	0.9018375
3.4	7.88168	0.5773503	0.1601288	0.8619733

Table 1 *Contd.*

$(3,5,3,1,5,3,4)$, $a = 2.3$, $N = 48$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.28125	1	0.04365079	0.9581749
1.3	1.339253	0.7692308	0.228485	0.8140107
1.6	1.4548	0.625	0.5794822	0.6331189
1.9	1.657253	0.5773503	0.395148	0.7167698
2.2	1.98205	0.5773503	0.1017238	0.9076685
2.3	2.124503	0.5773503	0	1
2.5	2.470703	0.5773503	0.01226268	0.9878859
2.8	3.1708	0.5773503	0.0005861016	0.9994142
3.1	4.136003	0.5773503	0.01242347	0.987729
3.4	5.42605	0.5773503	0.02844485	0.9723419
$(4,12,3,3,9,3,4)$, $a = 1.861$, $N = 130$				
a	c	g	$R_v(D)$	$P_v(D)$
1	2.3125	1	0.009433922	0.9906542
1.3	2.428506	0.7692308	0.04489015	0.9570384
1.6	2.6596	0.625	0.06212661	0.9415073
1.861	2.999662	0.5373455	0	0.9999999
1.9	3.064506	0.5263158	0.005700927	0.9943314
2.2	3.7141	0.4545455	1.306151	0.433623
2.5	4.691406	0.4	11.01483	0.08323048
2.8	6.0916	0.3571429	47.87253	0.02045302
3.1	8.022006	0.3225806	149.993	0.006622821
3.4	10.6021	0.2941176	1384.8678	0.002591561
$(5,5,1,2,5,5,4)$, $a = 2.736$, $N = 101$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.28125	1	0.1769823	0.8496305
1.3	1.339253	0.7692308	0.9263933	0.5191048
1.6	1.4548	0.625	2.349513	0.2985509
1.9	1.657253	0.5263158	3.359241	0.2293977
2.2	1.98205	0.4545455	2.794194	0.2635606
2.5	2.470703	0.4472136	0.3836349	0.722734
2.736	3.001114	0.4472136	9.177269e-07	0.9999991
2.8	3.1708	0.4472136	0.01833604	0.9819941
3.1	4.136003	0.4472136	0.3886651	0.720116
3.4	5.42605	0.4472136	0.8898899	0.5291313

Table 1 *Contd*.

$(6,6,2,1,4,4,2)$, $a=2$, $N=122$				
a	c	g	$R_v(D)$	$P_v(D)$
1	2.0625	1	0.0189447	0.9814075
1.3	2.178506	0.7692308	0.09644829	0.9120357
1.6	2.4096	0.625	0.1833427	0.8450637
1.9	2.814506	0.5263158	0.04318902	0.958599
2	3	0.5	0	1
2.2	3.4641	0.4545455	0.4736864	0.6785704
2.5	4.441406	0.4	6.513643	0.1330912
2.8	5.8416	0.3571429	31.66761	0.03061136
3.1	7.772006	0.3225806	103.0538	0.009610417
3.4	10.3521	0.2941176	268.5584	0.003709771
$(7,7,3,1,5,5,3)$, $a=2.378$, $N=136$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.708333	1	0.03871766	0.9627255
1.3	1.785671	0.7692308	0.2268935	0.8150667
1.6	1.939733	0.625	0.6365967	0.6110241
1.9	2.209671	0.5263158	0.8441001	0.5422699
2.2	2.642733	0.4545455	0.3022149	0.7679224
2.378	2.999071	0.4205214	0	1
2.5	3.294271	0.4	0.2922865	0.7738222
2.8	4.227733	0.3571429	6.364446	0.1357875
3.1	5.514671	0.3225806	30.80968	0.03143697
3.4	7.234733	0.2941176	95.92488	0.01031727
$(8,8,4,1,6,6,4)$, $a=3.13$, $N=282$				
a	c	g	$R_v(D)$	$P_v(D)$
1	1.515625	1	0.05028143	0.9521258
1.3	1.544627	0.7692308	0.3534161	0.7388711
1.6	1.6024	0.625	1.402633	0.4162101
1.9	1.703627	0.5263158	3.497783	0.2223317
2.2	1.866025	0.4545455	5.708419	0.1490664
2.5	2.110352	0.4	5.943253	0.1440247
2.8	2.4604	0.3571429	3.12934	0.2421694
3.1	2.943002	0.3225806	0.0445309	0.9573676
3.13	2.999676	0.3194888	0	0.9999985
3.4	3.588025	0.2941176	5.593379	0.1516673

4. Conclusion

In this paper, a measure of rotatability for second order response surface designs using BTD has presented which enables us to assess the degree of rotatability for a given second order response surface design. This measure $P_v(D)$ has the value one if and only if the design D is rotatable, and it is smaller than one for a nearly rotatable design.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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