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Research Article

On Nonlinearization of 3-parameter Eigenvalue Problems

Niranjan Bora*[1], Bikash Chutia[2], Rubul Moran[2] and Mukul Chandra Bora[3]

[1] *Department of Mathematics, Dibrugarh University Institute of Engineering and Technology, Dibrugarh, Assam, India*
[2] *Department of Mathematics, Dibrugarh University, Dibrugarh, Assam, India*
[3] *Dibrugarh University Institute of Engineering and Technology, Dibrugarh, Assam, India*
***Corresponding author:** niranjanbora11@gmail.com

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Abstract. In this paper, the linear 3-*parameter eigenvalue problem* (3PEP) in terms of matrix equations is considered. Using the Rayleigh Quotient iteration method, any one of the three parameters can be fixed to transform the problem into a linear 2-*parameter eigenvalue problem* (2PEP). This admits a family *nonlinear eigenvalue problems* (NEP) in one parameter. The transformation results from the elimination of the second equation of respective 2PEP, which is re-arranged as a generalized eigenvalue problem (GEP) of the form $Ey = \lambda Fy$, where E and F are matrices $n \times n$ over $\mathbb{C}$, $y \in \mathbb{C}^n$ is a non-zero vector and λ is a scalar. A review of some results of the condition number of NEP is also presented in this paper.

Keywords. Generalized eigenvalue problem, 3-parameter eigenvalue problem, Nonlinear eigenvalue problem, Condition number

Mathematics Subject Classification (2020). 15A18, 15A22, 47J10, 65H17

1. Introduction

Consider the following 3PEP in terms of matrix equations

$$E_i(\alpha)x_i := (M_{i0} - \alpha_1 M_{i1} - \alpha_2 M_{i2} - \alpha_3 M_{i3})x_i = 0, \tag{1.1}$$

where $a_i \in \mathbb{C}$ are spectral parameters, $x_i \in \mathbb{C}^{n_i}$ are non zero vectors and M_{ij} are $n_i \times n_i$ complex matrices, for $i,j := 1,2,3$. The objective of the problem (1.1) is to find tuples $(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3)$ with all $0 \neq x_i$ that satisfy $E_i(a)x_i = 0$. The 3-tuple $\alpha = (a_1,a_2,a_3)$ is called eigenvalue and the corresponding tensor product $x_1 \otimes x_2 \otimes x_3$ is called right eigenvector. The eigenvalue α is called simple if $\operatorname{Dim}\operatorname{Ker}(E_i(a) = 1$, for $i = 1,2,3$. The usual technique for spectral analysis of (1.1) is by converting it into joint GEPs in tensor product space. This transformation can be established by defining commuting 3-tuple of certain operator matrices and are given by

$$\Delta_1 z = \alpha_1 \Delta_0 z,\ \Delta_2 z = \alpha_2 \Delta_0 z,\ \Delta_3 z = \alpha_3 \Delta_0 z, \tag{1.2}$$

here

$$\Delta_0 = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix},\ \Delta_1 = \begin{vmatrix} M_{10} & M_{12} & M_{13} \\ M_{20} & M_{22} & M_{23} \\ M_{30} & M_{32} & M_{33} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} M_{11} & M_{10} & M_{13} \\ M_{21} & M_{20} & M_{23} \\ M_{31} & M_{30} & M_{33} \end{vmatrix} \text{ and}$$

$$\Delta_3 = \begin{vmatrix} M_{11} & M_{12} & M_{10} \\ M_{21} & M_{22} & M_{20} \\ M_{31} & M_{32} & M_{30} \end{vmatrix} \tag{1.3}$$

and $z = x_1 \otimes x_2 \otimes x_3$ is a decomposable tensor. Atkinson [2] investigated the problem of such kind carefully for $i,j := 1 : k$. The problem of this kind arises in diverse domain of science and engineering (e.g., [6]). Literature on the classical theory of multiparameter system has been reported in [2,9,25,35,36,38]. Operations of applied Linear algebra can be used to solve nonsingular system (1.4). But it is convenient to solve the problem consisting of matrices of low order only. The major computational disadvantages in the cost of computation of operator matrices Δ_i of size $N \times N$. Thus it is necessary to adopt numerical algorithm to find the solution of the problem (1.1). Method based on QZ-algorithm presented in [19] is suitable for this purpose. For 2PEP, various numerical methods are available in existing literature (e.g. [24,31,32]). For 3PEP method presented in [18], and for k-parameter problem methods developed in [12,34] are suitable to find the numerical solution of the problem.

Motivating Example. Consider the four-point boundary value problem from [17] which is represented by the differential equation

$$y''(x) + (\mu_1 + 2\mu_2 \cos(x) + 2\mu_3 \cos(2x))y(x) = 0 \tag{1.4}$$

subject to $y(0) = y(1) = y(2) = y(3) = 0$

The problem is to evaluate the 3-tuple (μ_1,μ_2,μ_3), for which the equation (1.4) agrees for the non zero solution $y(x)$. The problem (1.4) can be decomposed into a 3PEP consisting of three two point boundary value problems given by

$$y_i''(x_i) + (\mu_1 + 2\mu_2 \cos(x_i) + 2\mu_3 \cos(2x_i))y_i(x_i) = 0 \tag{1.5}$$

such that $y_i(x_i) = 0,\ y_i(i-1) = y_i(i) = 0,$

where $i = 1,2,3$. A smooth function $y(x)$ that satisfies the differential equation (1.4) can be generated from the functions $y_i(x_i)$, $i = 1,2,3$. The 3-parameter problem defined in equation (1.5) processes the Klein Oscillation property. That is for each 3-tuple of non-negative integers (n_1,n_2,n_3) there exist a 3-tuple (μ_1,μ_2,μ_3), such that the problem (1.4) processes a solution $y(x)$ that has n_1, n_2 and n_3 zeros on interval $(0,1)$, $(1,2)$ and $(2,3)$, respectively. Discretizing the equation (1.5) in the environment of Chebyshev collocation of 200 points yields a 3PEP of the form considered in equation (1.1).

2. Transformation of 3PEP into a 2PEP

Let Δ_0 defined by the equation (1.3) is nonsingular. Then, using the Rayleigh Quotient iteration method presented in [7], any one of three parameters a_i, $i = 1,2,3$ can be evaluated from the corresponding GEP. Let, a_3 is known. This a_3 can be calculated from the system $\Delta_3 z = a_3\Delta_0 z$ by using the Rayleigh Quotient iteration method. Let $z^* := x_1^* \otimes x_2^* \otimes x_3^*$ be the actual eigenvectors of the system $\Delta_3 z = a_3\Delta_0 z$. Then x_1^*, x_2^* and x_3^* satisfy the system $\Delta_3 z^* = a_3\Delta_0 z^*$. Thus the Tensor Rayleigh Quotient at the exact value of eigenvector becomes

$$\rho = \frac{(z^*)^T\Delta_3 z^*}{(z^*)^T\Delta_0 z^*} = \frac{(z^*)^T a_3\Delta_0 z^*}{(z^*)^T\Delta_0 z^*} = a_3\frac{(z^*)^T\Delta_0 z^*}{(z^*)^T\Delta_0 z^*} = a_3\,. \tag{2.1}$$

Since the problem considered in this paper is nonsingular so the Tensor Quotient in the equation (2.3) is well defined, where

$$\rho = \frac{(x_1^*\otimes x_2^*\otimes x_3^*)^T\Delta_3(x_1^*\otimes x_2^*\otimes x_3^*)}{(x_1^*\otimes x_2^*\otimes x_3^*)^T\Delta_0(x_1^*\otimes x_2^*\otimes x_3^*)}. \tag{2.2}$$

Although, it seems that complexity may be arose in computing ρ due to the presence of the operator matrices Δ_0 and Δ_3 and but it is possible to compute all the tensor quotient quite efficiently with decomposable tensor $z = x_1\otimes x_2\otimes x_3$ as the expressions present in (2.2) can be written as determinants which can be evaluated numerically. For example, the expression $(x_1\otimes x_2\otimes x_3)^T\Delta_0(x_1\otimes x_2\otimes x_3)$ can be written as [3]:

$$(x_1\otimes x_2\otimes x_3)^T\Delta_0(x_1\otimes x_2\otimes x_3) = \begin{vmatrix} x_1^T M_{11}x_1 & x_1^T M_{12}x_1 & x_1^T M_{13}x_1 \\ x_2^T M_{21}x_2 & x_2^T M_{22}x_2 & x_2^T M_{23}x_2 \\ x_3^T M_{31}x_3 & x_1^T M_{32}x_3 & x_3^T M_{33}x_3 \end{vmatrix}. \tag{2.3}$$

Let x_1^n, x_2^n and x_3^n be some approximations to the eigenvectors x_1, x_2 and x_3 respectively, where $n \in \mathbb{Z}^+$. Then, the iterative scheme of Rayleigh Quotient iteration method for $\Delta_3 z = a_3\Delta_0 z$ presented in [7], and is given below:

Step 1: Select starting guess x_1^0, x_2^0, x_3^0 with $\|z^0\| = 1$.

Step 2: Calculate the Rayleigh quotient $\rho_n(z) = \frac{(z^n)^T\Delta_3 z^n}{(z^n)^T\Delta_0 z^n}$.

Step 3: Solve the system $(\Delta_3 - \rho_n\Delta_0)z^{n+1} = \Delta_0 z^n$.

Step 4: Normalize z^{n+1} such that $z^{n+1} = \frac{z^{n+1}}{\|z^{n+1}\|}$.

Using above algorithm approximate value of $a_3$can be obtained. Then, 3PEP reduces to following 2PEP with three equations

$$E_1(a)x_1 := (M_1 - a_1M_{11} - a_2M_{12})x_1 = 0, \tag{2.4a}$$

$$E_2(a)x_2 := (M_2 - a_1 M_{21} - a_2 M_{22})x_2 = 0, \tag{2.4b}$$

$$E_3(a)x_3 := (M_3 - a_1 M_{31} - a_2 M_{32})x_3 = 0, \tag{2.4c}$$

where $M_i := M_{i0} - a_3 M_{i3}$, $i := 1,2,3$. Define the matrices $L_{10}, L_{1i} : \mathbb{C} \to \mathbb{C}$, for $i := 1,2$ such that

$$L_{10} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}; \quad L_{1i} = \begin{pmatrix} M_{1i} & 0 \\ 0 & M_{2i} \end{pmatrix}.$$

The first two equations of the system of equations (2.4)(a,b,c) can be replaced by a single matrix pencil as follows

$$P_1(a)z_1 := (L_{10} - a_1 L_{11} - a_2 L_{12})z_1 = 0, \tag{2.5}$$

where $z_1 = (x_1, x_2)^T$. Consider the following system

$$P_1(a)z_1 := (L_{10} - a_1 L_{11} - a_2 L_{12})z_1 = 0, \tag{2.6}$$

$$E_3(a)x_3 := (M_{30} - a_1 M_{31} - a_2 M_{32})x_3 = 0 \tag{2.7}$$

which is equivalent to the linear system (2.4) and is a 2PEP. The corresponding system of joint GEPs are given by

$$\Delta_i Z = a_i \Delta_0 Z, \quad i = 1,2, \tag{2.8}$$

where $Z := z_1 \otimes x_3$ and the operator matrices are defined as

$$\Delta_0 := L_{11} \otimes M_{32} - L_{12} \otimes M_{31},$$

$$\Delta_1 := L_{10} \otimes M_{32} - L_{12} \otimes M_{30}; \quad \Delta_2 := L_{11} \otimes M_{30} - L_{10} \otimes M_{31}. \tag{2.9}$$

Consider that 2PEP is nonsingular i.e. $|\Delta_0| \neq 0$. In the next section the conversation of 2PEP into a NEP with single parameter will be presented.

3. Nonlinearization of 2PEP

The 2PEP can be transform into a NEP of one parameter by nonlinearizing the later. The matrix pencil defined in equation (2.7) can be rewritten as parametrized GEP in terms of a_1. A family of continuous functions $\{h_i(a_1)\}$ is generated, defined by the eigenvalues of the matrix pencil (2.7) due to perturbation theory of eigenvalue problems, such that a_2 is the eigenvalue of a GEP. That is, the functions $h_i(a_1)$ and $g_i(a_1)$ can be viewed, as the solution to

$$(M_{30} - a_1 M_{31} - h_i(a_1) M_{32}) g_i(a_1) = 0, \tag{3.1}$$

$$s^T g_i(a_1) = 1 \tag{3.2}$$

for a fixed value of a_1 and for any given vector $s \in \mathbb{C}^k$. Here, normalization conditions in equation (3.2) have been introduced explicitly to define uniquely the corresponding eigenvector.

Since the right hand side of the equation (3.2) is an analytic function and therefore the normalized conditions in equation (3.2) are preferable over usual Euclidean normalization. Substituting a_2 by $h_i(a_1)$ in matrix pencil (2.6), it follows that the solution of 2PEP will satisfy

$$N(a_1)z_1 := (L_{10} - a_1 L_{11} - h_i(a_1) L_{12})z_1 = 0. \tag{3.3}$$

Thus, a_2 and the matrix pencil of the equation (2.6) have been eliminated from 2PEP. The problem define by the equation (4.2) is a NEP. Here, $N(a_1) = L_{10} - a_1 L_{11} - h_i(a_1) L_{12}$. The problem $N(a_1)z_1 := 0$ in fact a family of NEPs due to different nonlinearity of the functions $h_1, h_2, \dots, h_k$. The scalar $a_1 \in \mathbb{C}$ is called the eigenvalue it satisfy $N(a_1)z_1 = 0$ for some nonzero vector z_1. Such a nonzero vector z_1 is called eigenvector of NEP. The eigenvalue a_1 can be obtained as a root of the equation

$$\det(N(a_1)) = 0\,. \tag{3.4}$$

The eigenvalue $a_1^{(0)}$ of $N(a_1)z_1 = 0$ is said to be simple, when $\det(N(a_1))$ has simple zero at $a_1 = a_1^{(0)}$ [17]. NEP has wide applications in applied research e.g., [10, 37, 39]. Both theoretical and numerical aspects of NEP were well investigated by the researchers. Various numerical methods such as classical Newton's method [30] block type Newton method [26], disguised and quasi-Newton method [20], LU-decomposition method [11], modified Newton's method [8], residual inverse iteration [29], Jacobi-Davidson type methods [4], implicit determinant method [37], method using contour integrals [1, 3], Riesz-projected-based method [5], FEAST algorithm [14], Broyden's method [21] etc. were successfully applied for numerical solution of NEP. Recently, a rational approximation method [13] and a selection technique [17] were applied to NEP to evaluate several eigenvalues. For state of the art of iterative methods can be found in [16, 27, 28].

4. Equivalence and Existence of Nonlinearization of the 2PEP

The existence nonlinearization of the 2PEP into a single NEP is summarized as follows:

2PEP can be characterized explicitly by the eliminating the matrices M_{2i}, $i := 0,2$ from equation (2.7). Existence of a nonlinearization is closely related to the Jordan structure of GEP derived from (2.9) and is given by

$$(M_{20} - a_1 M_{21})x_3 = a_2 M_{22} x_3\,. \tag{4.1}$$

The equation (5.2) is important as it is useful for calculation of $a_2 = h_i(a_1)$ for a given a_1. Define the Jacobian matrix

$$J(a_1, a_2) := \begin{pmatrix} E_3(a_1, a_2) & -M_{22}x_3 \\ s^T & 0 \end{pmatrix}. \tag{4.2}$$

The Jacobian matrix, $J(a_1, a_2)$, defined in the equation (4.1) is generally nonsingular. But it is singular in non generic situation only. In [33], Ringh and Jarlebring showed that the Jacobian $J(a_1, a_2)$ is singular if and only if the GEP (4.1) has at least Jordan chain of length two or more. Thus, nonlinearization technique works if a solution to 2PEP corresponds to a simple eigenvalue a_2 of the GEP represented by the equation (4.1). Again, in [33] it is also showed that the Jacobian $J(a_1, a_2)$ is nonsingualar if we take the simple eigenvalue a_1 of the GEP such that s is not orthogonal to the corresponding eigenvector, which confirms the existence of nonlinerization. Therefore, the nonlinearization exists simple eigenvalues of GEP given in equation (4.1).

Again, to determine nontrivial solutions, the matrix pencil defined in equation (2.7) must satisfy the equation $\det(E_3(a_1,a_2) = 0$, which is a bivariate polynomial in a_1. Hence, the functions $h_i(a_1)$ presented in the NEP of the equation (3.3) are roots of certain polynomial. The coefficients of this polynomial are also polynomials in a_1. Thus, the functions $h_i(a_1)$ are algebraic. It is to be noted that the functions $h_i(a_1)$ are not always entire functions. These functions may have branch-point singularities, which create obstacles in computation of eigenvalues by the iterative methods. Relevant methods for nearest singularity are reported in the papers [22, 23].

5. Analysis of Condition Number

Condition number plays important role in the sensitivity analysis of eigenvalue problem. Norm-wise condition number of NEP, based on L_{1i}-matrices, $i = 0,1,2$ are generally used to measure the sensitivity of a_1 and is defined as [33].

$$\text{Cond}_{L_{1i}}(a_1) := \lim_{\varepsilon\to 0}\sup\left\{\frac{|\Delta a_1|}{\varepsilon} : \|L_{1j}\| = \varepsilon\gamma_j,\ j = 0,1,2\right\}, \tag{5.1}$$

where γ_j are scalars for $j = 0,1,2$ and Δa_1 is such that

$$((L_{10}+\Delta L_{10}) - (a_1+\Delta a_1)(L_{11}+\Delta L_{11}) - f(a_1+\Delta a_1)(L_{12}+\Delta L_{12})(z_1+\Delta z_1) = 0. \tag{5.2}$$

Denote u and z_1 as the left and right eigenvectors of the NEP respectively, then it follows [33] that

$$\text{Cond}_{L_{1i}}(a_1) := \|u\|\ \|z_1\|\frac{\Delta_1+\Delta_1|a_1|+\Delta_3|h(a_1)|}{|u^H N'(a_1)z_1|}. \tag{5.3}$$

The quantity $|u^H N'(a_1)z_1|$ present in the equation (5.3) plays crucial rule in the selection criteria of eigenvalues of NEP and again, $|u^H N'(a_1)z_1| \neq 0$ for simple eigenvalue [17].

Formula defined in equation (5.3) is further improved in [33] by considering the perturbations with respect to M_{3i}-matrices, $i = 0,1,2$ as follows:

$$\text{Cond}(a_1) := \lim_{\varepsilon\to 0}\sup\left\{\frac{|\Delta a_1|}{\varepsilon} : \|L_{1j}\| = \varepsilon\gamma_j \ \text{ and }\ \|M_{3j}\| = \varepsilon\mu_j,\ j = 0,1,2\right\}, \tag{5.4}$$

where μ_j, $j = 0,1,2$ are scalars and $\|\Delta a_1\|$ satisfies the equation (5.2) with a perturbed h such that $a_2+\Delta a_2 = h(a_1+\Delta a_1)$ with

$$\begin{aligned}&(M_{30}+\Delta M_{20}) - (a_1+\Delta a_1)(M_{31}+\Delta M_{31}) - (a_2+\Delta a_2)(M_{32}+\Delta M_{32})(x_3+\Delta x_3) = 0,\\ &1 = s^T(x_3+\Delta x_3).\end{aligned} \tag{5.5}$$

The perturbation of a_2 is analyzed subject to perturbations with respect to M_{3i}-matrices and fixing perturbations in a_1 by

$$\text{Cond}_h(a_1) := \lim_{\varepsilon\to 0}\sup\left\{\frac{|\Delta a_2|}{\varepsilon} : \|\Delta a_1\| = \varepsilon\gamma \ \text{ and }\ \|M_{3j}\| = \varepsilon\mu_j \text{ for } j := 0,1,2\right\}, \tag{5.6}$$

where Δa_2 satisfies (5.8) for a given a_1 and γ is a scalar.

Let u and x_3 be the left and right eigenvectors of GEP defined in equation (4.1) associated with simple eigenvalue $a_2 = f(a_1)$ for given $a_1 \in \mathbb{C}$. Then, the following relation is true [33].

$$\text{Cond}_h(a_1) = \text{Cond}_{h,M_{3i}}(a_1) + \text{Cond}_{h,a_1}(a_1),$$

where

$$\text{Cond}_{h,M_{3i}}(a_1) := \|u\|\ \|x_3\| \frac{\mu_1 + \mu_2|a_1| + \mu_3|h(a_1)|}{|u^H M_{32} x_3|} \tag{5.7}$$

and

$$\text{Cond}_{h,a_1}(a_1) := \gamma \frac{|u^H M_{31} x_3|}{|u^H M_{32} x_3|}. \tag{5.8}$$

Proposition 5.1 ([15])**.** *Let M be $p \times p$ complex matrix such that* $\text{Rank}(M) = p-1$. *Then,* $\text{adj}(M) = yx^*$ *for some vectors $0 \neq x$ and $0 \neq y$ such that $My = 0$ and $x^*M = 0$.*

Theorem 5.1. *Let u and z_1 be the left and right eigenvectors corresponding to the simple eigenvalue a_1 of the NEP defined in equation* (3.3)*. Moreover, let v and x_3 be the left and right eigenvectors corresponding to the simple eigenvalue $a_2 = f(a_1)$ of the GEP defined in equation* (4.1)*. Then,*

$$\text{Cond}(a_1) = \text{Cond}_{L_{1i}}(a_1) + \text{Cond}_{h,M_{3i}}(a_1) \frac{|u^H L_{12} z_1|}{|tr[adj(N(a_1)N'(a_1)]|}. \tag{5.9}$$

Proof. Using the [33, Theorem 4.2], we have

$$\text{Cond}(a_1) = \text{Cond}_{L_{1i}}(a_1) + \text{Cond}_{h,M_{3i}}(a_1) \frac{|u^H L_{12} z_1|}{|u^H N'(a_1) z_1|}. \tag{5.10}$$

Since the eigenvalue a_1 is simple and therefore $\text{Rank}(N(a_1) := n_3 - 1$.

Therefore, there exist vectors $0 \neq z_1$, $u \in \mathbb{C}_3^n$ such that $\text{adj}(N(a_1) := z_1 u^*$ with $N(a_1)z_1 = 0$ and $u^* N(a_1) = 0$.

This implies,

$$\text{tr}(\text{adj}(N(a_1)N'(a_1) := \text{tr}(z_1 u^H N'(a_1) = u^H N'(a_1) z_1.$$

Substituting it in the equation (5.10) proves Theorem 5.1. □

Theorem 5.2. *Under the assumptions and conditions of Theorem* 5.1, $\text{Cond}(a_1)$ *satisfies*

$$\text{Cond}(a_1) = \text{Cond}_{L_{1i}}(a_1) + \text{Cond}_{h,M_{3i}}(a_1) \frac{|u^H L_{12} z_1|}{|\kappa_1 \overline{\kappa}_2 q^T N'(a_1) p|}, \tag{5.11}$$

where κ_i; $(i = 1,2)$ are nonzero scalars, $0 \neq p, q$ are some right and left eigenvectors of the NEP $N(a_1)z_1 = 0$, respectively.

Proof. Let the eigenvalue a_1 be simple. Then, it is well known that $\text{Dim}(\text{Ker}(N(a_1) = 1$. Therefore, there exists scalars $0 \neq \kappa_i$; $i = 1,2$ such that $z_1 = \kappa_1 p$ and $u = \kappa_2 q$. This implies,

$$u^H N'(a_1) z_1 = (\kappa_2 q)^H N'(a_1)(\kappa_1 p) = \overline{\kappa}_2 \kappa_1 q^H N'(a_1) p.$$

Substituting it in the equation (5.10) proves Theorem 5.2.

Define the sequence $\{a_1^k\}_{k=0}^{\infty}$ such that it converges to some simple eigenvalue $a_1 \in \mathbb{C}$ which is computed by using iterative method of order $t \in \mathbb{N}$. Set the factor $c_n = \frac{a_1^{k+1}-a_1}{(a_1^k-a_1)^t}$. Then, c is termed as convergent factor and is defined as $c = \lim_{k\to\infty} c_k$, if the limit exists. Let, the matrix $N(a_1)$ is twice differentiable such that $u^H N''(a_1)z_1 \neq 0$, then convergent factor c is given by [39]

$$c = \frac{1}{2}\frac{u^H N''(a_1)z_1}{u^H N'(a_1)z_1}. \tag{5.12}$$

Using the equation (5.13), the expression $|u^H N'(a_1)z_1|$ present in the denominator part of the equation (5.10) can be expressed in terms of c. Thus, the equation (5.10) becomes

$$\operatorname{Cond}(a_1) = \operatorname{Cond}_{L_{1i}}(a_1) + 2c\operatorname{Cond}_{h,M_{3i}}(a_1)\frac{|u^H L_{12}z_1|}{|u^H N''(a_1)z_1|}. \tag{5.13}$$

□

5.1 Bounds of Cond(a_1, N)

Recall that, the matrix valued function $N(a_1) = L_{10} - a_1 L_{11} - h_i(a_1)L_{12}$. Here, $N(a_1)$ is an analytic function having Lipschitz derivative, which is continuous in real case. Assume that $(\hat{a}_1, \hat{z}_1, \hat{u})$ be the solution of GEP. Let $\hat{a}_1$ be simple eigenvalue. Consider the following neighborhood

$$M(a_1,\sigma) = \{a_1 \in \mathbb{C} : |a_1 - \hat{a}_1| < \sigma\}$$

Also, set the following:

$$\chi_\xi(z_1) = \{z_1 \in \mathbb{C}^N : \angle(\operatorname{Span}\{z_1\}, \operatorname{Span}\{\hat{z}_1\}) = \varepsilon\}$$

and

$$\chi_\xi(u) = \{u \in \mathbb{C}^N : \angle(\operatorname{Span}\{u\}, \operatorname{Span}\{\hat{u}\}) = \sigma\}.$$

Then, the functional defined by $r : \chi_\xi(z_1) \times \chi_\xi(u) \to M(a_1,\sigma)$, is said to be two sided [39] Rayleigh functional if the following conditions holds:

(i) $r(lz_1, mu) = r(z_1, u)$ for any non zero scalars $l, m \in \mathbb{C}$,

(ii) $u^H N(r(z_1,u))z_1 = 0$,

(iii) $u^H N'(r(z_1,u))z_1 \neq 0$,

for every $z_1 \in \chi_\xi(z_1)$ and $u \in \chi_\xi(u)$.

Theorem 5.3 ([39]). *Let $|\hat{u}^H N'(\hat{a}_1)\hat{z}_1| \neq 0$. Then, there exists $\varepsilon > 0$ and $\eta > 0$ such that for which the following inequality is true.*

$$|r(z_1,u) - \hat{a}_1| = \frac{8}{3}\frac{\|N(\hat{a}_1)\|}{|\hat{u}^H N'(\hat{a}_1)z_1|}\tan(\xi)\tan(\varepsilon), \tag{5.14}$$

where $\xi = \angle(\operatorname{Span}\{z_1\}, \operatorname{Span}\{\hat{z}_1\})$, $\eta = \angle(\operatorname{Span}\{u\}, \operatorname{Span}\{\hat{u}\})$ and $\hat{u}$, $\hat{z}_1$ are the left and right eigenvectors of the NEP corresponding to the eigenvalue $\hat{a}_1$ such that $\|\hat{u}\| = \|\hat{z}_1\| = 1$.

Theorem 5.4 ([39]). *Let $\hat{u}$, $\hat{z}_1$be the left and right eigenvectors of the NEP corresponding to the eigenvalue $\hat{a}_1$such that $\|\hat{u}\| = \|\hat{z}_1\| = 1$ and $|\hat{u}^H N'(\hat{a}_1)\hat{z}_1| \neq 0$. Assume that $\xi < \frac{p}{3}$ and $\eta < \frac{p}{3}$. Then,* *the following inequality holds.*

$$|r(z_1,u)-\widehat{a}_1| = \frac{32}{3}\frac{\|N(\widehat{a}_1)\|}{|\widehat{u}^H N'(\widehat{a}_1)z_1|}\|z_1-z_1^*\|\ \|u-u^*\|.$$

Remark 5.1. Let $\widehat{u}$, $\widehat{z}_1$ be the left and right eigenvectors corresponding to the eigenvalue $\widehat{a}_1$of the NEP defined in equation (3.3), such that $\|\widehat{u}\| = \|\widehat{z}_1\| = 1$ and $|\widehat{u}^H N'(\widehat{a}_1)\widehat{z}_1| \neq 0$. Then, Cond$(a_1)$ satisfy the following inequalities.

(i) There exists $\varepsilon > 0$ and $\sigma > 0$ such that

$$\text{Cond}(a_1) = \text{Cond}_{L_{1i}}(a_1) + \frac{3}{8}\frac{\text{Cond}_{h,M_{3i}}(a_1)|r(z_1,u)-a_1|\ |u^H L_{12}z_1|}{\|N(a_1)\|\tan(\xi).\tan(\eta)}$$

where $\xi = \angle(\text{Span}\{z_1\},\text{Span}\{\widehat{z}_1\})$ and $\eta = \angle(\text{Span}\{u\},\text{Span}\{\widehat{u}\})$,

(ii) For $\xi < \frac{p}{3}$ and $\eta < \frac{p}{3}$, we have

$$\text{Cond}(a_1) = \text{Cond}_{L_{1i}}(a_1) + \frac{3}{32}\frac{\text{Cond}_{h,M_{3i}}(a_1)|r(z_1,u)-a_1|\ |u^H L_{12}z_1|}{\|N(a_1)\|\ \|z_1-z_1^*\|\ \|u-u^*\|}.$$

Numerical Example. Consider the following nonsingular 3PEP given by the system of equations (5.15)-(5.17).

$$\begin{pmatrix}1&5&7\\6&10&4\\11&5&9\end{pmatrix}x_1 = \left\{a_1\begin{pmatrix}8&1&0\\9&11&0\\0&7&5\end{pmatrix}+a_2\begin{pmatrix}-4&-1&7\\6&-2&5\\-3&0&6\end{pmatrix}+a_3\begin{pmatrix}11&2&-3\\4&12&-6\\7&4&5\end{pmatrix}\right\}x_1, \tag{5.15}$$

$$\begin{pmatrix}7&0&0\\0&9&0\\0&0&-12\end{pmatrix}x_2 = \left\{a_1\begin{pmatrix}13&5&-2\\1&9&8\\6&-17&4\end{pmatrix}+a_2\begin{pmatrix}7&17&4\\8&10&15\\3&5&4\end{pmatrix}+a_3\begin{pmatrix}21&6&-5\\9&10&3\\8&6&9\end{pmatrix}\right\}x_2, \tag{5.16}$$

$$\begin{pmatrix}10&12&-7\\-4&11&-6\\5&13&2\end{pmatrix}x_3 = \left\{a_1\begin{pmatrix}23&6&4\\-3&5&2\\2&3&-10\end{pmatrix}+a_2\begin{pmatrix}5&9&12\\-3&11&4\\-7&8&2\end{pmatrix}+a_3\begin{pmatrix}7&25&3\\12&-5&4\\4&9&8\end{pmatrix}\right\}x_3. \tag{5.17}$$

The operator matrix Δ_0 is nonsingular and hence the problem considered above is nonsingular. Here $a_3 = 0.4213$ by Rayleigh Quotient iteration method. Now, 3PEP reduces to following 2PEP with three equations

$$\begin{pmatrix}-3.6343&4.1574&8.2639\\4.3148&4.9444&6.5278\\8.0509&3.3148&6.8935\end{pmatrix}x_1 = \left\{a_1\begin{pmatrix}8&1&0\\9&11&0\\0&7&5\end{pmatrix}+\ a_2\begin{pmatrix}-4&-1&7\\6&-2&5\\-3&0&6\end{pmatrix}\right\}x_1, \tag{5.18}$$

$$\begin{pmatrix}-1.8473&-2.5278&2.1065\\-3.7917&4.7870&-1.2639\\-3.3704&-2.5278&-15.7917\end{pmatrix}x_2 = \left\{a_1\begin{pmatrix}13&5&-2\\1&9&8\\6&-17&4\end{pmatrix}+a_2\begin{pmatrix}7&17&4\\8&10&15\\3&5&4\end{pmatrix}\right\}x_2, \tag{5.19}$$

$$\begin{pmatrix}7.0509&1.4675&-8.2639\\-9.0556&13.1063&-7.6852\\3.3148&9.2083&-1.3704\end{pmatrix}x_3 = \left\{a_1\begin{pmatrix}23&6&4\\-3&5&2\\2&3&-10\end{pmatrix}+a_2\begin{pmatrix}5&9&12\\-3&11&4\\-7&8&2\end{pmatrix}\right\}x_3. \tag{5.20}$$

Combining the first two equations (5.18) and (5.19), the following single matrix equation of the form (2.6) can be obtained

$$\begin{pmatrix} -3.6343 & 4.1574 & 8.2639 & 0 & 0 & 0 \\ 4.3148 & 4.9444 & 6.5278 & 0 & 0 & 0 \\ 8.0509 & 3.3148 & 6.8935 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.8473 & -2.5278 & 2.1065 \\ 0 & 0 & 0 & -3.7917 & 4.7870 & -1.2639 \\ 0 & 0 & 0 & -3.3704 & -2.5278 & -15.7917 \end{pmatrix} z_1$$

$$= a_1 \begin{pmatrix} 8 & 1 & 0 & 0 & 0 & 0 \\ 9 & 11 & 0 & 0 & 0 & 0 \\ 0 & 7 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13 & 5 & -2 \\ 0 & 0 & 0 & 1 & 9 & 8 \\ 0 & 0 & 0 & 6 & -17 & 4 \end{pmatrix} + a_2 \begin{pmatrix} -4 & -1 & 7 & 0 & 0 & 0 \\ 6 & -2 & 5 & 0 & 0 & 0 \\ -3 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 17 & 4 \\ 0 & 0 & 0 & 8 & 10 & 15 \\ 0 & 0 & 0 & 3 & 5 & 4 \end{pmatrix} z_1, \tag{5.21}$$

where $z_1 = (x_1, x_2)^T$. Equation (5.20) together with equation (5.21) yields a 2PEP of the form defined in equations (2.6) and (2.7). Respective NEP can be obtained using nonlinearization technique presented in Section 3.

6. Conclusion

In this paper, a general framework of nonlinearizing technique of the 3PEPs into a single parameter have been discussed. Moreover, an estimation of bounds of condition number Cond(a_1) of simple eigenvalue is also shown using two sided Rayleigh functional. The nonlinearization techniques presented in this paper have some limitations. The existence of the nonlinearization depends on the fact that the chosen eigenvalue of the GEP defined by the equation (4.1) need to be simple. Moreover, the nonlinear functions h_i involved in the study are algebraic functions, and therefore may contain singularities which may create problem in the implementation of numerical scheme. Again, to find the general solution of the problem (1.1) including the singular case gives us a new direction to develop other novel direct methods. This may be considered as future prospect in the research area of k-parameter eigenvalue problem.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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