Communications in Mathematics and Applications
Vol. 13, No. 3, pp. 1047–1059, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i3.1775

http://www.rgnpublications.com

Research Article

Triple Invariant Point Theorems with PPF Dependence for Contractive Type Mappings

Savita Rathee and Neelam Kumari*

Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, Haryana, India
***Corresponding author:** neelamjakhar45@gmail.com

Received: December 28, 2021 **Accepted:** September 21, 2022

Abstract. In this paper, some results concerning the existence and uniqueness of triple invariant point with PPF dependence for non linear mapping in partially ordered complete metric spaces using the domain space $C[[a,b],E]$ that is distinct from the range E. Our results generalize and extend recent coupled invariant point theorems with PPF dependence founded by Drici *et al*. (Fixed point theorems in partially ordered metric spaces for operators with PPF dependence, *Nonlinear Analysis* **67** (2007), 641 – 647).

Keywords. Triple invariant point, PPF dependence, Existence and uniqueness, Metric space

Mathematics Subject Classification (2020). 47H10, 54H25, 54E50

1. Introduction

Many problems in several branches of mathematics are well known to be transformed into invariant point problems in the form $Tx = x$ for self mapping T. Ran and Reurings [10] investigated the existence of invariant point in partially ordered sets. This study was continued by Bhaskar and Lakshmikantham in [3]. In partially ordered metric space, they proved some interesting coupled invariant point theorems. The idea of tripled invariant point for nonlinear mapping in partially ordered complete metric spaces was introduced by Berinde and Borcut [2].

Bernfeld *et al*. [1], on the other hand, presented the idea of PPF (*Past-Present-Future*) dependent invariant point which is one form of invariant points for nonself mapping. In 2007, Drici *et al*. [5] developed invariant point theorems of a nonlinear operator, in which the domain space is different from range space. $E_0 = C[[a,b],E]$ is the domain space and E is the range, which is partial order metric space. After that, they further extend the results of invariant point with PPF dependence in coupled invariant point with PPF dependence in [6].

In this article, we extend and generalize the outcomes of Dric *et al*. [6], and Vasile Berinde and Marin Borcut [2] and we will prove the results for existence and uniqueness of triple invariant point with PPF dependence in Partially ordered complete metric spaces.

2. Preliminaries

Here, we provide the relevant definitions and findings for different spaces that will be helpful for further explanation.

Definition 2.1 ([2]). A point $\phi \in E_0$ is said to be *PPF dependent invariant point* or an invariant point with PPF dependence of a nonself mapping $T : E_0 \to E$ if $T(\phi) = \phi(c)$ for some $c \in I$.

Definition 2.2 ([6]). Assume $H : E_0 \times E_0 \to E$ is such that $H(\phi,\phi) = T\phi$, where $\phi \in E_0$. If for $\phi_1, \phi_2 \in E_0$, $H(\phi_1,\psi) \le H(\phi_2,\psi)$ whenever $\phi_1 \le \phi_2$, and for $\psi_1, \psi_2 \in E_0$, $H(\phi,\psi_1) \ge H(\phi,\psi_2)$ whenever $\psi_1 \le \psi_2$, we say that H has the mixed monotone property.

Definition 2.3 ([6]). Let $H : E_0 \times E_0 \to E$. An element $(\phi^*,\psi^*) \in E_0 \times E_0$ is said to be a coupled invariant point with PPF dependence of H if $H(\phi^*,\psi^*) = \phi^*(c)$ and $H(\psi^*,\phi^*) = \psi^*(c)$ for some for some $c \in I$.

Now, we mention the existence outcomes in [6].

Theorem 2.4 ([6]). *Suppose $H : E_0 \times E_0 \to E$ is a continuous mapping having the mixed monotone property.*
Assume that there exist a $k \in [0,1)$ with $d[H(\phi,\psi), H(\psi,\phi)] \le k d_0(\phi,\psi)$.
If there exist $\alpha_0, \beta_0 \in E_0$ such that

$$\alpha_0(c) \le H(\alpha_0,\beta_0) \text{ and } \beta_0(c) \ge H(\beta_0,\alpha_0)$$

then there exist $\phi^*, \psi^* \in E_0$ such that $\phi^*(c) = H(\phi^*,\psi^*)$ and $\psi^*(c) = H(\psi^*,\phi^*)$.

Theorem 2.5 ([6]). *Assume that $H : E_0 \times E_0 \to E$ is a mapping having the mixed monotone property. If there exist a $k \in [0,1)$ with $d[H(\phi,\psi), H(\psi,\phi)] \le k d_0(\phi,\psi)$ and $\alpha_0, \beta_0 \in E_0$ such that*

$$\alpha_0(c) \le H(\alpha_0,\beta_0) \quad \text{and} \quad \beta_0(c) \ge H(\beta_0,\alpha_0).$$

Suppose further that $E_0 \times E_0$ has the following property:
(ϕ_n,ψ_n) is a sequence in $E_0 \times E_0$ such that ϕ_n is a nondecreasing and converges to ϕ and ψ_n is a non increasing and converges to ψ implies $\phi_n \le \phi$, $\psi \le \psi_n$ for all n. Then H has a coupled invariant point.

Theorem 2.6 ([6]). *In addition to the assumption of Theorem 2.4 or Theorem 2.5, suppose that every pair of elements in $E_0 \times E_0$ has either an upper bound or a lower bound, i.e., for every $(\phi_1,\psi_1), (\phi_2,\psi_2) \in E_0 \times E_0$ there exist a $(\gamma_1,\gamma_2) \in E_0 \times E_0$ which is comparable to the given vectors.*

Furthermore, if

$$\Omega_{\binom{\phi^*}{\psi^*}} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in E_0 \ : \ \begin{pmatrix} d_0(\phi,\phi^*) \\ d_0(\psi,\psi^*) \end{pmatrix} = \begin{pmatrix} d(\phi(c),\phi^*(c)) \\ d(\psi(c),\psi^*(c)) \end{pmatrix},$$

where $\binom{\phi^*}{\psi^*}$ *is a coupled invariant point of* H*, then* $\binom{\phi^*}{\psi^*}$ *is the only coupled invariant point of* H *in* $\Omega_0\binom{\phi^*}{\psi^*}$.

3. Main Results

Consider the partially ordered metric space (E,d). Suppose $E_0 = C[[a,b],E]$ is the set of all continuous from $[a,b]$ to E. Let T be a non self mapping from E_0 to E. Then the term "*invariant point of* T" refers to a point $\phi \in E_0$ where $T\phi = \phi(c)$ for some $c \in [a,b]$. Consider on the product space $E_0 \times E_0 \times E_0$ the following partial order hold:
For $(\phi,\psi,\xi),(f,g,h) \in E_0 \times E_0 \times E_0$,

$$(f,g,h) \le (\phi,\psi,\xi) \iff \phi \ge f,\ \psi \le g,\ \xi \ge h.$$

Definition 3.1. Consider (E,b) is a partially ordered metric space and $H : E_0 \times E_0 \times E_0 \to E$ where

$$H(\phi,\phi,\phi) = T\phi, \quad \phi \in E_0.$$

As any $\phi,\psi,\xi \in E_0$,

$$\phi_1,\phi_2 \in E_0, \quad \text{if } \phi_1 \le \phi_2 \text{ then } H(\phi_1,\psi,\xi) \le H(\phi_2,\psi,\xi),$$

$$\psi_1,\psi_2 \in E_0, \quad \text{if } \psi_1 \le \psi_2 \text{ then } H(\phi,\psi_1,\xi) \ge H(\phi,\psi_2,\xi)$$

and

$$\xi_1,\xi_2 \in E_0, \quad \text{if } \xi_1 \le \xi_2 \text{ then } H(\phi,\psi,\xi_1) \le H(\phi,\psi,\xi_2)$$

then we say that H has the mixed monotone property.

Definition 3.2. Let $H : E_0 \times E_0 \times E_0 \to E$. An element $\begin{pmatrix} \phi^* \\ \psi^* \\ \xi^* \end{pmatrix}$ is called a triple invariant point with PPF dependence of H if

$$H(\phi^*,\psi^*,\xi^*) = \phi^*(c),\ H(\psi^*,\phi^*,\psi^*) = \psi^*(c) \text{ and } H(\xi^*,\psi^*,\phi^*) = \xi^*(c) \quad \text{for some } c \in [a,b].$$

Theorem 3.3. *Consider* (E,d) *is a partially ordered complete metric space.* T *is a non self mapping from* E_0 *to* E*. Suppose* $H : E_0 \times E_0 \times E_0 \to E$*. Assume that*

(i) *H is continuous*

(ii) *H satisfies the mixed monotone property*

(iii) *$\exists$ constants $j,k,l \in [0,1)$ pleasing $j+k+l \le 1$ for which*

$$d(H(\phi,\psi,\xi),H(f,g,h)) \le jd(\phi(c),f(c)) + kd(\psi(c),g(c)) + ld(\xi(c),h(c)),$$
$$\forall\ \phi \ge f,\ \psi \le g,\ \xi \ge h. \tag{3.1}$$

(iv) *If* $\exists\ \phi_0,\psi_0,\xi_0 \in E_0$ *such that*

$$\phi_0(c) \le H(\phi_0,\psi_0,\xi_0),\ \psi_0(c) \ge H(\psi_0,\phi_0,\psi_0) \ \textit{and}\ \ \xi_0(c) \le H(\xi_0,\psi_0,\phi_0).$$

Then, $\exists\ \phi_0,\psi_0,\xi_0 \in E_0$ *as in*

$$\phi^*(c) = H(\phi^*,\psi^*,\xi^*),\ \psi^*(c) = H(\psi^*,\phi^*,\psi^*) \ \textit{and}\ \ \xi^*(c) = H(\xi^*,\psi^*,\phi^*) \quad \textit{for some } c \in [a,b].$$

Proof. Suppose $T\phi_0 = \phi_1(c)$, $c \in [a,b]$ for any $\phi_1 \in E_0$.
Let us denote

$$\phi_1(c) = H(\phi_0,\psi_0,\xi_0) = T\phi_0 \ge \phi_0(c),$$
$$\psi_1(c) = H(\psi_0,\phi_0,\psi_0) = T\psi_0 \le \psi_0(c),$$

and

$$\xi_1(c) = H(\xi_0,\psi_0,\phi_0) = T\xi_0 \ge \xi_0(c).$$

For $n \ge 1$, denote

$$\phi_n(c) = H(\phi_{n-1},\psi_{n-1},\xi_{n-1}),\ \psi_n(c) = H(\psi_{n-1},\phi_{n-1},\psi_{n-1}) \ \text{ and }\ \xi_n(c) = H(\xi_{n-1},\psi_{n-1},\phi_{n-1}). \quad (3.2)$$

Due to the mixed monotone property we can easily show that

$$\phi_2(c) = H(\phi_1,\psi_1,\xi_1) \ge H(\phi_0,\psi_0,\xi_0) = \phi_1(c),$$
$$\psi_2(c) = H(\psi_1,\phi_1,\psi_1) \le H(\psi_0,\phi_0,\psi_0) = \psi_1(c),$$
$$\psi_2(c) = H(\xi_1,\psi_1,\phi_1) \le H(\xi_0,\psi_0,\phi_0) = \xi_1(c).$$

Then, we obtain the following conditions

$$\phi_0(c) \le \phi_1(c) \le \ldots \le \phi_n(c) \le \ldots,$$
$$\psi_0(c) \ge \psi_1(c) \ge \ldots \ge \psi_n(c) \le \ldots,$$
$$\xi_0(c) \le \xi_1(c) \le \ldots \le \xi_n(c) \le \ldots.$$

For simplification we denote

$$D_n^{\phi} = d(\phi_{n-1}(c),\phi_n(c)),\ \ D_n^{\psi} = d(\psi_{n-1}(c),\psi_n(c)),\ \ D_n^{\xi} = d(\xi_{n-1}(c),\xi_n(c)).$$

By inequality (3.1) we have

$$\begin{aligned} D_2^{\phi} = d(\phi_1(c),\phi_2(c)) &= d(H(\phi_0,\psi_0,\xi_0),(\phi_1,\psi_1,\xi_1)) \\ &\le jd(\phi_0(c),\psi_0(c)) + kd(\psi_0(c),\psi_1(c)) + ld(\xi_0(c),\xi_1(c)) \\ &= jD_1^{\phi} + kD_1^{\psi} + lD_1^{\xi}. \end{aligned}$$

Similarly, we obtain

$$D_2^{\psi} \le (j+l)D_1^{\psi} + kD_1^{\phi} + 0.D_1^{\xi},$$
$$D_2^{\xi} \le jD_1^{\xi} + kD_1^{\psi} + lD_1^{\phi}$$

and

$$D_3^{\phi} \le (j^2+k^2+l^2)D_1^{\phi} + (2jk+2kl)D_1^{\psi} + 2jlD_1^{\xi},$$
$$D_3^{\psi} \le (kl+2jk)D_1^{\phi} + ((j+l)^2+k^2))D_1^{\psi} + klD_1^{\xi},$$

$$D_3^{\xi} \le (2jl+k^2)D_1^{\phi} + (2jk+2kl)D_1^{\psi} + (j^2+l^2)D_1^{\xi}.$$

To make writing easier, suppose

$$A = \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix}$$

represented by $\begin{pmatrix} x_1 & y_1 & z_1 \\ u_1 & v_1 & w_1 \\ s_1 & y_1 & t_1 \end{pmatrix}$ and

$$A^2 = \begin{pmatrix} j^2+k^2+l^2 & 2jk+2kl & 2jl \\ kl+2jk & (j+l)^2+k^2 & kl \\ 2jl+k^2 & 2jk+2kl & j^2+l^2 \end{pmatrix} = \begin{pmatrix} x_2 & y_2 & z_2 \\ u_2 & v_2 & w_2 \\ s_2 & y_2 & t_2 \end{pmatrix},$$

where $x_2+y_2+z_2 = s_2+y_2+t_2 = u_2+v_2+w_2 = (j+k+l)^2 < 1$ because $j+k+l<1$, and then by mathematical induction we will show that

$$A^n = \begin{pmatrix} x_n & y_n & z_n \\ u_n & v_n & w_n \\ s_n & y_n & t_n \end{pmatrix},$$

where

$$x_n+y_n+z_n = u_n+v_n+w_n = s_n+y_n+t_n = (j+k+l)^n < 1. \tag{3.3}$$

For this, if inequality (3.3) holds for n, then

$$\begin{aligned} A^{n+1} &= A^n A \\ &= \begin{pmatrix} x_n & y_n & z_n \\ u_n & v_n & w_n \\ s_n & y_n & t_n \end{pmatrix} \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \\ &= \begin{pmatrix} jx_n+ky_n+lz_n & kx_n+(j+l)y_n+kz_n & lx_n+jz_n \\ ju_n+kv_n+lw_n & ku_n+(j+l)v_n+kw_n & lu_n+jw_n \\ js_n+ky_n+lt_n & ks_n+(j+l)y_n+kt_n & ls_n+jt_n \end{pmatrix}. \end{aligned}$$

We have

$$\begin{aligned} x_{n+1}+y_{n+1}+z_{n+1} &= x_nj+y_nk+z_nl+x_nk+y_nj+z_nk+x_nl+y_nl+z_nj \\ &= x_n(j+k+l)+y_n(j+k+l)+z_n(j+k+l) \\ &= (x_n+y_n+z_n)(j+k+l) \\ &= (j+k+l)^n(j+k+l) \\ &= (j+k+l)^{n+1} \\ &< j+k+l < 1. \end{aligned}$$

Likewise, we have

$$u_{n+1}+v_{n+1}+w_{n+1} = s_{n+1}+y_{n+1}+t_{n+1} = (j+k+l)^{n+1} < j+k+l < 1.$$

Hence, we get

$$\begin{pmatrix} D^{\phi}_{n+1} \\ D^{\psi}_{n+1} \\ D^{\xi}_{n+1} \end{pmatrix} \le \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \begin{pmatrix} D^{\phi}_{1} \\ D^{\psi}_{1} \\ D^{\xi}_{1} \end{pmatrix}$$

that is

$$D^{\phi}_{n+1} \le x_n D^{\phi}_1 + y_n D^{\psi}_1 + z_n D^{\xi}_1, \tag{3.4}$$

$$D^{\psi}_{n+1} \le u_n D^{\phi}_1 + v_n D^{\psi}_1 + w_n D^{\xi}_1, \tag{3.5}$$

$$D^{\xi}_{n+1} \le s_n D^{\phi}_1 + y_n D^{\psi}_1 + t_n D^{\xi}_1. \tag{3.6}$$

By using these three inequalities, it is simple to prove that ϕ_n, ψ_n and ξ_n are Cauchy sequences. For $m > n$ we have

$$\begin{aligned} d(\phi_m,\phi_n) &\le d(\phi_m,\phi_{m-1}) + \ldots + d(\phi_{n+1},\phi_n) \\ &= D^{\phi}_m + D^{\phi}_{m-1} + \ldots + D^{\phi}_{n+1} \\ &\le x_{m-1}D^{\phi}_1 + y_{m-1}D^{\psi}_1 + z_{m-1}D^{\xi}_1 + x_{m-2}D^{\phi}_1 + y_{m-2}D^{\psi}_1 + z_{m-2}D^{\xi}_1 + \ldots + x_nD^{\phi}_1 + y_nD^{\psi}_1 + z_nD^{\xi}_1 \\ &= (x_n + x_{n+1} + \ldots + x_{m-1})D^{\phi}_1 + (y_n + y_{n+1} + \ldots + y_{m-1})D^{\psi}_1 + (z_n + z_{n+1} + \ldots + z_{m-1})D^{\xi}_1 \\ &\le (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1})D^{\phi}_1 + (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1})D^{\psi}_1 + (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1})D^{\xi}_1 \\ &= (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1})(D^{\phi}_1 + D^{\psi}_1 + D^{\xi}_1) \\ &= \beta^n \frac{1-\beta^{m-n}}{1-\beta}(D^{\phi}_1 + D^{\psi}_1 + D^{\xi}_1), \end{aligned}$$

where $\beta = j+k+l < 1$, which implies ϕ_n is a Cauchy sequence.
On the same way we can show that ψ_n and ξ_n are also Cauchy sequences.
Due to the completeness of E_0, there exist $\phi,\psi,\xi \in E_0$ such that

$$\lim_{n\to\infty} \phi_n = \phi, \quad \lim_{n\to\infty} \psi_n = \psi, \quad \lim_{n\to\infty} \xi_n = \xi \tag{3.7}$$

and

$$\lim_{n\to\infty} T\phi_n = \lim_{n\to\infty} \phi_{n+1} = \phi(c), \quad \lim_{n\to\infty} T\psi_n = \lim_{n\to\infty} \psi_{n+1} = \psi(c), \quad \lim_{n\to\infty} T\xi_n = \lim_{n\to\infty} \xi_{n+1} = \xi(c).$$

Now, we claim that

$$\phi(c) = H(\phi,\psi,\xi), \quad \psi(c) = H(\psi,\phi,\psi) \text{ and } \xi(c) = H(\xi,\psi,\phi).$$

Let $\epsilon > 0$. Because of continuity of H at (ϕ,ψ,ξ) for a given $\frac{\epsilon}{3} > 0$, $\exists$ a $\delta > 0$ such that

$$d(\phi(c), f(c)) + d(\psi(c), g(c)) + d\xi(c), h(c)) < \delta \Rightarrow d(H(\phi,\psi,\xi), H(f,g,h)) < \frac{\epsilon}{3}.$$

Then by (3.7) it follows that for $\zeta = \min\left(\frac{\epsilon}{3}, \frac{\delta}{3}\right)$, there exist q_0, r_0, s_0 such that, for $q \ge q_0$, $r \ge r_0$, $s \ge s_0$ we get

$$d(\phi_n(c),\phi(c)) < \zeta, \quad d(\psi_n(c),\psi(c)) < \zeta, \quad d(\xi_n(c),\xi(c)) < \zeta.$$

Now let $t_0 = \max(q_0, r_0, s_0)$.
For any $n \ge t_0$, we have

$$d(H(\phi,\psi,\xi),\phi(c)) \le d(H(\phi,\psi,\xi),\phi_{n+1}(c)) + d(\phi_{n+1}(c),\phi(c))$$

$$= d(H(\phi,\psi,\xi),H(\phi_n,\psi_n,\xi_n)) + d(\phi_{n+1}(c),\phi(c))$$
$$< \frac{\epsilon}{3} + \zeta \le \epsilon.$$

Hence $H(\phi,\psi,\xi) = \phi(c)$. Similarly, we can prove that $\psi(c) = H(\psi,\phi,\psi)$ and $\xi(c) = H(\xi,\psi,\phi)$. □

Theorem 3.4. *Consider (E,d) is a partially ordered complete metric space and T is a continuous mapping from E_0 to E. Suppose $H : E_0 \times E_0 \times E_0 \to E$. Assume that*

(i) *H satisfies the mixed monotone property.*

(ii) *Assume that E_0 possesses the following characteristics:*

  (a) *for a nondecreasing sequence $\{\phi_n\} \to \phi$, $\phi_n \le \phi$, $\forall\ n \in \mathbb{N}$,*

  (b) *for a non increasing sequence $\{\psi_n\} \to \psi$, $\psi_n \ge \psi$, $\forall\ n \in \mathbb{N}$.*

(iii) *$\exists$ constants $j,k,l \in [0,1)$ where $j+k+l \le 1$ as well as*

$$d(H(\phi,\psi,\xi),H(f,g,h)) \le jd(\phi(c),f(c)) + kd(\psi(c),g(c)) + ld(\xi(c),h(c)),$$
$$\forall\ \phi \ge f, \psi \le g, \xi \ge h. \tag{3.8}$$

(iv) *If there exist $\phi_0,\psi_0,\xi_0 \in E_0$ such that*

$$\phi_0(c) \le H(\phi_0,\psi_0,\xi_0),\quad \psi_0(c) \ge H(\psi_0,\phi_0,\psi_0) \ \text{and}\ \ \xi_0(c) \le H(\xi_0,\psi_0,\phi_0).$$

Then there exist $\phi_0,\psi_0,\xi_0 \in E_0$ such that

$$\phi^*(c) = H(\phi^*,\psi^*,\xi^*),\quad \psi^*(c) = H(\psi^*,\phi^*,\psi^*) \ \text{and}\ \ \xi^*(c) = H(\xi^*,\psi^*,\phi^*) \ \text{for some}\ c \in [a,b].$$

Proof. For this theorem, we only have to prove $\phi(c) = H(\phi,\psi,\xi)$, $\psi(c) = H(\psi,\phi,\psi)$ and $\xi(c) = H(\xi,\psi,\phi)$.

Let $\epsilon > 0$. Since

$$\lim_{n\to\infty} H^n(\phi_0,\psi_0,\xi_0) = \phi(c),\quad \lim_{n\to\infty} H^n(\psi_0,\phi_0,\psi_0) = \psi(c),\quad \lim_{n\to\infty} H^n(\xi_0,\psi_0,\phi_0) = \xi(c).$$

There exist $n_1,n_2,n_3 \in \mathbb{N}$ for some n,m,p such that $n \ge n_1$, $m \ge n_2$, $p \ge n_3$, we have

$$d(H^n(\phi_0,\psi_0,\xi_0),\phi(c)) < \frac{\epsilon}{4},\quad d(H^m(\psi_0,\phi_0,\psi_0),\psi(c)) < \frac{\epsilon}{4},\quad d(H^p(\xi_0,\psi_0,\phi_0),\xi(c)) < \frac{\epsilon}{4}.$$

Take $n \ge \{n_1,n_2,n_3\}$ and by using

$$H^n(\phi_0,\psi_0,\xi_0) \le \phi(c),\quad H^n(\psi_0,\phi_0,\psi_0) \ge \psi(c),\quad H^n(\xi_0,\psi_0,\phi_0,\xi(c) \le \xi_c,$$

we get

$$\begin{aligned}
d(H(\phi,\psi,\xi),\phi(c)) &\le d(H(\phi,\psi,\xi),H^{n+1}(\phi_0,\psi_0,\xi_0)) + d(H^{n+1}(\phi_0,\psi_0,\xi_0),\phi(c)) \\
&= d(H(\phi,\psi,\xi),H(H^n(\phi_0,\psi_0,\xi_0),(H^n(\psi_0,\phi_0,\psi_0),(H^n(\xi_0,\psi_0,\phi_0))) \\
&\quad + d(H^{n+1}(\phi_0,\psi_0,\xi_0),\phi(c)) \\
&\le jd(\phi(c),H^n(\phi_0,\psi_0,\xi_0)) + kd(\psi(c),H^n(\psi_0,\phi_0,\psi_0)) + ld(\xi(c),H^n(\xi_0,\psi_0,\phi_0)) \\
&\quad + d(\phi(c),H^{n+1}(\phi_0,\psi_0,\xi_0)) \\
&\le d(\phi(c),H^n(\phi_0,\psi_0,\xi_0)) + d(\psi(c),H^n(\psi_0,\phi_0,\psi_0)) + d(\xi(c),H^n(\xi_0,\psi_0,\phi_0)) \\
&\quad + d(\phi(c),H^{n+1}(\phi_0,\psi_0,\xi_0))
\end{aligned}$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

This implies that $H(\phi,\psi,\xi) = \phi(c)$.
On the same way we can prove that $d(\psi(c), H(\psi,\phi,\psi)) < \epsilon$ and $d(\xi(c), H(\xi,\psi,\phi)) < \epsilon$.
Hence $H(\psi,\phi,\psi) = \psi(c)$ and $H(\xi,\psi,\phi) = \xi(c)$. □

Now we can prove that tripled PPF dependent invariant point is unique by adding some extra property in above two Theorems.

Theorem 3.5. *With the hypothesis of Theorem* 3.3 *let us suppose the following condition: For every* $(\phi,\psi,\xi),(\phi_1,\psi_1,\xi_1) \in E_0 \times E_0 \times E_0$, $\exists$ *a* $(f,g,h) \in E_0 \times E_0 \times E_0$ *that is comparable to* (ϕ,ψ,ξ) *and* (ϕ_1,ψ_1,ξ_1)*, we find the uniqueness of triple PPF dependent invariant point of* H.

Proof. If possible consider $(\phi^*,\psi^*,\xi^*) \in E_0 \times E_0 \times E_0$ is any other tripled PPF dependent invariant point of H. For this we will prove $d((\phi(c),\psi(c),\xi(c)),(\phi^*(c),\psi^*(c),\xi^*(c))) = 0$.
By previous theorem

$$\lim_{n\to\infty} H^n(\phi_0,\psi_0,\xi_0) = \phi(c), \quad \lim_{n\to\infty} H^n(\psi_0,\phi_0,\psi_0) = \psi(c), \quad \lim_{n\to\infty} H^n(\xi_0,\psi_0,\phi_0) = \xi(c).$$

Two cases are considered:

Case (a). If (ϕ,ψ,ξ) is comparable to $(\phi^*,\psi^*\xi^*)$ as regards the ordering in $E_0 \times E_0 \times E_0$ then for all $n = 0,1,2,\dots$, $H^n(\phi,\psi,\xi),H^n(\psi,\phi,\psi),H^n(\xi,\psi,\phi) = (\phi,\psi,\xi)$ is comparable to $H^n(\phi^*,\psi^*,\xi^*),H^n(\psi^*,\phi^*,\psi^*),H^n(\xi^*,\psi^*,\phi^*) = (\phi^*,\psi^*,\xi^*)$.
Also

$$\begin{aligned}
&d((\phi(c),\psi(c),\xi(c)),(\phi^*(c),\psi^*(c),\xi^*(c)))\\
&\quad = d(\phi(c),\phi^*(c)) + d(\psi(c),\psi^*(c)) + d(\xi(c),\xi^*(c))\\
&\quad = d(H^n(\phi,\psi,\xi),H^n(\phi^*,\psi^*,\xi^*)) + d(H^n(\psi,\phi,\psi),H^n(\psi^*,\phi^*,\psi^*)) + d(H^n(\xi,\psi,\phi),H^n(\xi^*,\psi^*,\phi^*))\\
&\quad = \alpha^n[d(\phi(c),\phi^*(c)) + d(\psi(c),\psi^*(c)) + d(\xi(c),\xi^*(c))]\\
&\quad = \alpha^n d((\phi(c),\psi(c),\xi(c)),(\phi^*(c),\psi^*(c),\xi^*(c))),
\end{aligned}$$

where $\alpha = j + K + l < 1$.
Hence $d((\phi(c),\psi(c),\xi(c)),(\phi^*(c),\psi^*(c),\xi^*(c))) = 0$.

Case (b). If (ϕ,ψ,ξ) is not comparable to $(\phi^*,\psi^*\xi^*)$, then there exists a lower bound or an upper bound f,g,h of (ϕ,ψ,ξ) and (ϕ^*,ψ^*,ξ^*). Then $\forall\ n = 0,1,2,\dots$,

$$(H^n(f,g,h),H^n(g,f,g),H^n(h,g,f))$$

is comparable to

$$(H^n(\phi,\psi,\xi),H^n(\psi,\phi,\psi),H^n(\xi,\psi,\phi)) = (\phi,\psi,\xi)$$

and

$$(H^n(\phi^*,\psi^*,\xi^*),H^n(\psi^*,\phi^*,\psi^*),H^n(\xi^*,\psi^*,\phi^*)) = (\phi^*,\psi^*,\xi^*).$$

$$d\left(\begin{pmatrix}\phi(c)\\ \psi(c)\\ \xi(c)\end{pmatrix},\begin{pmatrix}\phi^*(c)\\ \psi^*(c)\\ \xi^*(c)\end{pmatrix}\right) \le d\left(\begin{pmatrix}H^n(\phi,\psi,\xi)\\ H^n(\psi,\phi,\psi)\\ H^n(\xi,\psi,\phi)\end{pmatrix},\begin{pmatrix}H^n(\phi^*,\psi^*,\xi^*)\\ H^n(\psi^*,\phi^*,\psi^*)\\ H^n(\xi^*,\psi^*,\phi^*)\end{pmatrix}\right)$$

$$\leq d\left(\begin{pmatrix} H^n(\phi,\psi,\xi) \\ H^n(\psi,\phi,\psi) \\ H^n(\xi,\psi,\phi) \end{pmatrix}, \begin{pmatrix} H^n(f,g,h) \\ H^n(g,f,g) \\ H^n(h,g,f) \end{pmatrix}\right) + d\left(\begin{pmatrix} H^n(f,g,h) \\ H^n(g,f,g) \\ H^n(h,g,f) \end{pmatrix}, \begin{pmatrix} H^n(\phi^*,\psi^*,\xi^*) \\ H^n(\psi^*,\phi^*,\psi^*) \\ H^n(\xi^*,\psi^*,\phi^*) \end{pmatrix}\right)$$

$$\leq \alpha^n[d(\phi(c),f(c)) + d(\psi(c),g(c)) + d(\xi(c),h(c))]$$

$$+[d(f(c),\phi^*(c)) + d(g(c),\psi^*(c)) + d(h(c),\xi^*(c))]$$

which $\to \infty$ when $n \to \infty$.

So, $d\left(\begin{pmatrix} \phi(c) \\ \psi(c) \\ \xi(c) \end{pmatrix}, \begin{pmatrix} \phi^*(c) \\ \psi^*(c) \\ \xi^*(c) \end{pmatrix}\right) = 0.$ □

Theorem 3.6. *With the hypothesis of Theorem* 3.3 *or* (*Theorem* 3.4) *let us consider every triple elements of* E_0 *has a lower bound or an upper bound in* E_0. *Then* $\phi = \psi = \xi$.

Proof. For proving this, we consider two cases:

Case (a). If ϕ,ψ,ξ are comparable then

$$\phi(c) = H(\phi,\psi,\xi), \quad \psi(c) = H(\psi,\phi,\psi), \quad \xi(c) = H(\xi,\psi,\phi)$$

are comparable and we get

$$\begin{aligned} d(\phi(c),\xi(c)) &= d(H(\phi,\psi,\xi),H(\xi,\psi,\phi)) \\ &\leq jd(\phi(c),\xi(c)) + k.0 + ld(\xi(c),\phi(c)) \\ &\leq (j+k+l)d(\phi(c),\xi(c)) \\ &< d(\phi(c),\xi(c)) \end{aligned}$$

that means $d(\phi(c),\xi(c)) = 0$.

So,

$$\phi(c) = \xi(c) \quad \forall\ c \in [a,b]$$

that is $\phi = \xi$.

$$\begin{aligned} d(\phi(c),\xi(c)) &= d(H(\phi,\psi,\xi),H(\psi,\phi,\psi)) \\ &= d(H(\phi,\psi,\phi),H(\psi,\phi,\psi)) \\ &\leq jd(\phi(c),\psi(c)) + kd(\psi(c),\phi(c)) + ld(\phi(c),\psi(c)) \\ &= (j+k+l)d(\phi(c),\psi(c)) \\ &< d(\phi(c),\psi(c)). \end{aligned}$$

That means $d(\phi(c),\xi(c)) = 0$.

So,

$$\phi(c) = \psi(c) \quad \forall\ c \in [a,b].$$

Hence, $\phi = \psi$. So, $\phi = \psi = \xi$.

Case (b). If ϕ,ψ,ξ are not comparable then ϕ,ψ,ξ have a lower bound or an upper bound.

So, there exist a function $f \in H$ comparable to ϕ,ψ,ξ.

Let us suppose that $\phi \leq f$, $\psi \leq f$, $\xi \leq f$ hold.

Then, we have

$$H(\phi,\psi,\xi)\le H(f,\psi,\xi),\ H(\psi,\phi,\psi)\ge H(\psi,f,\psi)\ \text{ and }\ H(\xi,\psi,\phi)\le H(\xi,\psi,f),$$
$$H(f,\psi,\xi)\le H(f,\psi,f),\ H(\phi,\psi,\phi)\le H(f,\psi,f)\ \text{ and }\ H(\xi,\psi,f)\le H(f,\psi,f),$$
$$H(f,\psi,f)\ge H(\psi,f,\psi).$$

This implies that

$$\begin{aligned}H^2(\phi,\psi,\xi)&=H(H(\phi,\psi,\xi),H(\psi,\phi,\psi),H(\xi,\psi,\phi))\\&\le H(H(f,\psi,\xi),H(\psi,f,\psi),H(\xi,\psi,\phi))\\&=H^2(f,\psi,\xi)\end{aligned}$$

that means $H^2(\phi,\psi,\xi)\le H^2(f,\psi,\xi)$

$$\begin{aligned}H^2(\psi,\phi,\psi)&=H(H(\psi,\phi,\psi),H(\phi,\psi,\phi),H(\psi,\phi,\psi))\\&\ge H(H(\psi,f,\psi),H(f,\psi,f),H(\psi,f,\psi))\\&=H^2(\psi,f,\psi)\end{aligned}$$

that means $H^2(\psi,\phi,\psi)\ge H^2(\psi,f,\psi)$

$$\begin{aligned}H^2(\xi,\psi,\phi)&=H(H(\xi,\psi,\phi),H(\psi,\xi,\psi),H(\phi,\psi,\xi))\\&\le H(H(\xi,\psi,f),H(\psi,\xi,\psi),H(f,\psi,\xi))\\&=H^2(\xi,\psi,f)\end{aligned}$$

that means $H^2(\xi,\psi,\phi)\le H^2(\xi,\psi,f)$

$$\begin{aligned}H^2(f,\psi,\xi)&=H(H(f,\psi,\xi),H(\psi,f,\psi),H(\xi,\psi,f))\\&\le H(H(f,\psi,f),H(\psi,f,\psi),H(f,\psi,f))\\&=H^2(f,\psi,f)\end{aligned}$$

that means $H^2(f,\psi,\xi)\le H^2(f,\psi,f)$

$$\begin{aligned}H^2(\xi,\psi,f)&=H(H(\xi,\psi,f),H(\psi,\xi,\psi),H(f,\psi,\xi))\\&\le H(H(f,\psi,f),H(\psi,f,\psi),H(f,\psi,f))\\&=H^2(f,\psi,f)\end{aligned}$$

that means $H^2(\xi,\psi,f)\le H^2(f,\psi,f)$

$$\begin{aligned}H^2(\phi,\psi,\phi)&=H(H(\phi,\psi,\phi),H(\psi,\phi,\psi),H(\phi,\psi,\phi))\\&\le H(H(f,\psi,f),H(\psi,f,\psi),H(f,\psi,f))\\&=H^2(f,\psi,f)\end{aligned}$$

that means $H^2(\phi,\psi,\phi)\le H^2(f,\psi,f)$.
By mathematical induction we get that this relation applies for $n>2$ as well.
Now,

$$\begin{aligned}&d(\phi(c),\psi(c))\\&\quad=d(H^{n+1}(\phi,\psi,\xi),H^{n+1}(\psi,\phi,\psi))\end{aligned}$$

$$= d[H(H^n(\phi,\psi,\xi),H^n(\psi,\phi,\psi),H^n(\xi,\psi,\phi)),H(H^n(\psi,\phi,\psi),H^n(\phi,\psi,\phi),H^n(\psi,\phi,\psi))]$$
$$\le d[H(H^n(\phi,\psi,\xi),H^n(\psi,\phi,\psi),H^n(\xi,\psi,\phi)),H(H^n(f,\psi,\xi),H^n(\psi,f,\psi),H^n(\xi,\psi,f))]$$
$$+ d[H(H^n(f,\psi,\xi),H^n(\psi,f,\psi),H^n(\xi,\psi,f)),H(H^n(f,\psi,f),H^n(\psi,f,\psi),H^n(f,\psi,f))]$$
$$+ d[H(H^n(\psi,\phi,\psi),H^n(\phi,\psi,\phi),H^n(\psi,\phi,\psi)),H(H^n(\psi,f,\psi),H^n(f,\psi,f),H^n(\psi,f,\psi))]$$
$$+ d[H(H^n(\psi,f,\psi),H^n(f,\psi,f),H^n(\psi,f,\psi)),H(H^n(f,\psi,f),H^n(\psi,f,\psi),H^n(f,\psi,f))].$$

Because of contractive condition of H, we have

$$d(\phi(c),\psi(c)) \le jd(H^n(\phi,\psi,\xi),H^n(f,\psi,\xi)) + kd(H^n(\psi,\phi,\psi),H^n(\psi,f,\psi))$$
$$+ ld(H^n(\xi,\psi,\phi),H^n(\xi,\psi,f)) + \ldots + ld(H^n(\psi,f,\psi),H^n(f,\psi,\xi)).$$

On the same way, we finally get

$$d(\phi(c),\psi(c)) \le \alpha^{n+1}[d(\phi(c),f(c)) + d(\psi(c),f(c)) + d(\xi(c),f(c))]$$

which $\to 0$ as $n \to \infty$.
So, $d(\phi(c),\psi(c)) = 0$.
Similarly $d(\phi(c),\xi(c)) = 0$ and $d(\psi(c),\xi(c)) = 0$.
So,

$$\phi(c) = \psi(c) \text{ and } \psi(c) = \xi(c)$$

which implies that

$$\phi(c) = \psi(c) = \xi(c) \quad \forall\ c \in [a,b].$$

Hence $\phi = \psi = \xi$. □

Theorem 3.7. *With the hypothesis of Theorem* 3.3 *let us suppose that* $\phi_0,\psi_0,\xi_0 \in E_0$ *are comparable. Then* $\phi = \psi = \xi$.

Proof. Here $\phi_0,\psi_0,\xi_0 \in E_0$ are such that

$$\phi(c) \le H(\phi_0,\psi_0,\xi_0), \quad \psi_0(c) \ge H(\psi_0,\phi_0,\psi_0), \quad \xi_0(c) \le H(\xi_0,\psi_0,\phi_0).$$

Now we will show that if $\phi_0 \le \psi_0$ and $\xi_0 \le \psi_0$ then

$$\phi_n \le \psi_n \text{ and } \xi_n \le \psi_n \quad \forall\ n \in \mathbb{N}.$$

Because of mixed monotone property of H,

$$\phi_1(c) = H(\phi_0,\psi_0,\xi_0) \le H(\psi_0,\phi_0,\psi_0) = \psi_1(c)$$

and

$$\psi_1(c) = H(\xi_0,\psi_0,\phi_0) \le H(\psi_0,\phi_0,\psi_0) = \psi_1(c).$$

Now suppose that

$$\phi_n \le \psi_n \text{ and } \xi_n \le \psi_n \quad \forall\ n.$$

Then

$$\phi_{n+1}(c) = H^{n+1}(\phi_0,\psi_0,\xi_0)$$
$$= H(H^n(\phi_0,\psi_0,\xi_0),(H^n(\psi_0,\phi_0,\psi_0),(H^n(\xi_0,\psi_0,\phi_0))$$

$$= H(\phi_n,\psi_n,\xi_n)$$
$$\leq H(\psi_n,\phi_n,\psi_n) = \psi_{n+1}(c)$$

and similarly for ξ_n.
Now

$$\begin{aligned}d(\phi(c),\psi(c)) &\leq d(\phi(c),H^{n+1}(\phi_0,\psi_0,\xi_0)) + d(\psi(c),H^{n+1}(\phi_0,\psi_0,\xi_0))\\ &\leq d(\phi(c),H^{n+1}(\phi_0,\psi_0,\xi_0)) + d(H^{n+1}(\phi_0,\psi_0,\xi_0),H^{n+1}(\psi_0,\phi_0,\psi_0))\\ &\quad + d(\psi(c),H^{n+1}(\psi_0,\phi_0,\psi_0))\\ &= d(\phi(c),H^{n+1}(\phi_0,\psi_0,\xi_0)) + d[H(H^n(\phi_0,\psi_0,\xi_0),H^n(\psi_0,\phi_0,\psi_0),H^n(\xi_0,\psi_0,\phi_0)),\\ &\quad H(H^n(\psi_0,\phi_0,\psi_0),H^n(\phi_0,\psi_0,\phi_0),H^n(\psi_0,\phi_0,\psi_0))] + d(\psi(c),H^{n+1}(\psi_0,\phi_0,\psi_0))\\ &\leq d(\phi(c),H^{n+1}(\phi_0,\psi_0,\xi_0)) + \alpha^{n+1}[d(\phi_0(c),\psi_0(c)) + d(\psi_0(c),\xi_0(c))]\\ &\quad + d(\psi(c),H^{n+1}(\psi_0,\phi_0,\psi_0))\end{aligned}$$

which $\to 0$ as $n\to\infty$, which implies that $d(\phi(c),\psi(c)) = 0$.
So,

$$\phi(c) = \psi(c) \quad \forall\ c\in[a,b].$$

Hence $\phi = \psi$.
Similarly, we have $d(\phi(c),\xi(c)) = 0$ and $d(\psi(c),\xi(c)) = 0$.
On the same way we can prove other cases for ϕ_0,ψ_0,ξ_0.
Hence $\phi = \psi = \xi$. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. R. Bernfeld, V. Lakshmikantham and Y. M. Reddy, Fixed point theorems of operators with PPF dependence in Banach spaces, *Applicable Analysis* **6** (1977), 271 – 280, DOI: 10.1080/00036817708839165.

[2] V. Berinde and M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications* **74**(15) (2011), 4889 – 4897, DOI: 10.1016/j.na.2011.03.032.

[3] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: Theory, Methods & Applications* **65**(7) (2006), 1379 – 1393, DOI: /10.1016/j.na.2005.10.017.

[4] L. B. Ćirić, A generalization of Banach's contraction principle, *Proceedings of the American Mathematical Society* **45**(2) (1974), 267 – 273, DOI: 10.2307/2040075.

[5] Z. Drici, F. A. McRae and J. V. Devi, Fixed-point theorems in partially ordered metric spaces for operators with PPF dependence, *Nonlinear Analysis: Theory, Methods & Applications* **67**(2) (2007), 641 – 647, DOI: 10.1016/j.na.2006.06.022.

[6] Z. Drici, F. A. McRae and J. V. Devi, Fixed point theorems for mixed monotone operators with PPF dependence, *Nonlinear Analysis: Theory, Methods & Applications* **69**(2) (2008), 632 – 636, DOI: 10.1016/j.na.2007.05.044.

[7] N. Hussain, S. Khaleghizadeh, P. Salimi and F. Akbar, New fixed point results with PPF dependence in Banach spaces endowed with a graph, *Abstract and Applied Analysis* **2013** (2013), 1 – 9, DOI: 10.1155/2013/827205.

[8] V. Lakshamikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications* **70**(12) (2009), 4341 – 4349, DOI: 10.1016/j.na.2008.09.020.

[9] N. V. Luong and N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Analysis: Theory, Methods & Applications* **74**(3) (2011), 983 – 992, DOI: 10.1016/j.na.2010.09.055.

[10] A. C. M. Ran and M. C. R. Reurings, A fixed point theorems in partially ordered sets and some applications to metric equations, *Proceedings of the American Mathematical Society* **132**(5) (2004), 1435 – 1443, URL: https://www.jstor.org/stable/4097222.