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Research Article

A Notes on Matrix Sequence of Pentanacci Numbers and Pentanacci Cubes

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Abstract. Pentanacci sequence is a fifth order recurrence relation and Pentanacci cubes $\Gamma_n^{(5)}$ are induced subgraphs of the hypercube, obtained by removing all the vertices that contain five or more consecutive 1's. In the present work, we give some properties related to matrix sequence of Pentanacci numbers and Pentanacci cubes $\Gamma_n^{(5)}$. Also, we investigate the generating function of matrix sequence and cube polynomials of Pentanacci cubes.

Keywords. Fibonacci cube, Pentanacci cube, Pentanacci sequence, Hypercube, Matrix sequence

Mathematics Subject Classification (2020). 40C99, 05C30

1. Introduction

Hypercubes Q_n are widely used as mathematical models in network analysis. The graph of hypercube is constructed, using all binary strings of length n, as vertices and the edges determine by the Hamming distance. Two vertices are connected by an edge if their Hamming distance is one. Liu *et al*. [6] introduced a Fibonacci cube which were obtained from hypercubes Q_n. Klavžar and Mollard [3] determine the cube polynomial of Fibonacci cubes and Lucas cubes and also explicitly determined the zeros of cube polynomials.

Belbachir and Ould-Mohamed [1] gave some properties related to Tribonacci cubes. The main difference between Fibonacci cubes and Tribonacci cubes, is that the vertices do not contain two consecutive 1's and three consecutive 1's, respectively. They investigated the cube polynomials of Tribonacci cubes and determined the corresponding generating function.

Soykan [8,9] investigated the matrix sequences associated with Tribonacci and Tribonacci-Lucas numbers. Also they determined the relation between Tribonacci and Tribonacci-Lucas matrix sequences. Construction of the k-Fibonacci cubes, the structure of k-Lucas cubes and diameter of the k-Fibonacci cubes and k-Lucas cubes are studied in [2]. We are interested in the study of the matrix sequence associated with Pentanacci numbers and cube polynomial of Pentanacci cube motivated by the results in [4,5].

This paper is organised as follows. In Section 2, we recall the definition of Pentanacci numbers and its extension to negative subscripts. In Section 3, we define matrix sequence of Pentanacci numbers and some results are presented. In Sections 4 and 5, the cube polynomials of Pentanacci cube is investigated and some enumerative properties are obtained. In the last section, we give the definition of radius and diameter of Pentanacci cube.

2. Preliminaries

Pentanacci sequence P_n (sequence A001591), [7] is defined by the fifth-order relation

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \quad n \geq 5 \tag{2.1}$$

with initial conditions $P_0 = 0$, $P_1 = 1$, $P_2 = 1$, $P_3 = 2$, $P_4 = 4$.
Also, this sequence can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} - P_{-(n-2)} - P_{-(n-3)} - P_{-(n-4)} + P_{-(n-5)}, \quad n = 1,2,3,\dots. \tag{2.2}$$

The first few values of the Pentanacci numbers is given in Table 1.

Table 1. Pentanacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
P_n	0	1	1	2	4	8	16	31	61	120	236	$\cdots$
P_{-n}	0	0	0	0	1	−1	0	0	0	2	−3	$\cdots$

We observe that

$$P_{n+1} \leq \sum_{i=0}^{n} P_i < P_{n+2}, \quad n \geq 2.$$

3. The Matrix Sequence of Pentanacci Numbers

Soykan [8] studied matrix sequences of Tribonacci and Tetranacci numbers and obtained the generating function of that matrix sequence. We define the matrix sequence of Pentanacci numbers and obtain its generating function.

Definition 3.1. For any integer $n \geq 5$, the Pentanacci matrix $\mathscr{P}_n$ is defined by

$$\mathscr{P}_n = \mathscr{P}_{n-1} + \mathscr{P}_{n-2} + \mathscr{P}_{n-3} + \mathscr{P}_{n-4} + \mathscr{P}_{n-5} \tag{3.1}$$

with initial conditions

$$\mathscr{P}_0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\quad \mathscr{P}_1=\begin{bmatrix}1&1&1&1&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix},\quad \mathscr{P}_2=\begin{bmatrix}2&2&2&2&1\\1&1&1&1&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},$$

$$\mathscr{P}_3=\begin{bmatrix}4&4&4&3&2\\2&2&2&2&1\\1&1&1&1&1\\1&0&0&0&0\\0&1&0&0&0\end{bmatrix},\quad \mathscr{P}_4=\begin{bmatrix}8&8&7&6&4\\4&4&4&3&2\\2&2&2&2&1\\1&1&1&1&1\\1&0&0&0&0\end{bmatrix}.$$

The Pentanacci matrix sequence $\mathscr{P}_n$ can be extended to negative subscripts by defining

$$\mathscr{P}_{-n}=-\mathscr{P}_{-(n-1)}-\mathscr{P}_{-(n-2)}-\mathscr{P}_{-(n-3)}-\mathscr{P}_{-(n-4)}+\mathscr{P}_{-(n-5)},\quad n=1,2,3,\dots.$$

The following theorem gives the nth term of the matrix sequence of Pentanacci numbers.

Theorem 3.1. *For any integer $n\geq 0$, we have the following formulas of the matrix sequences:*

$$\mathscr{P}_n=\begin{bmatrix}P_{n+1}&P_n+P_{n-1}+P_{n-2}+P_{n-3}&P_n+P_{n-1}+P_{n-2}&P_n+P_{n-1}&P_n\\P_n&P_{n-1}+P_{n-2}+P_{n-3}+P_{n-4}&P_{n-1}+P_{n-2}+P_{n-3}&P_{n-1}+P_{n-2}&P_{n-1}\\P_{n-1}&P_{n-2}+P_{n-3}+P_{n-4}+P_{n-5}&P_{n-2}+P_{n-3}+P_{n-4}&P_{n-2}+P_{n-3}&P_{n-2}\\P_{n-2}&P_{n-3}+P_{n-4}+P_{n-5}+P_{n-6}&P_{n-3}+P_{n-4}+P_{n-5}&P_{n-3}+P_{n-4}&P_{n-3}\\P_{n-3}&P_{n-4}+P_{n-5}+P_{n-6}+P_{n-7}&P_{n-4}+P_{n-5}+P_{n-6}&P_{n-4}+P_{n-5}&P_{n-4}\end{bmatrix}. \tag{3.2}$$

Proof. We prove this theorem by strong mathematical induction on n.

If $n=0$, then we have

$$\mathscr{P}_0=\begin{bmatrix}P_1&P_0+P_{-1}+P_{-2}+P_{-3}&P_0+P_{-1}+P_{-2}&P_0+P_{-1}&P_0\\P_0&P_{-1}+P_{-2}+P_{-3}+P_{-4}&P_{-1}+P_{-2}+P_{-3}&P_{-1}+P_{-2}&P_{-1}\\P_{-1}&P_{-2}+P_{-3}+P_{-4}+P_{-5}&P_{-2}+P_{-3}+P_{-4}&P_{-2}+P_{-3}&P_{-2}\\P_{-2}&P_{-3}+P_{-4}+P_{-5}+P_{-6}&P_{-3}+P_{-4}+P_{-5}&P_{-3}+P_{-4}&P_{-3}\\P_{-3}&P_{-4}+P_{-5}+P_{-6}+P_{-7}&P_{-4}+P_{-5}+P_{-6}&P_{-4}+P_{-5}&P_{-4}\end{bmatrix},$$

$$\mathscr{P}_0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}.$$

If $n=1$, then we have

$$\mathscr{P}_1=\begin{bmatrix}P_2&P_1+P_0+P_{-1}+P_{-2}&P_1+P_0+P_{-1}&P_1+P_0&P_1\\P_1&P_0+P_{-1}+P_{-2}+P_{-3}&P_0+P_{-1}+P_{-2}&P_0+P_{-1}&P_0\\P_0&P_{-1}+P_{-2}+P_{-3}+P_{-4}&P_{-1}+P_{-2}+P_{-3}&P_{-1}+P_{-2}&P_{-1}\\P_{-1}&P_{-2}+P_{-3}+P_{-4}+P_{-5}&P_{-2}+P_{-3}+P_{-4}&P_{-2}+P_{-3}&P_{-2}\\P_{-2}&P_{-3}+P_{-4}+P_{-5}+P_{-6}&P_{-3}+P_{-4}+P_{-5}&P_{-3}+P_{-4}&P_{-3}\end{bmatrix},$$

$$\mathscr{P}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which is true. Assume that the equality holds for $n \le k$. For $n = k+1$, using fifth-order recurrence relation, we have the required results. □

Lemma 3.1. *The generating function for the Pentanacci sequence* $\{P_n\}_{n\ge 0}$ *is*

$$G(y) = \frac{y}{1-y-y^2-y^3-y^4-y^5}.$$

Proof. By definition of generating function,

$$\begin{aligned} G(y) &= \sum_{n=0}^{\infty} \mathscr{P}_n y^n \\ &= \mathscr{P}_0 + \mathscr{P}_1 y + \mathscr{P}_2 y^2 + \mathscr{P}_3 y^3 + \mathscr{P}_4 y^4 + \sum_{n=5}^{\infty} \mathscr{P}_n y^n \\ &= y + y^2 + 2y^3 + 4y^4 + \sum_{n=5}^{\infty} [\mathscr{P}_{n-1} + \mathscr{P}_{n-2} + \mathscr{P}_{n-3} + \mathscr{P}_{n-4} + \mathscr{P}_{n-5}] y^n \\ &= y + y^2 + 2y^3 + 4y^4 + y[G(y) - y - y^2 - 2y^3] \\ &\quad + y^2[G(y) - y - y^2] + y^3[G(y) - y] + y^4 G(y) + y^5 G(y). \end{aligned}$$

Grouping $G(y)$, the required results are obtained. □

Theorem 3.2. *The generating function for the Pentanacci matrix sequence is*

$$\sum_{n=0}^{\infty} \mathscr{P}_n y^n = \frac{\begin{bmatrix} 1 & y+y^2+y^3+y^4 & y+y^2+y^3 & y+y^2 & y \\ y & 1-y & y^2+y^3+y^4 & y^2+y^3 & y^2 \\ y^2 & y-y^2 & 1-y-y^2 & y^3+y^4 & y^3 \\ y^3 & y^2-y^3 & y-y^2-y^3 & 1-y-y^2-y^3 & y^4 \\ y^4 & y^3-y^4 & y^2-y^3-y^4 & y-y^2-y^3-y^4 & 1-y-y^2-y^3-y^4 \end{bmatrix}}{1-y-y^2-y^3-y^4-y^5}. \tag{3.3}$$

Proof. Suppose that $G(y) = \sum_{n=0}^{\infty} \mathscr{P}_n y^n$ is the generating function for the Pentanacci matrix sequence. Then using definition, we obtain

$$\begin{aligned} G(y) &= \sum_{n\ge 0}^{\infty} \mathscr{P}_n y^n \\ &= \mathscr{P}_0 + \mathscr{P}_1 y + \mathscr{P}_2 y^2 + \mathscr{P}_3 y^3 + \mathscr{P}_4 y^4 + \sum_{n=5}^{\infty} \mathscr{P}_n y^n \\ &= \mathscr{P}_0 + \mathscr{P}_1 y + \mathscr{P}_2 y^2 + \mathscr{P}_3 y^3 + \mathscr{P}_4 y^4 \\ &\quad + \sum_{n=5}^{\infty} (\mathscr{P}_{n-1} + \mathscr{P}_{n-2} + \mathscr{P}_{n-3} + \mathscr{P}_{n-4} + \mathscr{P}_{n-5}) y^n \\ G(y)(1-y-y^2-y^3-y^4-y^5) &= \mathscr{P}_0 + (\mathscr{P}_1 - \mathscr{P}_0) y + (\mathscr{P}_2 - \mathscr{P}_1 - \mathscr{P}_0) y^2 \\ &\quad + (\mathscr{P}_3 - \mathscr{P}_2 - \mathscr{P}_1 - \mathscr{P}_0) y^3 + (\mathscr{P}_4 - \mathscr{P}_3 - \mathscr{P}_2 - \mathscr{P}_1 - \mathscr{P}_0) y^4. \end{aligned}$$

Using initial matrix sequence conditions, the required results are obtained. □

4. Pentanacci Cube

We define a Pentanacci string of length n as a binary string $\alpha_n = a_1a_2\ldots a_n$ where α_n does not contain five or more consecutive 1's. Let V_n be the set of all Pentanacci string of length n including the empty string. The Pentanacci cube of dimension n is the graph with vertex set V_n, where two vertices are adjacent if and only if their Hamming distance is equal to one. The Pentanacci cube is denoted by $\Gamma_n^{(5)}$. The Pentanacci cube can be decomposed into the subgraphs induced by the vertices that starts with 0,10,110,1110 and 11110, respectively [1,3]. The vertices beginning with 0 constitute a graph isomorphic to $\Gamma_{n-1}^{(5)}$, the vertices beginning with 10 constitute a graph isomorphic to $\Gamma_{n-2}^{(5)}$, the vertices beginning with 110 constitute a graph isomorphic to $\Gamma_{n-3}^{(5)}$, the vertices beginning with 1110 constitute a graph isomorphic to $\Gamma_{n-4}^{(5)}$ and the vertices beginning with 11110 constitute a graph isomorphic to $\Gamma_{n-5}^{(5)}$. This can be written symbolically as $\Gamma_n^{(5)} = 0\Gamma_{n-1}^{(5)} + 10\Gamma_{n-2}^{(5)} + 110\Gamma_{n-3}^{(5)} + 1110\Gamma_{n-4}^{(5)} + 11110\Gamma_{n-5}^{(5)}$.

From the decomposition of Pentanacci cube $\Gamma_n^{(5)}$,the number of edges can be obtained by the following recurrence relation

$$\begin{aligned}|E(\Gamma_n^{(5)})| &= |E(\Gamma_{n-1}^{(5)})| + |E(\Gamma_{n-2}^{(5)})| + |E(\Gamma_{n-3}^{(5)})| + |E(\Gamma_{n-t}^{(5)})| + |E(\Gamma_{n-5}^{(5)})| \\ &\quad + P_n + 3P_{n-1} + 2P_{n-2} + 2P_{n-3}; \quad n \geq 5 \end{aligned} \tag{4.1}$$

with $|E(\Gamma_0^{(5)})| = 0$, $|E(\Gamma_1^{(5)})| = 1$, $|E(\Gamma_2^{(5)})| = 4$, $|E(\Gamma_3^{(5)})| = 12$, $|E(\Gamma_4^{(5)})| = 32$.

Proposition 4.1. *The generating function of the number of edges in* $\Gamma_n^{(5)}$ *is given by*

$$\sum_{n\geq 0} |E(\Gamma_n^{(5)})|y^n| = \frac{y + 2y^2 + 3y^3 + 4y^4 - y^8 - y^9}{(1 - y - y^2 - y^3 - y^4 - y^5)^2}. \tag{4.2}$$

Proof. As in Lemma 3.1,

$$G(y) = \sum_{n\geq 0} P_n y^n = \frac{y}{1 - y - y^2 - y^3 - y^4 - y^5}.$$

Let $E(y)$ denote the generating function of the sequence $E(\Gamma_n^{(5)})$.
Then, we have

$$\begin{aligned} E(y) &= \sum_{n\geq 0} |E(\Gamma_n^{(5)})|y^n \\ &= y + 4y^2 + 12y^3 + 32y^4 + \sum_{n\geq 5} |E(\Gamma_n^{(5)})|y^n \\ &= y + 4y^2 + 12y^3 + 32y^4 + \sum_{n\geq 5} [|E(\Gamma_{n-1}^{(5)})| + |E(\Gamma_{n-2}^{(5)})| + |E(\Gamma_{n-3}^{(5)})| \\ &\quad + |E(\Gamma_{n-4}^{(5)})| + |E(\Gamma_{n-5}^{(5)})|]y^n + \sum_{n\geq 5} [P_n + 3P_{n-1} + 2P_{n-2} + 2P_{n-3}]y^n \\ &= (y + y^2 + y^3 + y^4 + y^5)E(y) + (1 + 3y + 2y^2 + 2y^3)G(y) - y^2 + y^4. \end{aligned}$$

□

Proposition 4.2. *The weight w of a vertex u is the number of 1's in its corresponding binary string. For $n \geq 5$, and $4 \leq w \leq \left\lfloor \frac{4(2n+1)}{5} \right\rfloor$, let S_n^w be the set of all vertices of $\Gamma_n^{(5)}$ having weight w. Then the number of these vertices is given by the recurrence formula below*

$$|S_n^w| = |S_{n-1}^w| + |S_{n-2}^w| + |S_{n-3}^w| + |S_{n-4}^w| + |S_{n-5}^w|, \tag{4.3}$$

where

$$|S_n^0| = |S_4^4| = 1, \quad n \geq 0 \tag{4.4}$$

$$|S_n^1| = n, \qquad n \geq 0 \tag{4.5}$$

$$|S_n^w| = 0, \qquad n \geq 0,\ w \geq \left\lfloor \frac{4(2n+1)}{5} \right\rfloor. \tag{4.6}$$

Proof. A vertex of maximum weight can be obtained by concatenating the string 11110 p times, where $p = \lfloor \frac{n}{5} \rfloor$ and ending with 1^{n-5p}. Hence, a maximum weight of Pentanacci string is $\left\lfloor \frac{4(n+1)}{5} \right\rfloor$. The proof of (4.3) follows directly from the decomposition of $\Gamma_n^{(5)}$. □

Proposition 4.3. *Pentanacci cube is a bipartite graph.*

Proof. Let $\Gamma_n^{(5)}$ be the Pentanacci cube. Its vertex set is classified in to two groups based on weights. Let $X = \{u \mid \text{weight}(u) = \text{even or zero weight}\}$ and Let $Y = \{v \mid \text{weight}(v) = \text{odd}\}$. Clearly, there are no edges existing between two vertices with the same weight. We can assign colour 1 to a even weight vertex. Similarly, there are no edges existing between two vertices in Y. Therefore, we assign colour 2 for each vertices in Y. Also there exist edges between X and Y and which satisfy the colouring property. Hence Pentanacci cube is 2-colourable. □

5. Cube Polynomial

It is well known that the cube polynomial of hypercube is $C(Q_n, x) = (2+x)^n$ ([1,4,5]). The cube polynomial of Fibonacci, Lucas cube and Tribonacci cubes have already been studied and derived from hypercube. We study the cube polynomial of Pentanacci cubes. The first few cube polynomials are listed below

- $C(\Gamma_0^{(5)}, x) = 1$
- $C(\Gamma_1^{(5)}, x) = x + 2$
- $C(\Gamma_2^{(5)}, x) = x^2 + 4x + 4$
- $C(\Gamma_3^{(5)}, x) = x^3 + 6x^2 + 12x + 8$
- $C(\Gamma_4^{(5)}, x) = x^4 + 8x^3 + 24x^2 + 32x + 16$

Lemma 5.1. *For any $n \geq 5$, the cube polynomial of $\Gamma_n^{(5)}$ satisfies the following recurrence relation*

$$\begin{aligned} C(\Gamma_n^{(5)}, x) = {} & C(\Gamma_{n-1}^{(5)}, x) + (1+x)C(\Gamma_{n-2}^{(5)}, x) + (1+x)^2 C(\Gamma_{n-3}^{(5)}, x) \\ & + (1+x)^3 C(\Gamma_{n-4}^{(5)}, x) + (1+x)^4 C(\Gamma_{n-5}^{(5)}, x) \end{aligned}$$

with

$$C(\Gamma_n^{(5)}, x) = (x+2)^n, \quad 0 \leq n \leq 4. \tag{5.1}$$

Proof. For $n \leq 4$, $\Gamma_n^{(5)}$ isomorphic to Q_n, then $C(\Gamma_n^{(5)}, x) = C(Q_n, x)$.
For $n \geq 5$, an induced k-cube in $\Gamma_{n-1}^{(5)}, \Gamma_{n-2}^{(5)}, \Gamma_{n-3}^{(5)}, \Gamma_{n-4}^{(5)}$ and $\Gamma_{n-5}^{(5)}$ remains of dimension k in $\Gamma_n^{(5)}$. The number of these hypercubes is obtained by

$$C(\Gamma_{n-1}^{(5)}, x) + C(\Gamma_{n-2}^{(5)}, x) + C(\Gamma_{n-3}^{(5)}, x) + C\Gamma_{n-4}^{(5),x} + C(\Gamma_{n-5}^{(5)}, x).$$

Consider the following sets:

$$A_n = \{\alpha_n \in V(\Gamma_n^{(5)}) \mid a_1a_2a_3a_4a_5 = 11110\}$$

$$B_n = \{\alpha_n \in V(\Gamma_n^{(5)}) \mid a_1a_2a_3a_4 = 1110\}$$

$$C_n = \{\alpha_n \in V(\Gamma_n^{(5)}) \mid a_1a_2a_3 = 110\}$$
$$D_n = \{\alpha_n \in V(\Gamma_n^{(5)}) \mid a_1a_2 = 10\}$$
$$E_n = \{\alpha_n \in V(\Gamma_n^{(5)}) \mid a_1 = 0\}$$

The subgraphs $10\Gamma_{n-2}^{(5)}$ induced by the set D_n has a copy in $0\Gamma_{n-1}^{(5)}$ those two copies are joined by perfect matching. Thus an induced k-cube in $\Gamma_{n-2}^{(5)}$ is of dimension $k+1$ in $\Gamma_n^{(5)}$. The subgraphs $110\Gamma_{n-3}^{(5)}$ induced by the set C_n has a copy in $10\Gamma_{n-2}^{(5)}$ those two copies are joined by perfect matching. The subgraph $1110\Gamma_{n-4}^{(5)}$ induced by the set B_n has a copy in $110\Gamma_{n-3}^{(5)}$ those two copies are joined by perfect matching. The subgraph $11110\Gamma_{n-5}^{(5)}$ induced by the set A_n has a copy in $1110\Gamma_{n-4}^{(5)}$ those two copies are joined by perfect matching. Also, The subgraph $110\Gamma_{n-3}^{(5)}$ induced by the set C_n has a copy in $0\Gamma_{n-1}^{(5)}$. Then an induced k-cube in $\Gamma_{n-3}^{(5)}$ is of dimension $k+2$ in $\Gamma_n^{(5)}$. Similarly the subgraph $1110\Gamma_{n-4}^{(5)}$ induced by the set B_n has a copy in $10\Gamma_{n-2}^{(5)}$. The subgraph $11110\Gamma_{n-5}^{(5)}$ induced by the set A_n has a copy in $110\Gamma_3^{(5)}$. □

Theorem 5.1. *The Generating function of the cube polynomial* $C(\Gamma_n^{(5)},x)$ *is*

$$\sum_{n\ge 0} C(\Gamma_n^{(5)},x)y^n = \frac{1+(1+x)y+(1+x)^2y^2+(1+x)^3y^3+(1+x)^4y^4}{1-y-(1+x)y^2-(1+x)^2y^3-(1+x)^3y^4-(1+x)^4y^5}.$$

Proof. Let $F(x,y)$ denote the generating function of the sequence $C(\Gamma_n^{(5)},x)$

$$\begin{aligned} F(x,y) &= \sum_{n\ge 0} C(\Gamma_n^{(5)},x)y^n \\ &= \sum_{n=0}^{4} C(\Gamma_n^{(5)},x)y^n + \sum_{n\ge 5} C(\Gamma_n^{(5)},x)y^n \\ &= 1+(2+x)y+(2+x)^2y^2+(2+x)^3y^3+(2+x)^4y^4+\sum_{n\ge 5} C(\Gamma_n^{(5)},x)y^n. \end{aligned}$$

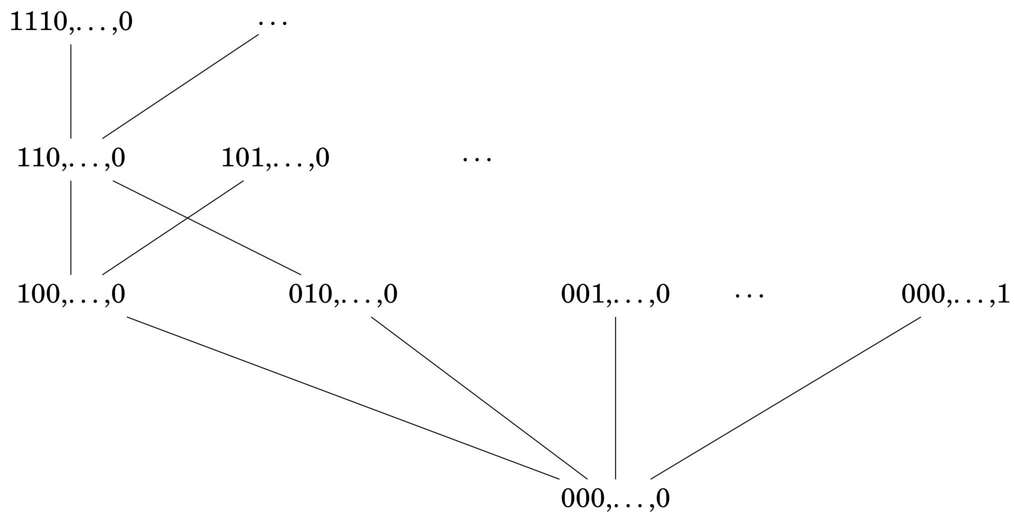
Using Lemma 5.1

$$\begin{aligned} F(x,y) = {} & 1+(2+x)y+(2+x)^2y^2+(2+x)^3y^3+(2+x)^4y^4 \\ & + y[F(x,y)-1-(2+x)y-(2+x)^2y^2-(2+x)^3y^3] \\ & + (1+x)y^2[F(x,y)-1-(2+x)y-(2+x)^2y^2] \\ & + (1+x)^2y^3[F(x,y)-1-(2+x)y] \\ & + (1+x)^3y^4[F(x,y)-1]+(1+x)^4y^5F(x,y). \end{aligned}$$

Grouping $F(x,y)$, we get the required results. □

6. Radius and Diameter

Now, we redraw the hybercube based on their weight. In Initial level we place the vertex having zero weight. The next level placed with vertices having weight one. Clearly, there is no edges between vertices having same weight and there exist n edges between initial level say level 0 and next level say level 1. In the third row having the vertices of weight 2 and so on. Clearly, each vertex in first row is adjacent to $n-1$ vertices in row 2. The total edges between each level is increased atleast $\lfloor \frac{n}{2} \rfloor$ level. The last row having the vertices of maximum weight $\left\lfloor \frac{4(n+1)}{5} \right\rfloor$. From this construction easily we observe that radius and diameter of the Pentanacci cube.



7. Conclusion

The study of hypercubes related to Fibonacci like number sequences has applications both computer science and theoretical chemistry. We present some enumerative properties related to matrix sequence of Pentanacci numbers and Pentanacci cubes. Also, we propose to investigate the application of Pentanacci matrices to coding theory.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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