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Research Article

Boosters and Filters in MS-Almost Distributive Lattices

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Abstract. This paper deals on boosters and β-filters in MS-almost distributive lattice M. It is proved that the set of all boosters in M is a bounded distributive lattice. Characterization of β-filters of M in terms of boosters is established and a dual homomorphism of M and the set of all boosters of M is derived. Further, it is shown that every filter in M is an e-filter and every maximal filter in M is a β-filter. Equivalent conditions on which the set of all boosters is a relatively complemented lattice are established.

Keywords. Booster, β-filter, ADL, MS-ADL, Maximal element

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1. Introduction

U. M. Swamy and G. C. Rao [10] introduced the concept of an *Almost Distributive Lattice* (ADL) as a common abstraction of lattice and ring theoretic generalizations of a Boolean algebra. In [10] it was proved that the commutativity of $\vee$, the commutativity of $\wedge$, the right distributivity of $\vee$ over $\wedge$ and the absorption law $(x \wedge y) \vee x = x$ are all equivalent to each other and whenever any one of these properties holds, an ADL becomes a distributive lattice. The notion of a pseudo-complementation of an ADL is introduced by Swamy *et al*. [11].

The concepts of Stone ADL and its characterization in terms of its ideals are studied by Swamy *et al*. [11]. Rafi *et al*. [6] studied δ-ideals in Pseudo-complemented ADLs.

A bounded distributive lattice that satisfies the property of dual endomorphism is called an Ockham algebra [3]. Berman in [2] introduced an Ockham algebras which contains an algebra called MS-algebras. The subclass of Ockham algebra that generalizes both de-Morgan algebras and Stone algebras is called an MS-algebra, which is introduced by Blyth and Varlet [3]. The class of all MS-algebras forms an equational class. The subvarieties of MS-algebras is characterized by Blyth and Varlet [4], and Rao [8] studied about β-filter of MS-Algebra. Furthermore, the concept of MS-ADL M is introduced by Addis [1].

Motivated by these results we introduce the notions of boosters and β-filters in M. The fact, the class of boosters in M is a bounded distributive lattice is proved. Characterization of β-filters of M in terms of boosters, and a dual homomorphism of M and the set of all boosters in M is established. Further, it is shown that any maximal filter in M is a β-filter. Finally, the conditions on which the lattice of boosters is a relatively complemented lattice are established.

2. Preliminaries

In the sequel, we use the following results:

Definition 2.1 ([10])**.** An almost distributive lattice is an algebra $A = (A, \vee, \wedge, 0)$ satisfying the following conditions:

(i) $0 \wedge r = 0$,

(ii) $r \vee 0 = r$,

(iii) $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$,

(iv) $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$,

(v) $(r \vee s) \wedge t = (r \wedge t) \vee (s \wedge t)$,

(vi) $(r \vee s) \wedge s = s$,

for all r, s and $t \in A$.

Let $x, y \in A$, we read x is less than or equal to y and we write $x \leq y$ if $x \wedge y = x$, equivalently $x \vee y = y$. An element $m \in A$ is maximal if $m \leq x$ implies $m = x$.

Lemma 2.2 ([7])**.** *Consider an ADL A and $m, s \in A$, where m is maximal. The following are equivalent:*

(i) *m is maximal with respect to $\leq$,*

(ii) *$m \vee s = m$,*

(iii) *$m \wedge s = s$,*

(iv) *$s \vee m$ is maximal.*

Definition 2.3 ([7]). Let F be a nonempty subset of an ADL A. Then F is called a filter of A, if for all $p,q \in F$ and $t \in A$; $p \wedge q \in F$ and $t \vee p \in F$.

Definition 2.4 ([3]). Let $(A, \vee, \wedge, 0, 1)$ be a bounded distributive lattice and $^\circ$ is a unary operation that satisfies:

(i) $1^\circ = 0$,

(ii) $(p \wedge q)^\circ = p^\circ \vee q^\circ$,

(iii) $p \leq p^{\circ\circ}$,

for all $p,q \in A$. Then an algebra $(A, \vee, \wedge, ^\circ, 0, 1)$ of type $(2,2,1.0.0)$ is called an MS-algebra.

Lemma 2.5 ([4]). *Let A be an MS-algebra and $p,q \in A$. Then*

(i) $0^\circ = 1$,

(ii) $p \leq q \Rightarrow q^\circ \leq p^\circ$,

(iii) $p^{\circ\circ\circ} = p^\circ$,

(iv) $(p \vee q)^\circ = p^\circ \wedge q^\circ$,

(v) $(p \vee q)^{\circ\circ} = p^{\circ\circ} \vee q^{\circ\circ}$,

(vi) $(p \wedge q)^{\circ\circ} = p^{\circ\circ} \wedge q^{\circ\circ}$.

Definition 2.6 ([1]). Let $(M, \vee, \wedge, 0)$ be an ADL with maximal elements and a unary operation $t \mapsto t^\circ$ on M satisfying the following:

(i) $p^{\circ\circ} \wedge p = p$,

(ii) $(p \vee q)^\circ = p^\circ \wedge q^\circ$,

(iii) $(p \wedge q)^\circ = p^\circ \vee q^\circ$,

(iv) $m^\circ = 0$

for a maximal element m and $p,q \in M$. Then the algebra $(M, \vee, \wedge, ^\circ, 0)$ of type $(2,2,1,0)$ is called an MS-almost distributive lattice (MS-ADL).

Lemma 2.7 ([1]). *For any elements r and s of an MS-ADL M the following conditions hold:*

(i) 0° *is maximal,*

(ii) $r \leq s \Rightarrow s^\circ \leq r^\circ$,

(iii) $r^{\circ\circ\circ} = r^\circ$,

(iv) $(r \wedge s)^{\circ\circ} = r^{\circ\circ} \wedge s^{\circ\circ}$,

(v) $(r \vee s)^{\circ\circ} = r^{\circ\circ} \vee s^{\circ\circ}$,

(vi) $(r \wedge m)^\circ = r^\circ$,

(vii) $(r \wedge s)^\circ = (s \wedge r)^\circ$.

Let D be a filter of an MS-algebra A. Rao in [8] defined that D is said to be an e-filter of A whenever $D = D^e$, where $D^e = \{s \in A \mid t^\circ \wedge s^\circ = s^\circ \text{ for some } t \in A\}$.

Definition 2.8. Let D be a filter in an MS-ADL M. Define $D^e = \{r \in M \mid s^o \wedge r^o = r^o$ for some $s \in M\}$.

Theorem 2.9. *Consider a filter D of an MS-ADL M. The following are equivalent:*

(i) *D is an e-filter,*

(ii) *For $r \in M, r^{oo} \in D$ implies $r \wedge m \in D$,*

(iii) *For all $r, s \in M$, $r^o = s^o$ and $r \in D$ implies $s \in D$.*

The concept of β-filter of MS-algebra is given by Rao [8]. Next, we extend the concept of this filter to β-filters of MS-ADLs. We consider an MS-ADL that contains more than one maximal elements and define its booster. In MS-ADL the commutativity of "$\vee$ and $\wedge$", and the right distributivity of "$\vee$" over "$\wedge$" do not hold. That is why even if some of the results seems to be similar, the proofs are not.

3. β-Filters of MS-ADLs

The idea of boosters and β-filters in MS-ADL M are introduced in this section. Characterization of β-filters of M in terms of boosters is also given.

Throughout the sequel M represents an MS-ADL.

Definition 3.1. For any $r \in M$, the booster of r is defined as follows:

$(r)^+ = \{s \in M \mid r^o \vee s$ is maximal in $M\}$.

Clearly, $(0)^+ = M$ and $(m)^+ = \{s \in M \mid s$ is maximal in $M\}$.

Lemma 3.2. *For any $t \in M$, $(t)^+$ is a filter of M.*

Proof. Assume n be a maximal element in M. Using Lemma 2.2(iv) for any $r \in M$, $r \vee n$ is a maximal element of M. Now, for any $t \in M$, $t^o \vee n$ is maximal. Hence $n \in (t)^+$. That is $(t)^+$ is non empty. Let $r, s \in (t)^+$, then $t^o \vee r$ and $t^o \vee s$ are maximal. Thus, we obtain that $t^o \vee (r \wedge s) = (t^o \vee r) \wedge (t^o \vee s) = t^o \vee s$ so that $t^o \vee (r \wedge s)$ is maximal. Hence $r \wedge s \in (t)^+$. Now, let $r, u \in M$, and $s \in (t)^+$. Consequently we get

$$\begin{aligned}
[t^o \vee (r \vee s)] \wedge u &= [(t^o \vee r) \vee s] \wedge u \\
&= [(t^o \vee r) \wedge u] \vee (s \wedge u) \\
&= [(r \vee t^o) \wedge u] \vee (s \wedge u) \\
&= [(r \vee t^o) \vee s] \wedge u \\
&= [r \vee (t^o \vee s)] \wedge u \\
&= u.
\end{aligned}$$

So that $t^o \vee (r \vee s)$ is maximal and hence $r \vee s \in (t)^+$. Therefore, $(t)^+$ is a filter in M. □

Lemma 3.3. *Let $r,s,t \in M$. Then the following conditions hold:*

(i) $(s \vee t)^+ = (t \vee s)^+$,

(ii) $(s \wedge t)^+ = (t \wedge s)^+$,

(iii) $s \leq t \Rightarrow (t)^+ \subseteq (s)^+$,

(iv) $s^o = t^o \Rightarrow (s)^+ = (t)^+$,

(v) $(s \vee t)^+ = (s)^+ \cap (t)^+$,

(vi) $(s)^+ = (t)^+ \Rightarrow (s \wedge r)^+ = (t \wedge r)^+$ *for all* $r \in M$,

(vii) $(s)^+ = (t)^+ \Rightarrow (s \vee r)^+ = (t \vee r)^+$ *for all* $r \in M$,

(viii) $(r)^+ = M$ *iff* $r = 0$.

Proof. (i) Let $p,q \in M$. Then

$$\begin{aligned}
p \in (s \vee t)^+ &\Leftrightarrow (s \vee t)^o \vee p \text{ is maximal}\\
&\Leftrightarrow [(s \vee t)^o \vee p] \wedge q = q\\
&\Leftrightarrow [(s^o \wedge t^o) \vee p] \wedge q = q\\
&\Leftrightarrow [(s^o \vee p) \wedge (t^o \vee p)] \wedge q = q\\
&\Leftrightarrow [(t^o \vee p) \wedge (s^o \vee p)] \wedge q = q\\
&\Leftrightarrow [(p \vee t^o) \wedge (p \vee s^o)] \wedge q = q\\
&\Leftrightarrow [p \vee (t^o \wedge s^o)] \wedge q = q\\
&\Leftrightarrow [(t^o \wedge s^o) \vee p] \wedge q = q\\
&\Leftrightarrow [(t \vee s)^o \vee p] \wedge q = q\\
&\Leftrightarrow [(t \vee s)^o \vee p] \text{ is maximal}\\
&\Leftrightarrow p \in (t \vee s)^+.
\end{aligned}$$

Hence $(s \vee t)^+ = (t \vee s)^+$.

(ii) Apply the same procedure as (i).

(iii) Let $s \leq t$ and $x \in (t)^+$. Then $t^o \leq s^o$ and $t^o \vee x$ is maximal. It is important to note that $t^o \leq s^o$ does not imply $t^o \vee x \leq s^o \vee x$. But $(s^o \vee x) \wedge (t^o \vee x) = ((s^o \vee x) \wedge t^o) \vee ((s^o \vee x) \wedge x) = ((s^o \wedge t^o) \vee (x \wedge t^o)) \vee x = (t^o \vee (x \wedge t^o)) \vee x = ((t^o \vee x) \wedge t^o) \vee x = ((x \vee t^o) \wedge t^o) \vee x = t^o \vee x$ which indicates that $s^o \vee x$ is maximal. Therefore $x \in (s)^+$.

(iv) It is direct.

(v) Clearly, $s \leq s \vee t$, but it is not necessarily true that $t \leq s \vee t$. For the former case by (iii) it is obvious that $(s \vee t)^+ \subseteq (s)^+$. Since $t \leq t \vee s$ we have $(t \vee s)^+ \subseteq t^+$. From (i) we obtain that $(s \vee t)^+ = (t \vee s)^+ \subseteq t^+$. Hence $(s \vee t)^+ \subseteq t^+$. Therefore $(s \vee t)^+ \subseteq (s)^+ \cap (t)^+$.

Let $x \in (s)^+ \cap (t)^+$. Then $s^o \vee x$ and $t^o \vee x$ are maximal elements of M.Then for any $y \in M$, we get $((s \vee t)^o \vee x) \wedge y = ((s^o \wedge t^o) \vee x) \wedge y = ((s^o \vee x) \wedge (t^o \vee x)) \wedge y = (t^o \vee x) \wedge y = y$. Hence $(s \vee t)^o \vee x$ is maximal and therefore $x \in (s \vee t)^+$. That indicates that $(s)^+ \cap (t)^+ \subseteq (s \vee t)^+$.

(vi) Let $(s)^+ = (t)^+$ and $x \in (s \wedge r)^+$. Then $(s \wedge r)^o \vee x$ is maximal. Thus, for any $y \in M$, $[(s \wedge r)^o \vee x] \wedge y = [(s^o \vee r^o) \vee x] \wedge y = y$. Next, for any $e, f, g, h \in M$ we apply one of the most important properties of an ADL, i.e., $[(e \vee f) \vee g] \wedge h = [e \vee (f \vee g)] \wedge h$ and we obtain $[s^o \vee (r^o \vee x)] \wedge y = [(s^o \vee r^o) \vee x] \wedge y = y$. This shows that $s^o \vee (r^o \vee x)$ is maximal and thus $r^o \vee x \in (s)^+ = (t)^+$. This in turn implies that $t^o \vee (r^o \vee x)$ is maximal. Now take $d \in M$, we get $[t^o \vee (r^o \vee x)] \wedge d = d$. Consequently, we get $[(t \wedge r)^o \vee x] \wedge d = [(t^o \vee r^o) \vee x] \wedge d = d$ which shows that $(t \wedge r)^o \vee x$ is maximal and $x \in (t \wedge r)^+$. Hence $(s \wedge r)^+ \subseteq (t \wedge r)^+$. Also, $(t \wedge r)^+ \subseteq (s \wedge r)^+$ so that $(s \wedge r)^+ = (t \wedge r)^+$.

(vii) Let $(s)^+ = (t)^+$ and $x \in (s \vee r)^+$. Then $(s \vee r)^o \vee x$ is maximal. So, for any $y \in M$, $[(s \vee r)^o \vee x] \wedge y = [(s^o \wedge r^o) \vee x] \wedge y = y$. Then $[(s^o \vee x) \wedge (r^o \vee x)] \wedge y = y$. Thus $(s^o \vee x) \wedge (r^o \vee x)$ is maximal. From the fact that $x \wedge y$ is maximal implies y is maximal we get $r^o \vee x$ is maximal. Also, as $[(r^o \vee x) \wedge (s^o \vee x)] \wedge y = [(s^o \vee x) \wedge (r^o \vee x)] \wedge y = y$, $(r^o \vee x) \wedge (s^o \vee x)$ is maximal and we get $s^o \vee x$ is also maximal. Thus $x \in (s)^+ = (t)^+$ which shows that $t^o \vee x$ is maximal. Hence for any $z \in M$, $[(t^o \vee x) \wedge (r^o \vee x)] \wedge z = z$. So $[(t^o \wedge r^o) \vee x] \wedge z = z$, which implies that $[(t \vee r)^o \vee x] \wedge z = z$. Thus $(t \vee r)^o \vee x$ is maximal and hence $x \in (t \vee r)^+$. Therefore, $(s \vee r)^+ \subseteq (t \vee r)^+$. Likewise, $(t \vee r)^+ \subseteq (s \vee r)^+$ so that $(s \vee r)^+ = (t \vee r)^+$.

(viii) It is clear. □

In the next theorem we showed that the collection $B_0(M)$ of all boosters in M is a distributive lattice.

Theorem 3.4. *$B_0(M)$ is a bounded distributive lattice.*

Proof. Obviously, $B_0(M)$ is a poset with respect to "$\subseteq$". Let $(e)^+$ and $(f)^+$ are boosters of M. From (v) of Lemma 3.3 we have $(e)^+ \cap (f)^+ = (e \vee f)^+$ which shows that $(e \vee f)^+$ is the lower bound of both $(e)^+$ and $(f)^+$ Now define the operations $\sqcup$ by $(e)^+ \sqcup (f)^+ = (e \wedge f)^+$. Let $(c)^+$ be a lower bound for both $(e)^+$ and $(f)^+$. Let $x \in (c)^+$. Clearly, $x \in (e)^+$ and $x \in (f)^+$. Then $e^o \vee x$ and $f^o \vee x$ are maximal. Hence for any $y \in M$, $(e^o \vee x) \wedge y = y$ and $(f^o \vee x) \wedge y = y$. Then $[(e \vee f)^o \vee x] \wedge y = [(e^o \wedge f^o) \vee x] \wedge y = [(e^o \vee x) \wedge (f^o \vee x)] \wedge y = y$. This shows that $(e \vee f)^o \vee x$ is maximal. Hence $x \in (e \vee f)^+$ so that $(c)^+ \subseteq (e \vee f)^+$. Therefore, $(e \vee f)^+$ is the infimum of $(e)^+$ and $(f)^+$. Clearly, $(e \wedge f)^+$ is the upper bound of both $(e)^+$ and $(f)^+$. Let $(c)^+$ be an upper bound for both $(e)^+$ and $(f)^+$. Let $(e)^+ \subseteq (c)^+$, $(f)^+ \subseteq (c)^+$ and $x \in (e \wedge f)^+$. Thus $(e^o \vee f^o) \vee x = (e \wedge f)^o \vee x$ is maximal. Hence for any $y \in M$, $[f^o \vee (e^o \vee x)] \wedge y = [(e^o \vee f^o) \vee x] \wedge y = y$. Consequently, $f^o \vee (e^o \vee x)$ is maximal which indicates that $e^o \vee x \in (f)^+ \subseteq (c)^+$. Thus $c^o \vee (e^o \vee x)$ is maximal and for any $y \in M$, $[e^o \vee (c^o \vee x)] \wedge y = [c^o \vee (e^o \vee x)] \wedge y = y$. Thus $e^o \vee (c^o \vee x)$ is maximal and so $c^o \vee x \in (e)^+ \subseteq (c)^+$. Hence $c^o \vee (c^o \vee x)$ is maximal. Then, for any $y \in M, (c^o \vee x) \wedge w = [c^o \vee (c^o \vee x)] \wedge w = w$ so that $c^o \vee x$ is maximal and $x \in (c)^+$. Hence $(e \wedge f)^+$ is supremum of $(e)^+$ and $(f)^+$ in $B_0(M)$. Therefore, $(B_0(M), \cap, \sqcup, (m)^+, M)$ is a bounded lattice. □

Consider elements m and t of an MS-ADL M, where m is maximal. Clearly, $t^o \vee m$ is maximal and hence $m \in t^+$. Following this, we have the next result.

Corollary 3.5. *An MS-ADL M has a maximal element if and only if $B_0(M)$ has a smallest element. Also, M is dual homomorphic to $B_0(M)$.*

Proof. Let m be a maximal element of M. Then $(m)^+ = \{m\} \subseteq (b)^+$ for any $(b)^+ \in B_0(M)$. Conversely, if we assume that $(s)^+$ is the smallest element of $B_0(M)$, there exists $t \in (s)^+$ such that $s^o \vee t$ is maximal in M. Conversely, assume that the class $B_0(M)$ of all boosters of an MS-ADL M has a smallest element. For any $b \in M$ define a map $\alpha : M \to B_0(M)$ by $\alpha(b) = (b)^+$. Clearly, $\alpha(0) = (0)^+ = M$ and for any $s, t \in M$, $\alpha(s \vee t) = (s \vee t)^+ = (s)^+ \cap (t)^+ = \alpha(s) \cap \alpha(t)$. Similarly, $\alpha(s \wedge t) = (s \wedge t)^+ = (s)^+ \sqcup (t)^+ = \alpha(s) \sqcup \alpha(t)$. Then α is a dual homomorphism. □

Definition 3.6. Let F be a filter of M and I be an ideal of $B_0(M)$. Define operators β and $\overleftarrow{\beta}$ respectively by

$$\beta(F) = \{(r)^+ \mid r \in F\} \quad \text{and} \quad \overleftarrow{\beta}(I) = \{r \in M \mid (r)^+ \in I\}.$$

Lemma 3.7. *For any filters D, J and K of M, and for any ideals I and J of $B_0(M)$, the following conditions hold:*

(i) $\beta(J \cap K) = \beta(J) \cap \beta(K)$,

(ii) $\overleftarrow{\beta}(I \cap J) = \overleftarrow{\beta}(I) \cap \overleftarrow{\beta}(J)$,

(iii) $\overleftarrow{\beta}\beta(J \cap K) = \overleftarrow{\beta}\beta(J) \cap \overleftarrow{\beta}\beta(K)$,

(iv) *$\beta(D)$ is an ideal of $B_0(M)$,*

(v) *$\overleftarrow{\beta}(I)$ is a filter of M,*

(vi) *β and $\overleftarrow{\beta}$ are isotone.*

Proof. (1) Suppose J and K are filters of M. For any $(t)^+ \in \beta(J) \cap \beta(K)$ we obtained that $(d)^+ = (t)^+ \in \beta(J)$ for some $d \in J$ and $(e)^+ = (t)^+ \in \beta(K)$ for some $e \in K$. Clearly, $d \vee e \in J \cap K$. Thus $(t)^+ = (d)^+ \cap (e)^+ = (d \vee e)^+ \in \beta(J \cap K)$. Hence $\beta(J) \cap \beta(K) \subseteq \beta(J \cap K)$. On the other hand, take $(t)^+ \in \beta(J \cap K)$ so that $(t)^+ = (d)^+$ for some $d \in J \cap K$. Thus $(T)^+ = (d)^+$ for some $d \in J$ and $d \in K$. So, $(t)^+ \in \beta(J)$ and $(t)^+ \in \beta(K)$. Hence $(t)^+ \in \beta(J) \cap \beta(K)$. Therefore $\beta(J \cap K) \subseteq \beta(J) \cap \beta(K)$.

(2) For any $t \in \overleftarrow{\beta}(I \cap J)$, $(t)^+ \in I \cap J$. Thus by (1) we obtain that $(t)^+ \in \beta(I \cap J) = \beta(I) \cap \beta(J)$ so that $(t)^+ = (i)^+$ and $(t)^+ = (j)^+$ for some $i \in I$ and $j \in J$. Then $(t)^+ \in \beta(I)$ and $(t)^+ \in \beta(J)$. Therefore $t \in \overleftarrow{\beta}(I)$ and $t \in \overleftarrow{\beta}(J)$ so that $t \in \overleftarrow{\beta}(I) \cap \overleftarrow{\beta}(J)$. The converse follows in a similar way.

(3) For any $p \in \overleftarrow{\beta}\beta(J \cap K)$, $(p)^+ \in \beta(J \cap K)$. From (1) we have $(p)^+ \in \beta(J) \cap \beta(K)$. Since $\beta(J)$ and $\beta(K)$ are ideals of $B_0(M)$, we have $p \in \overleftarrow{\beta}\beta(J)$ and $\overleftarrow{\beta}\beta(K)$. Then $p \in \overleftarrow{\beta}\beta(J) \cap \overleftarrow{\beta}\beta(K)$. Whence $\overleftarrow{\beta}\beta(J \cap K) \subseteq \overleftarrow{\beta}\beta(J) \cap \overleftarrow{\beta}\beta(K)$. Following the same procedure one can easily show the converse holds.

(4) Take a filter D in M. Since $m \in D$, $(m)^+ \in \beta(D)$. Take $(s)^+, (t)^+ \in \beta(D)$. Then $s, t \in D$ so that $s \wedge t \in D$. It follows that $(s)^+ \sqcup (t)^+ = (s \wedge t)^+ \in \beta(D)$. For any $r \in M$ we have $r \vee s \in D$ and $(r)^+ \in B_0(M)$. Then using Lemma 3.3(1), $(s)^+ \cap (r)^+ = (s \vee r)^+ = (r \vee s)^+ \in \beta(D)$. Therefore, $\beta(D)$ is an ideal in $B_0(M)$.

(5) Consider an ideal I of $B_0(M), r \in M$ and $s, t \in \overleftarrow{\beta}(I)$. Thus $(s)^+, (t)^+ \in I$. Then $(s \wedge t)^+ = (s)^+ \sqcup (t)^+ \in I$. Hence $s \wedge t \in \overleftarrow{\beta}(I)$. Also, $(r \vee s)^+ = (s \vee r)^+ = (s)^+ \cap (r)^+ \in I$. So $r \vee s \in \overleftarrow{\beta}(I)$. So, $\overleftarrow{\beta}(I)$ is a filter of M.

(6) Take two ideals J and K of M so that $J \subseteq K$. Let $j \in \overleftarrow{\beta}(J)$. Then $(j)^+ \in J \subseteq K$. It follows that $j \in \overleftarrow{\beta}(K)$ and hence $\overleftarrow{\beta}(J) \subseteq \overleftarrow{\beta}(K)$. Which shows that $\overleftarrow{\beta}$ is an isotone. The same is true for β. □

Lemma 3.8. *Consider the lattice of filters in MS-ADL M. Let D and E are filters in M. Then the map $D \mapsto \overleftarrow{\beta}\beta(D)$ is a closure operator. That is,*

(i) $D \subseteq \overleftarrow{\beta}\beta(D)$,

(ii) $F \subseteq E$ *implies* $\overleftarrow{\beta}\beta(D) \subseteq \overleftarrow{\beta}\beta(E)$,

(iii) $\overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(D)\} = \overleftarrow{\beta}\beta(D)$.

The intersection of filters is a filter and for any filter D of M, $r \vee s \in D$ if and only if $s \vee r \in D$. Following this we have the next theorem:

Theorem 3.9. *β is a homomorphism between the lattice of filters of M and the lattice of ideals of $B_0(M)$.*

Proof. Let D and E are filters of M. Clearly, from Lemma 3.7 $\beta(D \cap E) = \beta(D) \cap \beta(E)$ and $\beta(D) \sqcup \beta(E) \subseteq \beta(D \vee E)$. On the other hand, if $(s)^+ \in \beta(D \vee E)$, then $(s)^+ = (t)^+$ for some $t \in D \vee E$. Hence, $t = d \wedge e$ for some $d \in D$ and $e \in E$. Thus $(s)^+ = (t)^+ = (d \wedge e)^+ = (d)^+ \sqcup (e)^+ \in \beta(D) \sqcup \beta(E)$. So, $\beta(D \vee E) \subseteq \beta(D) \sqcup \beta(E)$. Therefore, β is a homomorphism between the lattice of filters(ideals) of $M(B_0(M))$. □

Definition 3.10. Any filter D in M is called β-filter whenever $\overleftarrow{\beta}\beta(D) = D$.

Theorem 3.11. *Every maximal filter is a β-filter.*

Proof. Let N be a maximal filter in M. Then from (1) of Lemma 3.8 we get $N \subseteq \overleftarrow{\beta}\beta(N)$. Now, let $y \in \overleftarrow{\beta}\beta(N)$. Then $(y)^+ \in \beta(N)$. Thus $(y)^+ = (n)^+$ for some $n \in N$. Suppose $y \notin N$. Then $N \vee [y) = M$ and hence $b \wedge y = 0$ for some $b \in N$. Thus $y \wedge b = 0$ so that $y^o \vee b^o = (y \wedge b)^o = 0^o$ is maximal, which yields that $b^o \in (y)^+ = (n)^+$. Therefore $n^o \vee b^o$ is maximal. Hence $b^{oo} \wedge n^{oo} = (b^o \vee n^o)^o = (n^o \vee b^o)^o = 0$. Since $b^{oo}, n^{oo} \in N$, we get $0 = b^{oo} \wedge n^{oo} \in N$, which is a contradiction. Hence $y \in N$. Thus $\overleftarrow{\beta}\beta(N) \subseteq N$. This proves the theorem. □

Corollary 3.12. *A maximal β-filter in M is a maximal filter.*

Proof. For a maximal β-filter D and any β-filter E, $D \subseteq E \Rightarrow D = E$. Hence D is a maximal filter. □

Theorem 3.13. *A filter D of M is a β-filter iff for all $s,t \in M, (s)^+ = (t)^+$ and $s \in D$ implies $t \in D$.*

Proof. If D is a β-filter of M and $s,t \in M$ be such that $(s)^+ = (t)^+$. Suppose $s \in D$. Then $(t)^+ = (s)^+ \in \beta(D)$. Since $\beta(D)$ is an ideal of $B_0(M)$, $t \in \overleftarrow{\beta}\beta(D) = D$. Conversely, for any $s,t \in M$, let $(s)^+ = (t)^+$ and $s \in D$ implies $t \in D$. Then $D \subseteq \overleftarrow{\beta}\beta(D)$. Now, let $s \in \overleftarrow{\beta}\beta(D)$. Then $(s)^+ \in \beta(D)$. Hence, $(s)^+ = (t)^+$ for some $t \in D$. From the given assumption, we get $s \in D$ so that $\overleftarrow{\beta}\beta(D) \subseteq D$. This shows that D is a β-filter of M. □

It is obvious that an MS-algebras are MS-ADLs. Next, we illustrate an e-filter of an MS-ADL is not necessarily a β-filter.

Example 3.14. Consider an MS-algebra $M = \{r,s,t,u,v,1\}$ on which the binary operations $\vee$ and $\wedge$ are defined by Table 1a, and Table 1b respectively, and the unary operation ^{o} is defined by Table 1c as follows:

$\vee$	0	r	s	t	u	v	1
0	0	r	s	t	u	v	1
r	r	r	s	t	u	v	1
s	s	s	s	t	u	v	1
t	t	t	t	t	v	v	1
u	u	u	u	v	u	v	1
v	v	v	v	v	v	v	1
1	1	1	1	1	1	1	1

(a)

$\wedge$	0	r	s	t	u	v	1
0	0	0	0	0	0	0	0
r	0	r	r	r	r	r	r
s	0	r	s	s	s	s	s
t	0	r	s	t	s	t	t
u	0	r	s	s	u	u	u
v	0	r	s	t	u	v	v
1	0	1	1	1	1	1	1

(b)

	0	r	s	t	u	v	1
$^\circ$	1	v	v	t	u	s	0

(c)

Table 1. Binary and unary operations defined on M

One can simply observe that, a set $A = \{1,v\}$, is e-filter of M, but not a β-filter of M.

Using the above Theorem, we show the relation between e-filters and β-filters of an MS-ADL.

Theorem 3.15. *Every β-filter in M is an e-filter.*

Proof. Let D be a β-filter in M. Then $\overleftarrow{\beta}\beta(D) = D$. *Claim*: D is e-filter in M. If $s^{\circ\circ} \in D$ then $s^{\circ\circ} \in \overleftarrow{\beta}\beta(D)$. This implies $(s^{\circ\circ})^+ = (t)^+$ for some $t \in D$. Since $(s \wedge m)^+ = (s^{\circ\circ})^+$, $(s \wedge m)^+ = (t)^+$ for some $t \in D$. Thus $s \wedge m \in D$. Hence D is an e-filters in M. □

Theorem 3.16. *Consider the set*

$$\mathfrak{F} = \{J \mid B \subseteq J, B \text{ is a } \beta\text{-filter, } J \text{ is a prime filter of } M\}.$$

If $J \in \mathfrak{F}$ is minimal, then J is a β-filter.

Proof. Let $J \in \mathfrak{F}$ be minimal and $B \subseteq J$ for some β-filter B of M. If J is not β-filter, then there exist $d, e \in M$ so that $(d)^+ = (e)^+, d \in J$ and $e \notin J$. Consider $I = (M - J) \vee (d \vee e]$. So $I \cap B = \emptyset$. On the other hand if we take $a \in I \cap B$. Then $a = b \vee c$ for some $b \in M - J$ and $c \in (d \vee e]$. Then $b \vee c = b \vee [(d \vee e) \wedge c] = (b \vee d \vee e) \wedge (b \vee c)$. Since $b \vee c \in B$ we have $b \vee d \vee e = (b \vee d \vee e) \vee (b \vee c) \in B$. From $(d)^+ = (e)^+$, we can get $(b \vee e)^+ = (b \vee d \vee e)^+$. Since B is a β-filter and $b \vee d \vee e \in B$, we get $b \vee e \in B \subseteq J$. It follows that $b \in J$ or $e \in J$, a contradiction. Thus $I \cap B = \emptyset$. Hence for some prime filter P, $I \cap P = \emptyset$ and $B \subseteq P$. Since $I \cap P = \emptyset$, one can get $P \subseteq J$. Further $e \vee d \notin P$ and $e \vee d \in J$. It shows that $P \subset J$. Hence J is not minimal in $\mathfrak{F}$ containing B, a contradiction. So J is a β-filter in M. □

Corollary 3.17. *The set $\mathscr{F}_\beta(M)$ of all β-filters of M forms a distributive lattice.*

Lemma 3.18. *For any ideal J of $B_0(M)$, $\beta\overleftarrow{\beta}(J) = J$.*

Proof. Let J be an ideal of $B_0(M)$. If $(s)^+ \in J$, then $s \in \overleftarrow{\beta}(J)$ so that $(s)^+ \in \beta\overleftarrow{\beta}(J)$. Thus $J \subseteq \beta\overleftarrow{\beta}(J)$. Conversely, if $(s)^+ \in \beta\overleftarrow{\beta}(J)$, then we get $(s)^+ = (t)^+$ for some $t \in \overleftarrow{\beta}(J)$. Also, $t \in \overleftarrow{\beta}(J)$ implies that $(s)^+ = (t)^+ \in J$ so that $\beta\overleftarrow{\beta}(J) \subseteq J$. Hence $\beta\overleftarrow{\beta}$ is a constant mapping on $B_0(M)$. □

Next we show that the set of β-filters of M (denoted by $\mathscr{F}_\beta(M)$) is isomorphic to the set of ideals of $B_0(M)$ (denoted by $I(B_0(M))$).

Theorem 3.19. *There is an isomorphism of $\mathscr{F}_\beta(M)$ onto $I(B_0(M))$.*

Proof. Assume α is the restriction of β to $\mathscr{F}_\beta(M)$. It follows that α is one-to-one. Take J be an ideal of $B_0(M)$ so that $\overleftarrow{\beta}(J)$ is a filter of M. From Lemma 3.18 we get $\overleftarrow{\beta}\beta\{\overleftarrow{\beta}(J)\} = \overleftarrow{\beta}\{\beta\overleftarrow{\beta}(J)\} = \overleftarrow{\beta}(J)$ which indicates $\overleftarrow{\beta}(J)$ is a β-filter of M. Thus $\alpha(\overleftarrow{\beta}(J)) = \beta\overleftarrow{\beta}(J) = J$ and hence α is onto. Let J and K are β-filters in M. We obtain that $\alpha(J \cap K) = \beta(J \cap K) = \beta(J) \cap \beta(K) = \alpha(J) \cap \alpha(K)$ and

$$
\begin{aligned}
\alpha\{\overleftarrow{\beta}\beta(J \vee K)\} &= \beta\{\overleftarrow{\beta}\beta(J \vee K)\} \\
&= \beta(J \vee K) \\
&= \beta(J) \sqcup \beta(K) \\
&= \alpha(J) \sqcup \alpha(K).
\end{aligned}
$$

Therefore α is an isomorphism of $\mathscr{F}_\beta(M)$ onto $B_0(M)$. □

Consequently, we have the next corollary.

Corollary 3.20. *Prime β-filters in M are in correspondence with the prime ideals in $B_0(M)$.*

Theorem 3.21. *Every proper β-filter in M is the intersection of all prime β-filters containing it.*

Proof. For a proper β-filter D in M consider the set

$$D_0 = \cap\{P \mid P \text{ is a prime } \beta\text{-filter and } D \subseteq P\}.$$

Obviously, $D \subseteq D_0$. Let $d \notin D$ and $\Gamma = \{E \mid E$ is a β-filter,$D \subseteq E, d \notin E\}$. One can simply observe that $D \in \Gamma$ and Γ contains a maximal element, say N. Let $a, b \in M$ be such that $a \notin N$ and $b \notin N$. Then $N \subset N \vee [a) \subseteq \overleftarrow{\beta}\beta\{N \vee [a)\}$ and $N \subset N \vee [b) \subseteq \overleftarrow{\beta}\beta\{N \vee [b)\}$. Since N is maximal, we obtain that $d \in \overleftarrow{\beta}\beta\{N \vee [a)\}$ and $d \in \overleftarrow{\beta}\beta\{N \vee [b)\}$ so that $d \in \overleftarrow{\beta}\beta\{N \vee [a)\} \cap \overleftarrow{\beta}\beta\{N \vee [b)\} = \overleftarrow{\beta}\beta\{[N \vee [a)] \cap [M \vee [b)]\} = \overleftarrow{\beta}\beta\{N \vee [a \vee b)\}$. If $a \vee b \in N$, then $d \in \overleftarrow{\beta}\beta(N) = N$, which is a contradiction. This shows that N is a prime β- filter such that $d \notin N$ and hence $d \notin D_0$. □

Next, we give equivalent conditions on which the lattice $B_0(M)$ is relatively complemented. First we need to remember the following result.

Lemma 3.22 ([7])**.** *An ADL is relatively complemented iff each prime ideal is maximal.*

Theorem 3.23. *The following conditions are equivalent on M.*

(i) *$B_0(M)$ is relatively complemented,*

(ii) *every prime β-filter is a maximal filter,*

(iii) *every prime β-filter is minimal.*

Corollary 3.24. *If $B_0(M)$ is relatively complemented, then each β-filter is an intersection of all maximal filters.*

4. Conclusion

The concepts of boosters and β-filters in MS-almost distributive lattices is introduced. We investigate the class of boosters in an MS-almost distributive lattice is a bounded distributive lattice. Characterization of β-filters of an MS-almost distributive lattice in terms of boosters is established. We derived a dual homomorphism between an MS-almost distributive lattice and the set of all boosters in MS-almost distributive lattice. Further, it is shown that any maximal filter in an MS-almost distributive lattice is a β-filter. Finally, the conditions on which the lattice of boosters is a relatively complemented lattice are given. We hope in the future, we study on boosters and filters on skew lattices and Heyting almost distributive lattices.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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