Communications in Mathematics and Applications
Vol. 13, No. 3, pp. 1097–1107, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i3.1708

http://www.rgnpublications.com

Research Article

D-hyponormal and *D*-quasi-hyponormal Operators

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Received: October 23, 2021 **Accepted:** March 8, 2022

Abstract. New classes of operators named D-hyponormal, and D-quasi-hyponormal are introduced in this paper. Some basic properties of these operators are presented. An investigation of extensions of the Fuglede-Putnam theorem for D-hyponormal operators is given.

Keywords. Drazin inverse, D-hyponormal operator, D-quasi-hyponormal operator, Fuglede-Putnam theorem

Mathematics Subject Classification (2020). 47B15, 47B20

1. Introduction

Let $\mathcal{H}$ represent a separable, complex and infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. We denote by A^*, $\sigma(A)$, $\mathcal{R}(A)$ and $\ker(A)$ the adjoint, the spectrum, the range and the kernel of an operator $A \in \mathcal{B}(\mathcal{H})$, respectively.

For an arbitrary $A \in \mathcal{B}(\mathcal{H})$, we have: $|A|^2 = (A^*A)$ (the absolute value of A) and $[A^*,A] = |A|^2 - |A^*|^2 = A^*A - AA^*$ (the self commutator of A).

$A \in \mathcal{B}(\mathcal{H})$ is called:

- *normal* if: $|A|^2 = |A^*|^2$,
- *hyponormal* if: $|A^*|^2 \le |A|^2$; let $[HN]$ denote the hyponormal operators class,
- *co-hyponormal* if: $|A|^2 \le |A^*|^2$. In other words, A is co-hyponormal if A^* is hyponormal,

- *quasihyponormal* if: $A^*(|A|^2-|A^*|^2)A \geq 0$; let $[QH]$ denote the quasihyponormal operators class.

In [7], Caradus introduced and studied the Drazin inverse for bounded linear operators. The Drazin inverse is useful in different fields, including: difference and differential equations, Markov chains and Cauchy problems ([3], [6]).

Definition 1. Let $A \in \mathcal{B}(\mathcal{H})$. A is Drazin invertible if there exists a unique operator $A^D \in \mathcal{B}(\mathcal{H})$ (A^D is the Drazin inverse of A), verifying:

$$AA^D = A^D A, \quad A^D AA^D = A^D, \quad A^{v+1}A^D = A^v, \quad \text{for some } v \in \mathbb{N}.$$

The index of A, denoted by $ind(A)$, is the smallest number $v \in \mathbb{N}$ satisfying the previous equation.

Let $\mathcal{B}(\mathcal{H})^D$ denote the set of all Drazin invertible operators in $\mathcal{B}(\mathcal{H})$.

It is known that if A is invertible then $ind(A) = 0$, i.e., $A^D = A^{-1}$. If $ind(A) = 1$, then $A^D = A^{\ddagger}$ (group inverse). If A is nilpotent, then it is Drazin invertible, $A^D = 0$ and $ind(A) = p$, where p denotes the nilpotent power of A.

For $A \in \mathcal{B}(\mathcal{H})$, it was observed that A^D satisfies $(A^*)^D = (A^D)^*$ and $(A^k)^D = (A^D)^k$ for $k \in \mathbb{N}$.

An operator A is called finite if it satisfies:

$$\|AX - XA - I\| \geq 1, \quad \forall\, X \in \mathcal{B}(\mathcal{H}).$$

Williams [20] proved that finite operators class, denoted by $\mathcal{F}(\mathcal{H})$, contains every normal and hyponormal operators. Mecheri [16], and Messaoudene [8] have generalized William's results to more classes containing normal and hyponormal operators classes.

The classes of operators introduced above are related to some well-known theorems in operator theory, such as the classical Fuglede-Putnam theorem. Since the papers of Fuglede [11] and then Putnam [19], there have been many extensions of this theorem to nonnormal operators (see [2], [1], [4], [12], [18]).

This theorem reads as follows:

Theorem 2 ([13])**.** *Let $A,B \in \mathcal{B}(\mathcal{H})$ be normal operators. If $AX = XB$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB^*$.*

In this paper, new classes of operators denoted by $[DH]$ and $[DQH]$, called D-hyponormal and D-quasi-hyponormal operators, respectively, associated with a Drazin invertible operator are introduced. Some properties of these operators are given. A D-hyponormal operator is proved to be finite. An investigation of extensions of the Fuglede-Putnam theorem for D-hyponormal operators is given.

2. Preliminaries

Lemma 3 ([6])**.** *For $A,B \in \mathcal{B}(\mathcal{H})^D$, the following properties hold.*

(a) *$AB \in \mathcal{B}(\mathcal{H})^D$ if and only if $BA \in \mathcal{B}(\mathcal{H})^D$. Moreover*

$$(AB)^D = A[(BA)^D]^2B \ \text{ and } \ ind(AB) \leq ind(BA)+1.$$

(b) *If A is idempotent, then $A^D = A$.*

(c) *If $AB = BA$, then $(AB)^D = A^D B^D = B^D A^D$, $BA^D = A^D B$ and $B^D A = AB^D$.*

(d) *If $BA = AB = 0$, then $A^D + B^D = (A+B)^D$.*

Remark 4. Let $A \in \mathcal{B}(\mathcal{H})^D$. Then:

(1) $A^\pi = I - AA^D$ is the spectral idempotent of A that corresponds to $\{0\}$.

(2) $A = A_1 \oplus A_2$, where A_1 is invertible and A_2 is nilpotent, is the matrix form of A according to the decomposition $\mathcal{H} = \overline{\mathcal{R}(A^\pi)} \oplus \ker(A^\pi)$ ($\overline{\mathcal{R}(A^\pi)}$ is the closure of $\mathcal{R}(A^\pi)$).

Lemma 5 ([6])**.** *If $A \in \mathcal{B}(\mathcal{H})^D$ and $B \in \mathcal{B}(\mathcal{K})^D$ with $ind(A) = m$ and $ind(B) = n$, then $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is also Drazin invertible and*

$$T^D = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix},$$

where

$$X = \sum_{i=0}^{n-1} (A^D)^{i+2} C B^i B^\pi + A^\pi \sum_{i=0}^{m-1} A^i C (B^D)^{i+2} - A^D C B^D. \tag{2.1}$$

Definition 6 ([9])**.** Let $A \in \mathcal{B}(\mathcal{H})^D$. A is called:

(1) D-*normal* if: $A^D A^* = A^* A^D$.

(2) D-*quasi-normal* if: $A^D A^* A = A^* A A^D$.

Let $[DN]$ and $[DQN]$ denote the classes constituting of D-normal and D-quasi-normal operators.

These classes were firstly introduced by Dana and Yousefi [9]. From the definitions above, we can easily verify that:

$$[N] \subset [DN] \subset [DQN].$$

Definition 7. Let $\lambda \in \mathbb{C}$. If there exists a normed sequence $\{x_n\} \in \mathcal{H}$ verifying $\lim_n (A - \lambda I)x_n = 0$, then λ is said to be in the approximate spectrum $\sigma_a(A)$ of A. If in addition, $\lim_n (A - \lambda I)^* x_n = 0$, then λ belongs to the approximate reduced spectrum $\sigma_{ar}(A)$ of A.

3. *D*-hyponormal Operators

Definition 8. Let $A \in \mathcal{B}(\mathcal{H})^D$. A is D-hyponormal if:

$$A^* A^D - A^D A^* \geq 0.$$

The class of D-hyponormal operators is denoted by $[DH]$.

D-hyponormal operators provide an extension of hyponormal operators because in general the D-hyponormal operator is different from hyponormal operator.

Example 9. Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^4)$. Then:

$$A^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $A \in [DH]$ but it is not hyponormal.

In the next remark we give a condition that $[DH]$ class coincide with $[HN]$ class.

Remark 10. Let $A \in [DH]$. If $ind(A) \le 1$, then $A \in [HN]$.

Proposition 11. *Let $A \in [DH]$. Then A^* is D-co-hyponormal operator.*

Proof. Since A is a D-hyponormal operator, then:

$$\begin{aligned} A^*A^D \ge A^D A^* &\Longrightarrow (A^*A^D)^* \ge (A^D A^*)^* \\ &\Longrightarrow (A^D)^* A \ge A(A^D)^*. \end{aligned}$$

Hence, A^* is a D-co-hyponormal operator. □

Proposition 12. *If $S, A \in \mathcal{B}(\mathcal{H})^D$ such that S is unitary equivalent to A and if A is D-hyponormal operator, then so is S.*

Proof. Let $A \in [DH]$ and $S \in \mathcal{B}(\mathcal{H})^D$ which is unitary equivalent to A. Thus there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ satisfying $S = U^*AU$. So $S^* = U^*A^*U$ and $S^D = U^*A^DU$.

We have:

$$\begin{aligned} S^*S^D &= U^*A^*UU^*A^DU \\ &= U^*A^*A^DU \\ &\ge U^*A^DA^*U \\ &\ge U^*A^DUU^*A^*U \\ &= S^DS^*. \end{aligned}$$

Thus, $S^*S^D - S^DS^* \ge 0$. □

Theorem 13. *If A, A^* are two D-hyponormal operators, then A is a D-normal operator.*

Proof. First let $A^* \in [DH]$. Then $A(A^*)^D \ge (A^*)^D A$. Since $(A^*)^D = (A^D)^*$, we have

$$\begin{aligned} A(A^D)^* \ge (A^D)^* A &\Longrightarrow (A(A^D)^*)^* \ge ((A^D)^* A)^* \\ &\Longrightarrow A^D A^* \ge A^* A^D. \end{aligned}$$

On the other hand, $A \in [DH]$ implies $A^*A^D \ge A^DA^*$. Hence $A^*A^D = A^DA^*$, which completes the proof. □

Recall that a pair $(A,B) \in \mathcal{B}(\mathcal{H})^2$ is called a doubly commuting pair if (A,B) satisfies $BA = AB$ and $A^*B = BA^*$.

Theorem 14. *Let $A,B \in [DH]$. If (A,B) is a doubly commuting pair, then the following assertions hold.*

(1) *AB is D-hyponormal.*

(2) *If $BA = AB = 0$, then $A+B$ is D-hyponormal operator.*

Proof. (1) Since $BA = AB$ and $A^*B = BA^*$, it follows that:

$$\begin{aligned}(AB)^*(AB)^D &= A^*B^*A^DB^D = A^*A^DB^*B^D \\ &\geq A^DA^*B^DB^* \\ &= A^DB^DA^*B^* \\ &= (AB)^D(AB)^*.\end{aligned}$$

Hence, AB is D-hyponormal.

(2) Under the assumptions that A and B are D-hyponormal, it follows by taking into account the statements of Lemma 3 that:

$$\begin{aligned}(A+B)^*(A+B)^D &= (A^*+B^*)(A^D+B^D) \\ &= A^*A^D + A^*B^D + B^*A^D + B^*B^D \\ &\geq A^DA^* + B^DA^* + A^DB^* + B^DB^* \\ &= (A+B)^D(A+B)^*.\end{aligned}$$

Hence, $A+B$ is D-hyponormal. □

Proposition 15. *If $A,B \in [DH]$, then $(A \oplus B) \in [DH]$ and $(A \otimes B) \in [DH]$.*

Proof. Let $A,B \in [DH]$, then:

$$\begin{aligned}(A \oplus B)^*(A \oplus B)^D &= (A^* \oplus B^*)(A^D \oplus B^D) \\ &= A^*A^D \oplus B^*B^D \\ &\geq A^DA^* \oplus B^DB^* \\ &= (A^D \oplus B^D)(A^* \oplus B^*) \\ &= (A \oplus B)^D(A \oplus B)^*.\end{aligned}$$

Hence $(A \oplus B)$ is of class $[DH]$. Now, for $x_1, x_2 \in \mathcal{H}$:

$$\begin{aligned}(A \otimes B)^*(A \otimes B)^D(x_1 \otimes x_2) &= (A^* \otimes B^*)(A^D \otimes B^D)(x_1 \otimes x_2) \\ &= A^*A^Dx_1 \otimes B^*B^Dx_2 \\ &\geq A^DA^*x_1 \otimes B^DB^*x_2 \\ &= (A^D \otimes B^D)(A^* \otimes B^*)(x_1 \otimes x_2) \\ &= (A \otimes B)^D(A \otimes B)^*(x_1 \otimes x_2).\end{aligned}$$

Thus $(A \otimes B)$ is of class $[DH]$. □

Theorem 16. *If $A \in [DH]$, then*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(A^D)} \oplus \ker(A^D),$$

where A_1 is of class $[HN]$ and $A_3^k = 0$ ($k = ind(A)$).

Proof. Suppose $A \in [DH]$, then $\ker(A^D) = \ker(A^{*D})$. If $\mathcal{R}(A^D)$ is not dense and A has the matrix representation:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{\mathcal{R}(A^D)} \oplus \ker(A^D)$, then

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = PAP = PA = AP,$$

(P denotes the orthogonal projection onto $\mathcal{R}(A^D)$). Thus

$$PA^*A^DP = \begin{pmatrix} A_1^*A_1^D & 0 \\ 0 & 0 \end{pmatrix} \text{ and } PA^DA^*P = \begin{pmatrix} A_1^DA_1^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $A \in [DH]$, $PA^*A^DP \geq PA^DA^*P$ implies $A_1^*A_1^D \geq A_1^DA_1^*$. Hence $A_1 \in [DH]$. Furthermore, by Remark 4, A_1 is invertible. So, by Remark 10, $A_1 \in [HN]$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}$. Then

$$\begin{aligned} \langle A_3^D x_2, x_2 \rangle &= \langle (A^D - A^DP)x, (I-P)x \rangle \\ &= (I-P)x, A^{D*}(I-P)x \rangle \\ &= 0. \end{aligned}$$

So, $A_3^D = 0$. Hence A_3 is a nilpotent operator. □

Lemma 17. *If $A \in [DH]$, then the restriction $A_{|\mathcal{M}}$ of A to a closed subspace $\mathcal{M}$ of $\mathcal{H}$ reducing A is also of class $[DH]$.*

Proof. Let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$ with $A_1 = A_{|\mathcal{M}}$. Now we can write the matrix representation of A as:

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

Then

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = AP = PAP.$$

Since $A \in [DH]$, we have:

$$A^*A^D - A^DA^* \geq 0.$$

Hence

$$\begin{pmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{pmatrix}\begin{pmatrix} A_1^D & X \\ 0 & A_3^D \end{pmatrix} - \begin{pmatrix} A_1^D & X \\ 0 & A_3^D \end{pmatrix}\begin{pmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{pmatrix} \geq 0.$$

Therefore,

$$\begin{pmatrix} A_1^*A_1^D - A_1^D A_1^* - XA_2^* & E \\ F & A_3^*A_3^D - A_3^D A_3^* \end{pmatrix} \geq 0,$$

for some operators E,F and X is defined by (2.1). Hence

$$A_1^*A_1^D - A_1^D A_1^* \geq XA_2^* \geq 0.$$

This implies that $A_1 = A_{|\mathcal{M}} \in [DH]$. □

Proposition 18. *Let* $A \in [DH]$. *If* $(A-\lambda)x = 0$, $\lambda \neq 0$, *then* $(A-\lambda)^* x = 0$, *for some* $x \in \mathcal{H}$.

Proof. If $x = 0$, then the proof is obvious. If $x \neq 0$, let $\mathcal{M} = span\{x\}$. Hence $\mathcal{M}$ is an invariant subspace of A. Suppose

$$A = \begin{pmatrix} \lambda & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp. \tag{3.1}$$

Let Q be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, where $A|_{\mathcal{M}} = \lambda$. Hence $A_1 = AQ = QAQ$ and $A_1^* = QA^* = QA^*Q$.

For the proof, it suffices to show that $A_2 = 0$ in (3.1).

Since $A \in [DH]$,

$$Q(A^*A^D - A^D A^*)Q \geq 0,$$

$$\begin{pmatrix} \frac{\bar{\lambda}}{\lambda} & 0 \\ 0 & 0 \end{pmatrix} = Q(A^*A^D)Q \geq Q(A^D A^*)Q = \begin{pmatrix} \frac{\bar{\lambda}}{\lambda} + XA_2^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $A_2 = 0$. □

Lemma 19 ([16])**.** *Let* $A \in \mathcal{B}(\mathcal{H})$. *If* $\sigma_{ar}(A) \neq \phi$, *then* A *is finite.*

Lemma 20. *If* $A \in [DH]$, *then* $\sigma_{ar}(A) \neq \phi$.

Proof. Let A be a D-hyponormal operator, we have: $\sigma_{ar}(A) \subset \sigma_a(A)$. Since $\sigma_a(A)$ is never empty, it suffices to prove that $\sigma_a(A) \subset \sigma_{ar}(A)$.

Let $\lambda \in \sigma_a(A)$, then there is a normed sequence $\{x_n\} \in \mathcal{H}$ satisfying: $\lim_n (A - \lambda I)x_n = 0$. Using Proposition 18 we obtain $\lim_n (A - \lambda I)^* x_n = 0$ and $\lambda \in \sigma_{ar}(A)$. This completes the proof. □

Theorem 21. *Let* $A \in [DH]$, *then* $A \in \mathcal{F}(\mathcal{H})$.

Proof. Let $A \in [DH]$. Then $\sigma_{ar}(A) \neq \phi$ by Lemma 20 and so A is finite by Lemma 19. □

Let $\mathcal{C}_2(\mathcal{H})$ denote the Hilbert-Schmidt operators class. $\mathcal{C}_2(\mathcal{H})$ is itself a Hilbert space with the inner product:

$$\langle A, B \rangle = tr(AB^*) = tr(B^*A)$$

where $tr(.)$ denotes trace (.).

For given operators $A, B \in \mathcal{B}(\mathcal{H})$, the operator $\mathcal{K}$ defined on $\mathcal{C}_2(\mathcal{H})$ via the formula $\mathcal{K}X = AXB$ has been studied in [4].

From the basic property of Hilbert-Schmidt norms, we have: $\mathcal{K}^*X = A^*XB^*$. Moreover, $\mathcal{K}^D X = A^D X B^D$, where $\mathcal{K}^D$ is the Drazin inverse of $\mathcal{K}$.

Lemma 22. *If $A \in [DH]$ and $B \in [DN]$, then $\mathcal{K} \in [DH]$.*

Proof. Since $A^*A^D - A^DA^* \geq 0$ and $B^*B^D - B^DB^* = 0$, we have

$$\begin{aligned}(\mathcal{K}^*\mathcal{K}^D - \mathcal{K}^D\mathcal{K}^*)X &= \mathcal{K}^*\mathcal{K}^D X - \mathcal{K}^D\mathcal{K}^* X \\ &= \mathcal{K}^*(A^D X B^D) - \mathcal{K}(A^* X B^*) \\ &= A^*A^D X B^D B^* - A^D A^* X B^* B^D \\ &\geq A^D A^* X B^D B^* - A^D A^* X B^* B^D \\ &= A^D A^* X B^D B^* - A^D A^* X B^D B^* \\ &= 0.\end{aligned}$$

Hence, $\mathcal{K} \in [DH]$. □

Theorem 23. *Let $A \in [DH]$ and B an invertible D-normal operator. If $AX = XB$, for some $X \in \mathcal{C}_2(\mathcal{H})$, then $A^*X = XB^*$.*

Proof. Let $\mathcal{K}$ be a Hilbert-Schmidt operator defined by $\mathcal{K}X = AXB^{-1}$, for all $X \in \mathcal{C}_2(\mathcal{H})$. Since $A \in [DH]$ and $B \in [DN]$, by Lemma 22, $\mathcal{K}$ is of class $[DH]$. Moreover,

$$\mathcal{K}X = AXB^{-1} = XBB^{-1} = X,$$

that is, X is an eigenvector of $\mathcal{K}$. Hence $\mathcal{K}^*X = X$ by Proposition 18 and so $A^*X = XB^*$ as desired. □

Corollary 24. *Let $A, B \in [DN]$ such that B is invertible. If $AX = XB$, for some $X \in \mathcal{C}_2(\mathcal{H})$, then $A^*X = XB^*$.*

4. *D*-quasi-hyponormal Operators

Definition 25. Let $A \in \mathcal{B}(\mathcal{H})^D$. A is D-quasi-hyponormal if:

$$A^*AA^D \geq A^DA^*A.$$

Let $[DQH]$ denote the class of all D-quasi-hyponormal operators.

Remark 26. Let $A \in \mathcal{B}(\mathcal{H})^D$. A is D-quasi-hyponormal if and only if:

$$|A|^2A^D \geq A^D|A|^2.$$

Obviously, $[DQH]$ includes classes of quasihyponormal operators and D-hyponormal operators, we have:

$$[HN] \subset [QH] \subset [DQH] \text{ and } [HN] \subset [DH] \subset [DQH].$$

we give some sufficient conditions for a D-quasi-hyponormal operator to be quasi-hyponormal.

Remark 27. Let $A \in [DQH]$. If $ind(A) < 1$, then $A \in [HN]$.

Remark 28. Let $A \in [DH]$. If $ind(A) = 1$, then $A \in [QH]$.

Theorem 29. *If $A \in [DQH]$, then the following statements hold.*

(1) *If $S \in \mathcal{B}(\mathcal{H})^D$ and unitary equivalent to A, then $S \in [DQH]$.*

(2) *If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces A, then $A_{|\mathcal{M}} \in [DQH]$.*

(3) *If A has a dense range in $\mathcal{H}$, $A \in [DH]$.*

(4) *If $B \in [DQH]$ with $[A,B] = [A,B^*] = 0$, then $AB \in [DQH]$.*

(5) *If $B \in [DQH]$ with $BA = AB = A^*B = B^*A = 0$, then $B + A$ is of class $[DQH]$.*

Proof. (1) and (2) are trivial.

(3) Since $A \in [DQH]$, we have for $y \in \mathcal{R}(A) : y = Ax$, $x \in \mathcal{H}$,

$$\begin{aligned} \|(A^*A^D - A^DA^*)y\| &= \|(A^*A^D - A^DA^*)Ax\| \\ &= \|(A^*AA^D - A^DA^*A)x\| \\ &\geq 0. \end{aligned}$$

Hence, $A \in [DH]$.

(4) Let $A, B \in [DQH]$ such that $[A,B] = [A,B^*] = 0$. Then, by Lemma 3(c), we get that $[A,B^D] = [A^D,B] = [A^D,B^*] = [A^*,B^D] = 0$. Thus

$$\begin{aligned} (AB)^*(AB)(AB)^D &= B^*A^*ABB^DA^D = B^*BA^*AB^DA^D \\ &= B^*BA^*B^DAA^D = B^*BB^DA^*AA^D \\ &\geq B^DB^*BA^*AA^D = B^DB^*A^*BAA^D \\ &= B^DB^*A^*ABA^D = B^DB^*A^*AA^DB \\ &\geq B^DB^*A^DA^*AB = B^DA^DB^*A^*AB \\ &= (AB)^D(AB)^*(AB). \end{aligned}$$

Hence, $AB \in [DQH]$.

(5) Let $B \in [DQH]$ with $BA = AB = A^*B = B^*A = 0$. Then:

$$\begin{aligned} (B+A)^*(B+A)(B+A)^D &= (B^* + A^*)(BB^D + AA^D) \\ &= B^*BB^D + A^*AA^D \\ &\geq B^DB^*B + A^DA^*A \\ &= (B+A)^D(B+A)^*(B+A). \end{aligned}$$

Hence $B + A$ is of class $[DQH]$. □

Proposition 30. *The tensor product and the direct sum of two operators in $[DQH]$ are in $[DQH]$.*

Proof. The proof of this proposition is formally the same as the proof of Proposition 15 with suitable changes and thus we omit the details. □

5. Conclusion

In this paper, we have introduced new classes of operators denoted by $[DH]$ and $[DQH]$, called D-hyponormal and D-quasi-hyponormal operators, respectively. We have presented some properties of these operators. We also proved that the Fuglede-Putnam theorem holds for D-hyponormal operators.

Acknowledgment

This work would not have been possible without the financial support of the Directorate General for Scientific Research and Technological Development (DGRSDT).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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