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Research Article

A Generalised Balancing Sequence and Solutions of Diophantine Equations $x^2 \pm pxy + y^2 \pm x = 0$

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Abstract. We consider a generalization of balancing sequences and investigate some properties of the generalised balancing sequences in this paper. For a positive integer p we solve for the Diophantine equations, $x^2 \pm pxy + y^2 \pm x = 0$ and express its solutions in terms of generalised balancing sequences.

Keywords. Diophantine equation, Balancing sequences

Mathematics Subject Classification (2020). 94A60, 11T74

1. Introduction

Behera and Panda [2] defined balancing numbers and balancers, respectively as the natural numbers n and r that satisfy the Diophantine equation

$$1+2+3+\ldots+(n-1) = (n+1)+(n+2)\ldots+(n+r).$$

This equation when simplified gives $n^2 = \frac{(n+r)(n+r+1)}{2}$. This leads to the conclusion that if n^2 is a triangular number then n is a balancing number and vice versa. B_n is denoted as the nth balancing number. It satisfies the recurrent relation

$$B_{n+1} = 6B_n - B_{n-1}$$

with $B_0 = 0$ and $B_1 = 1$. Hence they are also called as balancing sequences. Panda and Rout [7] generalized the above recurrent relation to

$$B_{n+1} = pB_n - qB_{n-1}$$

with $B_0 = 0$, $B_1 = 1$ for p a positive integer, they also proved that all properties of balancing number studies are also true for this generalised sequence when $q = 1$. Many authors have discussed the conics whose equations are satisfied by terms of balancing numbers. In this paper we investigate some properties of the generalised balancing sequences. We prove that there are infinitely many positive integer solutions for the Diophantine equation $x^2 \pm pxy + y^2 \pm x = 0$ for a positive integer p, that are positive and can be written in the terms of the above generalised sequence obtained by using continued fractions. Marlewski and Zarzycki [4] had proved that there exists infinitely many integer solutions (x, y), that are positive, for the Diophantine equation $x^2 - pxy + y^2 + x = 0$ if and only if $p = 3$.

Bahramain and Daghigh [1] proved that for a positive integer p the Diophantine equation $x^2 \pm pxy - y^2 \pm x = 0$ has positive solutions (x, y) that are infinitely many and they expressed these solutions in terms of Fibonacci sequences. We adapt a similar approach for the Diophantine equations $x^2 \pm pxy + y^2 \pm x = 0$ and show that there are infinitely many solutions in each case and express the solutions in terms of generalised balancing sequences.

2. Some Preliminaries on Generalised Balancing Sequences

Generalised balancing sequences are numbers satisfying the recurrent relation

$$B_{n+1} = pB_n - B_{n-1}$$

with $B_0 = 0$, $B_1 = 1$ for p, a positive integer. In this section we investigate some properties of the above balancing sequences for $q = 1$. The equation $B_{n+1} = pB_n - B_{n-1}$ can be expressed as a matrix equation given as

$$\begin{bmatrix} B_{n+1} \\ B_n \end{bmatrix} = \begin{bmatrix} p & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} B_n \\ B_{n-1} \end{bmatrix}$$

and the matrix $\begin{bmatrix} p & -1 \\ 1 & 0 \end{bmatrix}$ is denoted as Q_{B_p}. Basing on this matrix we prove some results on the above Generalised Balancing Sequence in the following subsection.

2.1 Some Results on Generalised Balancing Sequences

Theorem 2.1 ([1])**.** $Q_{B_p}^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$, $n \geq 1$.

Proof. Follows by induction. □

Remark 2.1 ([1])**.** The definition of the Balancing sequence $B_{n+1} = pB_n - B_{n-1}$ can be extended to all integers.

Theorem 2.2 ([1])**.** $B_{-n} = -B_n$ *for all* $n \geq 1$.

Proof. For $Q_{B_p} = \begin{bmatrix} p & -1 \\ 1 & 0 \end{bmatrix}$, we have $Q_{B_p}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & p \end{bmatrix}$. Now by taking $n = -q$ we get

$$B_{-(q-1)} = pB_{-q} - B_{-(q+1)}$$

then by matrix representation we have

$$\begin{bmatrix} B_{-q} \\ B_{-(q+1)} \end{bmatrix} = Q_{B_p}^{-1} \begin{bmatrix} B_{(-q+1)} \\ B_{-q} \end{bmatrix}$$

and by induction we have

$$Q_{B_p}^{-q} = \begin{bmatrix} B_{-(q-1)} & -B_{-q} \\ B_{-q} & -B_{-(q+1)} \end{bmatrix}$$

Now from Remark 2.1, we get

$$Q_{B_p}^{q} = \begin{bmatrix} B_{q+1} & -B_{q} \\ B_{q} & -B_{q-1} \end{bmatrix}$$

for all $s \geq 1$, then

$$(Q_{B_p}^{q})^{-1} = \begin{bmatrix} -B_{q-1} & -B_{q} \\ -B_{q} & B_{q+1} \end{bmatrix}.$$

Now as $Q_{B_p}^{-q} = (Q_{B_p}^{q})^{-1}$, we have

$$\begin{bmatrix} -B_{q-1} & B_{q} \\ -B_{q} & B_{q+1} \end{bmatrix} = \begin{bmatrix} B_{-(q-1)} & -B_{-q} \\ B_{-q} & -B_{-(q+1)} \end{bmatrix}$$

giving $B_{-q} = -B_q$ for all $q \geq 1$. □

Theorem 2.3 ([1]). *$pB_nB_{n+1} - B_n^2 + 1 = B_{n+1}^2$ $\forall$ integers n.*

Proof. Follows from $\det(Q_{B_p}^n) = 1$ i.e. $-B_{n+1}B_{n-1} + B_n^2 = 1$. □

Theorem 2.4 ([1]).

$$B_{m+n} = -B_{m+1}B_n + B_mB_{n-1} = B_mB_{n+1} - B_{m-1}B_n$$
$$B_{2m} = B_mB_{m+1} - B_{m-1}B_m$$
$$B_{2m-1} = B_m^2 - B_{m-1}^2 \quad \forall \text{ integers } m \text{ and } n.$$

Proof. Follows from the equality $Q_{B_p}^{m+n} = Q_{B_p}^m \cdot Q_{B_p}^n$. □

2.2 Some Relations of Generalised Balancing Sequences and Convergents of A Continued Fraction

For any positive integer D, if $\sqrt{D}$ can be written as continued fraction that is infinite and simple, given as $\sqrt{D} = a_1 + \frac{1}{a_2 + \frac{1}{a_3} + \dots}$ then it is denoted as $\sqrt{D} = [a_1, a_2, a_3, \dots]$. If $\sqrt{D}$ can be written as continued fraction that is infinite and simple, given as $\sqrt{D} = a_1 - \frac{1}{a_2 - \frac{1}{a_3} - \dots}$ then it is denoted as $\sqrt{D} = (a_1, a_2, a_3, \dots)$.

For a non-negative integer n, the nth convergent of the continued fraction $[a_1, a_2, a_3, \dots]$ is the real number $[a_1, a_2, a_3, \dots a_n] = h_n/k_n$.
Then

$$h_{-1} = 0, h_0 = a_0; \quad k_{-1} = 0,\ k_0 = 1;$$
$$h_{n+1} = a_{n+1}h_n + h_{n-1},\ n \geq 0; \quad k_{n+1} = a_{n+1}k_n + k_{n-1},\ n \geq 0.$$

For n, non-negative integer, the nth convergent of the continued fraction $(a_1, a_2, a_3, \dots)$ is the real number $(a_1, a_2, a_3, \dots a_n) = h_n/k_n$.
Then

$$h_{-1} = 0, h_0 = a_0; \quad k_{-1} = 0,\ k_0 = 1$$
$$h_{n+1} = a_{n+1}h_n - h_{n-1},\ n \geq 0; \quad k_{n+1} = a_{n+1}k_n - k_{n-1},\ n \geq 0.$$

Let $p \geq 1$ be any integer then, p^2-4 is a real number and the infinite simple continued fraction of p^2-4 is given as

$$\sqrt{p^2-4} = \begin{cases} (p, \overline{(p+1)/2, 2, (p+1)/2, 2p}) & \text{when } p \text{ is odd,} \\ (p, \overline{p/2, 2p}) & \text{when } p \text{ is even.} \end{cases}$$

The next two theorems give the convergents of $\sqrt{p^2-4}$ in terms of the generalised balancing sequence B_n. We prove these theorems in both cases p odd and p even. We first assume p is odd then by the continued fraction $(p, \overline{(p+1)/2, 2, (p+1)/2, 2p})$ we have for $a_0 = p$, $a_1 = p+1/2$, $a_2 = 2$, $a_3 = p+1/2$, $a_4 = 2p$, $n \geq 1$.

$$\begin{bmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_{n-1} & k_{n-1} \\ h_{n-2} & k_{n-2} \end{bmatrix}, \quad n \geq 1.$$

Now taking $A_n = \begin{bmatrix} a_n & -1 \\ 1 & 0 \end{bmatrix}$ and $P_n = \begin{bmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{bmatrix}$ note we have $P_n = A_n P_{n-1}$, $n \geq 1$, and setting $N = A_4 A_3 A_2 A_1$, we prove the following lemma by using the formula in Theorem 2.4.

Lemma 2.1. *For any positive integer t,*

$$N^t = \begin{bmatrix} B_{3t+1} & -2B_{3t} \\ 1/2 B_{3t} & -B_{3t-1} \end{bmatrix}.$$

Proof. Followed by induction. □

Theorem 2.5. *For an odd positive integer p, if the nth convergent of the continued fraction $\sqrt{p^2-4}$ is h_n/k_n then for every non-negative integer n the following holds:*

(i) $h_{8n} = B_{6n+2} - B_{6n}$,

(ii) $k_{8n} = B_{6n+1}$,

(iii) $h_{8n+3} = 1/2[B_{6n+4} - B_{6n+2}]$,

(iv) $k_{8n+3} = 1/2 B_{6n+3}$,

(v) $h_{8n+6} = B_{6n+6} - B_{6n+4}$,

(vi) $k_{8n+6} = B_{6n+5}$.

Proof. We have

$$\begin{aligned} P_{4n} &= A_{4n} P_{4n-1} \\ &= A_{4n} A_{4n-1} P_{4n-2} \\ &= A_{4n} A_{4n-1} A_{4n-2} P_{4n-3} \\ &= A_{4n} A_{4n-1} A_{4n-2} A_{4n-3} P_{4n-4} \\ &= N P_{4n-4} \end{aligned}$$

therefore, we have

$$\begin{aligned} P_{8n} &= N^{2n} P_0 \\ &= \begin{bmatrix} B_{6n+1} & -2B_{6n} \\ 1/2 B_{6n} & -B_{6n-1} \end{bmatrix} \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Now by definition of P_n as

$$P_{8n}=\begin{bmatrix} h_{8n} & k_{8n}\\ h_{8n-1} & k_{8n-1}\end{bmatrix},$$

We have

$$h_{8n}=pB_{6n+1}-2B_{6n}=B_{6n+2}-B_{6n}$$

and

$$k_{8n}=B_{6n+1}.$$

Also

$$\begin{aligned}P_{8n+3}&=A_{8n+3}A_{8n+2}A_{8n+1}N^{2n}P_0\\ &=\begin{bmatrix}1/2B_3 & -B_2\\ B_2 & -2\end{bmatrix}\begin{bmatrix}B_{6n+1} & -2B_{6n}\\ 1/2B_{6n} & -B_{6n-1}\end{bmatrix}P_0\\ &=\begin{bmatrix}1/2B_{6n+3} & -B_{6n+2}\\ B_{6n+2} & -2B_{6n+1}\end{bmatrix}\begin{bmatrix}p & 1\\ 1 & 0\end{bmatrix}\end{aligned}$$

therefore $h_{8n+3}=1/2[B_{6n+4}-B_{6n+2}]$ and $k_{8n+3}=1/2B_{6n+3}$.
Finally, $P_{8n+6}=A_{8n+6}A_{8n+5}N^{2n+1}P_0$.
Therefore $h_{8n+6}=B_{6n+6}-B_{6n+4}$ and $k_{8n+6}=B_{6n+5}$. □

Now, we prove theorem that give convergents of $\sqrt{p^2-4}$ in terms of Generalised Balancing Sequence B_n in the case when p is even. Now for p even, we have the continued fraction for p^2-4 given as $(p,\overline{p/2,2p})$ with $a_0=p$, $a_1=p/2$, $a_2=2p$, then note $P_n=A_nP_{n-1}$, $\forall\ n\geq 1$ with $A_{2n}=\begin{bmatrix}2p & -1\\ 1 & 1\end{bmatrix}$ and $A_{2n-1}=\begin{bmatrix}p/2 & -1\\ 1 & 0\end{bmatrix}$. We have $P_{2n}=A_{2n}P_{2n-1}$ $=A_{2n}A_{2n-1}P_{2n-2}$. Now setting $M=A_2A_1$ we prove the following lemma by using Theorem 2.4.

Lemma 2.2. For all positive integer t,

$$M^t=\begin{bmatrix}B_{2t+1} & -2B_{2t}\\ 1/2B_{2t} & -B_{2t-1}\end{bmatrix}.$$

Proof. Follows by induction. □

Theorem 2.6. *For an even positive integer p, if the nth convergent of the continued fraction $\sqrt{p^2-4}$ is h_n/k_n then for all non-negative integer n the following holds:*

(i) $h_{2n}=B_{2n+2}-B_{2n}$,

(ii) $k_{2n}=B_{2n+1}$.

Proof. We have

$$\begin{aligned}P_{2n}&=MP_{2n-2}\\ &=M^nP_0,\end{aligned}$$

$$P_n=\begin{bmatrix}B_{2n+1} & -2B_{2n}\\ 1/2B_{2n} & -B_{2n-1}\end{bmatrix}\begin{bmatrix}p & 1\\ 1 & 0\end{bmatrix}$$

implies

$$h_{2n}=pB_{2n+1}-2B_{2n}=B_{2n+2}-B_{2n}\quad\text{and}\quad k_{2n}=B_{2n+1}.$$ □

3. Solutions of Diophantine Equations $x^2 \pm pxy + y^2 \pm x = 0$ in Terms of Generalized Balancing Sequences

3.1 Existence of Solutions

The following section we show that the Diophantine equations $x^2 \pm pxy + y^2 \pm x = 0$ are solvable in integers for all positive p and express the solutions in terms of generalized balancing sequence.

Lemma 3.1. *If the solution of $x^2 - pxy + y^2 - x = 0$ is (x,y) then $(x, px-y)$ and $(py-x+1, y)$ are also solutions of the equation.*

Proof. Follows by the simple verification. □

Theorem 3.1. *For all non-negative integer n, the following pairs satisfy the equation $x^2 - pxy + y^2 - x = 0$*

$(B_{2n}^2, B_{2n}B_{2n-1})$
$(B_{2n}^2, B_{2n}B_{2n+1})$
$(B_{2n+1}^2, B_{2n}B_{2n+1})$
$(B_{2n+1}^2, B_{2n+1}B_{2n+2})$.

Proof. First note $(B_{2n}^2, B_{2n}B_{2n-1})$ satisfies $x^2 - pxy + y^2 - x = 0$. On substituting $(B_{2n}^2, B_{2n}B_{2n-1})$ in $x^2 - pxy + y^2 - x$, we get by $\det Q_{B_p}^{2n}$

$$\begin{aligned}(B_{2n}^2)^2 - pB_{2n}^3 B_{2n-1} + B_{2n}^2 B_{2n-1}^2 - B_{2n}^2 &= B_{2n}^2[B_{2n}^2 - pB_{2n}B_{2n-1} + B_{2n-1}^2 - 1]\\ &= B_{2n}^2[B_{2n}^2 - B_{2n-1}(pB_{2n} - B_{2n-1}) - 1]\\ &= B_{2n}^2[B_{2n}^2 - B_{2n-1}B_{2n+1} - 1]\\ &= B_{2n}^2.0 = 0.\end{aligned}$$

Therefore $(x,y) = (B_{2n}^2, B_{2n}B_{2n-1})$ is a solution of $x^2 - pxy + y^2 - x = 0$.
Now, by above lemma note $(x, px-y)$ satisfies the Diophantine equation $x^2 - pxy + y^2 - x = 0$ and we have

$$\begin{aligned}(x, px-y) &= (B_{2n}^2, pB_{2n}^2 - B_{2n}B_{2n-1})\\ &= (B_{2n}^2, B_{2n}[pB_{2n} - B_{2n-1}])\\ &= (B_{2n}^2, B_{2n}B_{2n+1}).\end{aligned}$$

Therefore $(x,y) = (B_{2n}^2, B_{2n}B_{2n+1})$ is a solution of $x^2 - pxy + y^2 - x = 0$.
Now for $(x,y) = (B_{2n}^2, B_{2n}B_{2n+1})$ as it is a solution of $x^2 - pxy + y^2 - x = 0$, again by above lemma $(py - x + 1, y)$ is also a solution of the Diophantine equation $x^2 - pxy + y^2 - x = 0$ and we have $(py - x + 1, y) = (B_{2n+1}^2, B_{2n}B_{2n+1})$. Therefore $(x,y) = (B_{2n+1}^2, B_{2n}B_{2n+1})$ is a solution of $x^2 - pxy + y^2 - x = 0$.

Similarly, as $(B_{2n+1}^2, B_{2n}B_{2n+1})$ satisfies $x^2 - pxy + y^2 - x = 0$ then $(x, px-y)$ is also a solution of the Diophantine equation $x^2 - pxy + y^2 - x = 0$ and we have $(x, px-y) = (B_{2n+1}^2, B_{2n+1}B_{2n+2})$. Therefore $(x,y) = (B_{2n+1}^2, B_{2n+1}B_{2n+2})$ is a solution of $x^2 - pxy + y^2 - x = 0$. □

Each of the four formulas for solution of $x^2 - pxy + y^2 - x = 0$, as in the above theorem, defines a class of solutions. We prove that these four classes of solutions are the only solutions for the Diophantine equation $x^2 - pxy + y^2 - x = 0$ in the following section.

3.2 The Four Classes of Solutions for Each of the Diophantine Equations $x^2 \pm pxy + y^2 \pm x = 0$

In this section, we prove that the four classes of solutions obtained in the above are the only solutions of the Diophantine equation $x^2 - pxy + y^2 - x = 0$. We recall some properties of convergents in the following theorems.

Theorem 3.2. *If the integer M satisfies $|M| < \sqrt{D}$ then any positive integer solution (s,t) of $x^2 - Dy^2 = M$ with $gcd(s,t) = 1$ satisfies $s = h_n$, $t = k_n$ where the nth convergent of the infinite simple continued fraction, $\sqrt{D} = (a_0, a_1, a_2, \ldots)$ is h_n/k_n for n a positive integer.*

Proof. See ([5, Theorem 7.22]). □

Theorem 3.3. Let the infinite simple continued fraction of $\sqrt{D}$ be $(a_0, a_1, a_2, \ldots)$ and suppose that m_n and q_n are two sequences given by

$$m_0 = 0, \quad q_0 = 1,$$
$$m_{n+1} = a_n q_n + m_n,$$
$$q_{n+1} = (D - m_{n+1}^2)/q_n.$$

Then

(i) m_n and q_n are integers for any positive integers n,

(ii) $h_n^2 - Dk_n^2 = (-1)^{n+1} q_{n+1}$ for any integer $n \geq -1$.

Proof. See ([5, Theorem 7.24]). □

Theorem 3.4. If positive integers p, x and y satisfy the equations $x^2 - pxy + y^2 - x = 0$ then there exists c,e such that $(x,y) = (c^2, ce)$ with $\gcd(c,e) = 1$, where c and e are positive integers.

Proof. See ([4, Theorem 1]). □

Theorem 3.5. *For an odd positive integer p, every positive solution of $x^2 - pxy + y^2 - x = 0$ is of the form $\left(B_{2n}^2, B_{2n-1}B_{2n}\right)$.*

Proof. Consider p to be an odd positive integer. Let (x,y) be any solution of $x^2 - pxy + y^2 - x = 0$ which is positive then by Theorem 3.4 above note that there exists c and e, positive integers such that $(x,y) = (c^2, ce)$ with $gcd(c,e) = 1$. Then on substituting (c^2, ce), we have

$$c^4 - pc^3e + c^2e^2 - c^2 = 0,$$
$$c^2 - pce + e^2 - 1 = 0.$$

This equation has integer solutions if and only if

$$\begin{aligned} \triangle &= p^2e^2 - 4(e^2 - 1) \\ &= e^2(p^2 - 4) + 4 \ \text{ is a square.} \end{aligned}$$

Therefore there is an integer t satisfying

$$\triangle = t^2 = (p^2 - 4)e^2 + 4,$$
$$t^2 - (p^2 - 4)e^2 = 4,$$

then we obtain

$$c = \frac{pe \pm t}{2},$$

by solving for (t,e) from the equation $t^2-(p^2-4)e^2=4$. Now considering the continued fraction of $\sqrt{p^2-4}$ given as $\sqrt{p^2-4}=\big(p,\overline{(p+1)/2,2,(p+1)/2,2p}\big)$ with $a_0=p$, $a_{4n-3}=\frac{(p+1)}{2}$, $a_{4n-2}=2$, $a_{4n-1}=\frac{p+1}{2}$, $a_{4n}=2p$, for $n\geq 0$. Now by the above theorem we have the periodic sequence give as

$$\{(-1)^{n+1}q_{n+1}\}_{n=-1}^{\infty}=\{1,\overline{4,p+2,4,1}\}.$$

Assuming (t,e) is a positive solution of $t^2-(p^2-4)e^2=4$ we have $(t,e)=(h_n,k_n)$ for some positive integers n by Theorem 3.2 and as $h_n^2-Dk_n^2=(-1)^{n+1}q_{n+1}$ by Theorem 3.3, we have by periodicity we have

$$h_{8n}^2-(p^2-4)k_{8n}^2=(-1)^{8n+1}q_{4n+1}=4,$$
$$h_{8n+3}^2-(p^2-4)k_{8n+3}^2=(-1)^{8n+4}q_{8n+4}=1,$$
$$h_{8n+6}^2-(p^2-4)k_{8n+6}^2=(-1)^{8n+7}q_{8n+7}=4$$

for all $n\geq 0$.

Therefore, all the solutions (t,e) of $t^2-(p^2-4)e^2=4$ are

$$\begin{aligned}(t,e)&=(h_{8n},k_{8n})\\&=(2h_{8n+3},2k_{8n+3})\\&=(h_{8n+6},k_{8n+6}),\quad n\geq 0.\end{aligned}$$

Now for $c=\frac{pe\pm t}{2}$ the solutions (c,e) are

$$\Big(\frac{pk_{8n}+h_{8n}}{2},k_{8n}\Big)$$
$$\big(pk_{8n+3}+h_{8n+3},2k_{8n+3}\big)$$
$$\Big(\frac{pk_{8n+6}+h_{8n+6}}{2},k_{8n+6}\Big),\quad n\geq 0.$$

Using the theorem and rearranging, we get

$$(c,e)=\big(B_{6n+2},B_{6n+1}\big)$$
$$(c,e)=\big(B_{6n+4},B_{6n+3}\big)$$
$$(c,e)=\big(B_{6n+6},B_{6n+5}\big)$$

and finally as $(x,y)=(c^2,ce)$ we obtain

$$(x,y)=\big(B_{6n+2}^2,B_{6n+1}B_{6n+2}\big)$$
$$(x,y)=\big(B_{6n+4}^2,B_{6n+3}B_{6n+4}\big)$$
$$(x,y)=\big(B_{6n+6}^2,B_{6n+5}B_{6n+6}\big)$$

and therefore

$$(x,y)=\big(B_{2n}^2,B_{2n-1}B_{2n}\big),\quad n\geq 1.$$

Therefore, any solution (x,y) of $x^2-pxy+y^2-x=0$ is of the form $(x,y)=\big(B_{2n}^2,B_{2n-1}B_{2n}\big)$ for $p>2$ and also for $p=1$. □

Theorem 3.6. *For an even positive integer p, every positive solution (x,y) of $x^2-pxy+y^2-x=0$ is of the form $\big(B_{2n}^2,B_{2n-1}B_{2n}\big)$.*

Proof. Let p be an even positive integer. We have $\sqrt{p^2-4}=\big(p,\overline{\frac{p}{2},2p}\big)$. Let $a_0=p$, $a_{2n+1}=\frac{p}{2}$, $a_{2n+2}=2p$, $\forall\ n\geq 0$.

We have by periodic sequence

$$\{(-1)^{n+1}q_{n+1}\}_{n=-1}^{\infty} = \{\overline{1,4}\}$$

and

$$h_{2n}^2 - (p^2-4)k_{2n}^2 = 4, \quad \forall\ n \geq 0.$$

Moreover, in this case all solutions of

$$t^2 - (p^2-4)e^2 = 4$$

are

$$(t,e) = (h_{2n}, k_{2n}),$$

$$(c,e) = \left(\frac{pk_{2n}+h_{2n}}{2}, k_{2n}\right).$$

But from the theorem if p is even then

$$c = \frac{pk_{2n}+h_{2n}}{2} = B_{2n+2}$$

$$e = B_{2n+1}$$

$$(x,y) = (c^2, ce) = \left(B_{2n+2}^2, B_{2n+1}B_{2n+2}\right).$$

Replacing $n+1$ by n, we have

$$(x,y) = \left(B_{2n}^2, B_{2n-1}B_{2n}\right), \quad n \geq 1.$$

Therefore, for a positive even integer p, every positive solution of $x^2 - pxy + y^2 - x = 0$ is of the form $(x,y) = \left(B_{2n}^2, B_{2n-1}B_{2n}\right)$ for $p > 2$ and also for $p = 2$. □

Theorem 3.7. *For a positive integer p, all the solutions of $x^2 - pxy + y^2 - x = 0$ are*

(1) $\left(B_{2n}^2, B_{2n-1}B_{2n}\right)$,

(2) $\left(B_{2n}^2, B_{2n}B_{2n+1}\right)$,

(3) $\left(B_{2n+1}^2, B_{2n}B_{2n+1}\right)$,

(4) $\left(B_{2n+1}^2, B_{2n+1}B_{2n+2}\right)$, *for all integers $n \geq 0$.*

Proof. Let p be any integer and (x,y) be any solution of $x^2 - pxy + y^2 - x = 0$. Then (x,y) has to be $(0,0)$, $(1,0)$, positive solution or non-positive solution. If (x,y) is a solution that is non-positive then it is of the form

(i) $x > 0$, $y < 0$, or

(ii) $x < 0$, $y < 0$, or

(iii) $x < 0$, $y > 0$.

If the solution (x,y) is as in (i) with $x > 0$ and $y < 0$; note by taking $x' = x$ and $y' = -y$ we have (x', y') is a solution of the $x^2 + pxy + y^2 - x = 0$ that is positive. Similarly, if (x,y) is as in (ii) with $x < 0$ and $y < 0$; then by taking $x' = -x$ and $y' = -y$ we have (x', y') is a solution of $x^2 - pxy + y^2 + x = 0$ that is positive, and if (x,y) is as in (iii) with $x < 0$ and $y > 0$; then by taking $x' = -x$ and $y' = y$ we have (x', y') is a solution of $x^2 + pxy + y^2 + x = 0$ that is positive. Therefore, the solution (x,y), that are positive, are the only solution of $x^2 - pxy + y^2 - x = 0$. Note that if (x,y) is any positive solutions then by Theorem 3.6 and Theorem 3.1, we get that

(1) $\left(B_{2n}^2, B_{2n-1}B_{2n}\right)$,

(2) $\left(B_{2n}^2, B_{2n}B_{2n+1}\right)$,

(3) $\left(B^2_{2n+1}, B_{2n}B_{2n+1}\right)$,

(4) $\left(B^2_{2n+1}, B_{2n+1}B_{2n+2}\right)$ are all the solutions of $x^2 - pxy + y^2 - x = 0$. □

The above theorem in general classifies all the solutions of the equations $x^2 \pm pxy + y^2 \pm x = 0$ as shown in the following theorems.

Theorem 3.8. *For a positive integer p all the solutions of $x^2 + pxy + y^2 - x = 0$ are*

(1) $\left(B^2_{2n}, -B_{2n-1}B_{2n}\right)$,

(2) $\left(B^2_{2n}, -B_{2n}B_{2n+1}\right)$,

(3) $\left(B^2_{2n+1}, -B_{2n}B_{2n+1}\right)$,

(4) $\left(B^2_{2n+1}, -B_{2n+1}B_{2n+2}\right)$, *for all integers $n \geq 0$.*

Theorem 3.9. *For a positive integer p all the solutions of $x^2 - pxy + y^2 + x = 0$ are*

(1) $\left(-B^2_{2n}, B_{2n-1}B_{2n}\right)$,

(2) $\left(-B^2_{2n}, B_{2n}B_{2n+1}\right)$,

(3) $\left(-B^2_{2n+1}, B_{2n}B_{2n+1}\right)$,

(4) $\left(-B^2_{2n+1}, B_{2n+1}B_{2n+2}\right)$, *for all integers $n \geq 0$.*

Theorem 3.10. *For a positive integer p all the solutions of $x^2 + pxy + y^2 + x = 0$ are*

(1) $\left(-B^2_{2n}, -B_{2n-1}B_{2n}\right)$,

(2) $\left(-B^2_{2n}, -B_{2n}B_{2n+1}\right)$,

(3) $\left(-B^2_{2n+1}, -B_{2n}B_{2n+1}\right)$,

(4) $\left(-B^2_{2n+1}, -B_{2n+1}B_{2n+2}\right)$, *for all integers $n \geq 0$.*

4. Conclusion

We investigate some properties of the generalised balancing sequences $B_{n+1} = pB_n - B_{n-1}$. We describe the solutions of each of the Diophantine Equations $x^2 \pm pxy + y^2 \pm x = 0$ in four classes expressed in terms of generalised balancing sequences. It is observed that for any positive solution (x, y) of any of the equations $x^2 \pm pxy + y^2 \pm y = 0$, the interchanged pair (y, x) is a positive solution of the corresponding equations $x^2 \pm pxy + y^2 \pm x = 0$ and vice versa. Hence by the above arguments the solutions of $x^2 \pm pxy + y^2 \pm y = 0$ also can be expressed in terms of generalised balancing sequences.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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