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Research Article

Inverse Split Majority Dominating Set of a Graph

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Abstract. In this paper, we introduced an inverse split majority dominating set of a graph G. Inverse split majority domination number $\gamma_{SM}^{-1}(G)$ is determined for some classes of graphs. Some important results and characterization theorems on $\gamma_{SM}^{-1}(G)$ are established. Many Bounds on inverse split majority domination number and its relationship with other domination parameters are also obtained.

Keywords. Majority domination number, Inverse majority domination number, Split dominating (SD) set, Inverse Split Majority dominating (ISMD) set, Inverse Split majority domination number

Mathematics Subject Classification (2020). 05C69

1. Introduction

C. Berge presented domination as a graph theoretic notion [1] in 1958, and O. Ore [16] in 1962. In 1977, E. J. Cockayne and S. T. Hedetniemi produced a study on dominance [5], which was researched extensively in this article. T. W. Haynes and colleagues authored "Fundamentals of Domination in Graphs", has a variety of domination parameters [8]. Kulli and Sigarkanti [11] pioneered the unique parameter inverse domination in Graphs in 1991.

Graph theory may be used to depict any binary relationship. Both dominant sets and their inverses play key roles in domination. When D is a dominant set, $V-D$ is a dominating

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set as well. We often have a number of primary nodes in an information retrieval system to transmit on the information. In the event that the system breaks down, we have a backup set of auxiliary nodes to complete the work in the complement. Whenever the complement set is joined, communication flows among the complement's members. Furthermore, the components of the dominating sets as well as the elements of the inverse dominating sets can stand with each other to promote communication. They are extremely important in coding theory, computer science, operations research, switching circuits, electrical networks, and so on.

2. Inverse Split Majority Dominating Sets in Graphs

Definition 2.1. Let T be a minimum majority dominating (MD) set of a graph G with p vertices. Let T' be the inverse majority dominating (IMD) set of G with respect to D. Then $T'_1 \supseteq T'$ is called an inverse split majority dominating (ISMD) set of G if the induced subgraph $\langle V - T'_1 \rangle$ is disconnected. The inverse split majority domination number $\gamma^{-1}_{SM}(G)$ of G is the minimum cardinality of a minimal inverse split majority dominating (ISMD) set of a graph G.

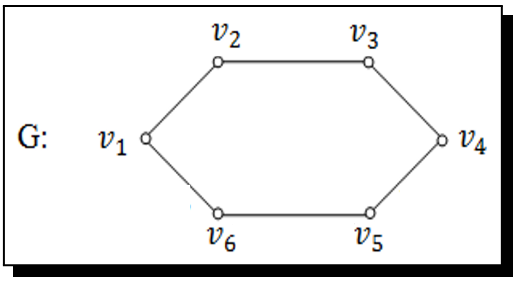

Figure 1

Example 2.2. Let $T = \{v_1\}$ be a MD set and $T' = \{v_4\} \subseteq V - T$ is an IMD-set with respect to T. Then $\gamma_M(G) = 1$ and $\gamma^{-1}_{SM}(G) = 1$. Since $\langle V - T' \rangle$ is connected, choose $T'_1 = \{v_4, v_6\} \subseteq V - T$ and $\langle V - T'_1 \rangle$ is disconnected. Then $\gamma_M(G) = 1 = \gamma^{-1}_{SM}(G)$ and $\gamma^{-1}_{SM}(G) = 2$.

Proposition 2.3. G can be any graph, $\gamma^{-1}_M(G) \le \gamma^{-1}_{SM}(G)$.

Proof. Since every ISMD-set is also an IMD-set, $\gamma^{-1}_M(G) \le \gamma^{-1}_{SM}(G)$. □

Proposition 2.4. If graph, $\gamma_{SM}(G)) \le \gamma^{-1}_{SM}(G)$, $\gamma_{SM}(G)$ is the split majority domination number.

Proposition 2.5. G can be any graph then $\gamma^{-1}_M(G) \le i^{-1}_M(G) \le \gamma^{-1}_{SM}(G)$, where $i^{-1}_M(G)$ is the inverse independent majority domination number.

Proposition 2.6. G can be any graph then $\gamma_M(G) \le \gamma^{-1}_M(G) \le \gamma^{-1}_{SM}(G)$.

Example 2.7. G can be any graph, $\gamma_M(G) < \gamma^{-1}_M(G) < \gamma^{-1}_{SM}(G)$.
Let $G = B_{5,6}$ with $p = 11$ vertices.

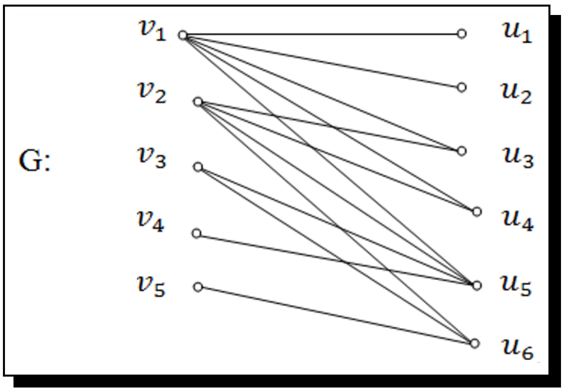

Figure 2

$V_1(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2(G) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ with $d(v_1) = 5$. Let $T = \{v_1\}$ and $\gamma_M(G) = 1$. Let $T' = \{v_2, v_3\} \subseteq (V - T)$ and $\gamma_M^{-1}(G) = 2$. Let $T_1' = \{v_2, v_3, v_5\}$ and $T_1' \supseteq T'$ such that $|N[T_1']| > \lceil \frac{p}{2} \rceil$ and the $\langle V - T' \rangle$ is disconnected. Then T_1' is an ISMD set of G and $\gamma_{SM}^{-1}(G) = |T_1'| = 3$. Thus $\gamma_M(G) < \gamma_M^{-1}(G) < \gamma_{SM}^{-1}(G)$.

3. Inverse Split MD Number for Some Classes of Graph

Result 3.1. (i) *Let $G = K_p$, $p \geq 2$. Then $\gamma_{SM}^{-1}(G) = 0$.*

(ii) *Let $G = K_{1,p-1}$ then $\gamma_{SM}^{-1}(G) = 0$.*

(iii) *Let $G = C_p$, $p \geq 3$,* $\gamma_{SM}^{-1}(G) = \begin{cases} 2, & \text{if } 3 \leq p \leq 6 \\ \lceil \frac{p}{6} \rceil, & \text{if } p > 7. \end{cases}$

(iv) *For a corona graph $G = K_{po}, K_1$, $\gamma_{SM}^{-1}(G) = 1$, where K_p is complete graph.*

(v) *For the Petersen graph with $p = 10$ and $q = 15$, $\gamma_{SM}^{-1}(G) = 3$.*

(vi) *Let $G = K_p - \{e\}$, where e is any edge in G, $\gamma_{SM}^{-1}(G) = p - 2$.*

(vii) *Let $G = D_{r,s}$, $r \leq s$ and $r, s \geq 2$,* $\gamma_{SM}^{-1}(G) = \begin{cases} 1, & \text{if } s = r, r+1, r+2 \\ |e_i| + 1, & \text{if } s \geq r + 3. \end{cases}$.

Proposition 3.2. For a path P_p, $p \geq 2$. Then $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{6} \rceil$.

Proof. G be a path with $p \geq 2$. Let $T = \{u_2, u_5, u_8, \dots, u_t\}$ be a MD-dominating set with $d(u_i, u_j) \geq 3$, for $\forall\ i \neq j$ and $|T| = t = \lceil \frac{p}{6} \rceil$. Choose $T' = \{u_3, u_6, u_9, \dots, u_t\} \subseteq V - D$ such that $d(u_i, u_j) \geq 3$ and $u_i, u_j \in T'$ with $|T'| = t = \lceil \frac{p}{6} \rceil$. $|N[T']| \geq \sum_{i=1}^{t} d(u_i) + t = 3t$. $|N[T']| \geq 3 \lceil \frac{p}{6} \rceil \geq \lceil \frac{p}{2} \rceil$. Since $T' \subseteq V - T$ and $|N[T']| \geq \lceil \frac{p}{2} \rceil$, T' is an IMD-set of G since path P_p is minimally connected, $\langle V - T' \rangle$ is disconnected and $\langle V - T' \rangle$ is splitted into many components. T' is called an ISMD set of G and

$$\gamma_{SM}^{-1}(G) \leq |T'| = \left\lceil \frac{p}{6} \right\rceil. \tag{3.1}$$

Let T' be a $\gamma_{SM}^{-1}(G)$-set with $|T'| = t = \gamma_{SM}^{-1}(G)$. Then

$$|N[T']| \geq \left\lceil \frac{p}{2} \right\rceil. \tag{3.2}$$

Since $T' \subseteq V - T$ and $\langle V - T' \rangle$ is disconnected, $|N[T']| \leq \sum_{i=1}^{t} d(u_i) + \gamma_{SM}^{-1}(G)$

$$\gamma_{SM}^{-1}(G) \geq \left\lceil \frac{p}{6} \right\rceil \qquad \left(\text{since } \frac{1}{3}\left\lceil \frac{p}{2} \right\rceil = \left\lceil \frac{p}{6} \right\rceil, \text{ if } p = 2r, 2r+1\right). \tag{3.3}$$

From (3.1) and (3.3), we get $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{6} \rceil$. □

Proposition 3.3. For a wheel graph $G = W_p$. Then $\gamma_{SM}^{-1}(G) = 0$.

Proof. Let $V(G) = \{u_1, u_2, \dots, u_{p-1}, u_p\}$, where $\Delta(G) = |u_1|$ and $d(u_i) = 3$, for all i. $T = \{u_1\}$ be a MD set of G. Choose a set $T' = \{u_2, u_5, \dots, u_{p-1}\}) \subseteq (V - T)$ and T' is an IMD-set of G such that $d(u_i, u_j) \geq 3$ for $\forall\ i \neq j$, $\gamma_M^{-1}\ (W_p) = \left\lceil \frac{p-2}{6} \right\rceil$. Since the graph $G = W_P$, $\Delta(G) = u_1$, and $|u_1| = p - 1$. Therefore, the $\langle V - T' \rangle$ is not disconnected. In any way, one could not find a set T' such that $\langle V - T' \rangle$ is disconnected. Hence $\gamma_{SM}^{-1}(G) = 0$. □

Proposition 3.4. Let $G = K_{r,s}$ complete bipartite graph, then $\gamma_{SM}^{-1}(G) = \begin{cases} s, & \text{if } r = s, \\ s = \max(r,s), & \text{if } r < s. \end{cases}$

Proof. **Case (i):** When $r = s$.

Let $V_1(G) = \{u_1, u_2, \dots, u_m\}$ and $V_2(G) = \{v_1, v_2, \dots, v_n\}$, $T = \{u_1\}$ is a MD-set and $T' = \{v_1\} \subseteq V - T$ is an IMD-set but the $\langle V - T' \rangle$ remains connected since the graph G is a complete bipartite. Now, there exists a subset $T_1' = \{v_1, v_2, \dots, v_n\} \subseteq V - T$ such that $|N[T_1']| = p > \frac{p}{2}$ and the $\langle V - T_1' \rangle$ is disconnected. T_1' is a minimal IMSD set. Thus, $\gamma_{SM}^{-1}(G) = |T_1'| = s$.

Case (ii): When $r < s$.

Let $T = \{u_1\}$ be a MD-set of G. Let $T' = \{u_2\} \subseteq V - T$ where, $u_2 \in V_1(G)$. Then $|N[T']| = n + 1 > \lceil \frac{P}{2} \rceil$ and $T' \subseteq V - T$. Therefore, T' is a minimum IMD-set but the $\langle V - T' \rangle$ is connected. Now choose a subset $T_1' = \{v_1, v_2, \dots, v_n\} \subseteq V - T$ where $v_i \in V_2(G)$ such that the $\langle V - T_1' \rangle$ is disconnected with m components and $|N[T_1']| > \lceil \frac{p}{2} \rceil$. Hence T_1' is an ISMD-set and $\gamma_{SM}^{-1}(G) = |T_1'| = n$. In general, if $r < s$ then $\gamma_{SM}^{-1}(G) = \max\{r, s\} = s$. □

Example 3.5. Let $G = K_{3,10}$ and $V_1(G) = \{x_1, x_2, x_3\}$, $V_2(G) = \{y_1, y_2, \dots, y_{10}\}$. Then $T = \{x_1\}$, $T' = \{x_2\}$ and $T_1' = \{y_1, y_2, \dots, y_n\}$ are the MD-set, an ISMD set them $\gamma_{SM}^{-1}(G) = |T_1'| = 10$.

Theorem 3.6. *Let $G = S(K_{1,p-1})$ then $\gamma_{SM}^{-1}(G) = \left\lceil \frac{p-2}{4} \right\rceil$.*

Proof. $V(G) = \{x, x_1, x_2, x_3, \dots, x_{\lfloor \frac{p}{2} \rfloor}, y_1, y_2, \dots, y_{\lfloor \frac{p}{2} \rfloor}\}$ where x is a central vertex, $y_1, y_2, \dots, y_{\lfloor \frac{p}{2} \rfloor}$ are pendants and $x_1, x_2, x_3, \dots, x_{\lfloor \frac{p}{2} \rfloor}$ are middle vertices of each edge of G. Since $d(u) = \lceil \frac{p}{2} \rceil - 1$, $T = \{u\}$ is a MD-set of G. Choose $T' = \{x_1, x_2, x_3, \dots, x_t\} \subseteq V - T$ with $|T'| = t = \left\lceil \frac{p-2}{4} \right\rceil \ni d(u_i, u_j) \geq 2$ for $\forall\ i \neq j$. Then $|N[T']| \geq 2t + 1$. Therefore, $|N[T']| \geq 2\left\lceil \frac{p-2}{4} \right\rceil + 1 \geq \lceil \frac{p}{2} \rceil$ and $T' \subseteq V - T$. Hence T'

is an IMD-set. Since the graph $S(K_{1,p-1})$ has cut vertices, $\langle V-T'\rangle$ is disconnected. Hence T' is an ISMD-set and $\gamma_{SM}^{-1}(G)=|T'|=\left\lceil\frac{p-2}{4}\right\rceil$. □

Theorem 3.7. *For a binary tree t, $\gamma_{SM}^{-1}(G)=\lceil\frac{p}{8}\rceil$ and $\gamma_{SM}^{-1}(G)=\gamma_M^{-1}(G)$.*

Proof. By induction on the level we proved the result 'k' of a tree t. Here t be a binary tree with p vertices such that $d(u_1)=2$, $d(u_i)=3$, where u_i's are intermediate vertices and others are pendants. Let T and T' be a MD-set and IMD-set of G. At level $1=2^1$, $p=3=1+2$ and $\gamma_M^{-1}(G)=\gamma_{SM}^{-1}(G)=1$. At level $2=2^2$, $p=3+4=7$. Since u_1 and u_2 are intermediate vertices, $\gamma_M(G)=|T|=\{u_1\}=1$, $\gamma_M^{-1}(G)=|T'|=\{u_2\}=1$. Since u_2 is a cut vertex of t, $\langle V-T'\rangle$ is disconnected, T' is also an ISMD set. Hence, $\gamma_{SM}^{-1}(G)=1$. At level $3=2^3$, $p=7+8=15$, there are 4 intermediate vertices $\{u_1,u_2,u_3,u_4\}$. Let $T=\{u_1,u_3\}$ and $T'=\{u_2,u_4\}\ni|N[T]|=|NT']|=8=\lceil\frac{p}{2}\rceil$. Hence T is MD-set of G and since, $T'\subseteq V-T$, T' is an IMD-set of G. Since T' has cut vertex, $\langle V-T'\rangle$ is disconnected and T' is also an ISMD-set. Then $\gamma_M(G)=\gamma_M^{-1}(G)=\gamma_{SM}^{-1}(G)=2$. This result is true for $(k-1)$ level. If $l=k=2^k$, then $p=[V(l_{k-1}]+2^k$, and there are 2^k pendants and $(p-2^k-1)$ inter mediate vertices in t. The set $T'=\{u_1,u_3,u_4\}\subseteq V-T$ such that $|T'|=t=\lceil\frac{p}{8}\rceil$ with $d(u_i,u_j)=2$. Then $|N[T']|\geq\sum_{i=1}^{t}d(u_i)+t=4\lceil\frac{p}{8}\rceil\geq\lceil\frac{p}{2}\rceil$. Hence T' is an IMD-set. Since $(V-T)$ contains cut vertex, $\langle V-T'\rangle$ is disconnected. Hence, T' is an ISMD-set and $\gamma_{SM}^{-1}(G)=\lceil\frac{p}{8}\rceil=|T'|$. □

Theorem 3.8. *G be a uniform caterpillar with p vertices and t pendants attached to each vertex of G. Then $\gamma_{SM}^{-1}(G)=\left\lceil\frac{p}{2(t+3)}\right\rceil$.*

4. Main Results on $\gamma_{SM}^{-1}(G)$

Theorem 4.1. *G has atleast one full degree vertex then $\gamma_{SM}^{-1}(G)=0$.*

Proof. By an induction on the number of full degree vertices u_i we proved the theorem. Suppose G has exactly one vertex $\nabla(G)=|u_1|$. Then $T=\{u_1\}$ be a MD-set of G and $\gamma_M(G)=1$. Since $\delta\geq 1$, $T'\subseteq V-T$ is an IMD-set of G with cardinality $|T'|\geq 2$. Then the $\langle V-T'\rangle$ is not disconnected. Since u_1 is adjacent to all vertices of G and $T'\subseteq V-T$, $u_1\in(V-T)$. Either $\langle V-T'\rangle$ has a cut vertex or $\langle V-T'\rangle$ has no cut vertex, $\langle V-T'\rangle$ is still connected to the full degree vertex u_1. Hence one could not find an ISMD-set for G and $\gamma_{SM}^{-1}(G)=0$. If G has two full degree vertices u_1 and u_2 then the subset $T=\{u_1\}$ and $T'=\{u_2\}\subseteq(V-T)$ are the MD-set and an IMD-set of G respectively the $\langle V-T'\rangle$ contains the full degree vertex $u_1\in D$, u_1 is adjacent to all the vertices of $(V-T')$ and $\langle V-T'\rangle$ is not disconnected for any IMD-set D' of G. Hence $\gamma_{SM}^{-1}(G)=0$.

This result is true for $(p-1)$ full degree vertices. Suppose all vertices of G are full degree then graph G is complete. Then $\gamma_M(G)=|T|=1$ and $\gamma_M^{-1}(G)=|T'|=1$. Since every vertex is adjacent to all vertices of G, the induced subgraph $\langle V-T'\rangle$ is not disconnected for any IMD set $T'\subseteq V-T$. Hence $\gamma_{SM}^{-1}(G)=0$. Thus the theorem. □

Theorem 4.2. *Let T be a γ_M-set of G. If the induced subgraph $\langle V-T\rangle$ contains a cut vertex then $\gamma_{SM}^{-1}(G)=\gamma_M^{-1}(G)$.*

Proof. Let T be a minimum MD-set of G. Let $u\in V-T$ be a cut vertex of G.

Case (i): When $d(u)\ge\lceil\frac{p}{2}\rceil-1$. Then $T'=\{u\}$ is an IMD set of G and $\gamma_{SM}^{-1}(G)=|T'|=\gamma_M^{-1}(G)=1$.

Case (ii): When $d(u)\le\lceil\frac{p}{2}\rceil-2$. Then $\gamma_M^{-1}(G)\ge 2$. Since the γ_M^{-1}-set T' contains a cut vertex u, the $\langle V-T'\rangle$ is disconnected $\gamma_{SM}^{-1}(G)\ge 2$. Since the cut vertex u of degree $d(u)\ge 2$, u is not a pendant. If $d(u)=2$ and $u\in V-T$, the set $T'=\{u,u_1,\dots,u_t\}\subseteq V-T$ such that $d(u_i,u_j)\ge 3$, for $i\ne j$ and $|T'|=t+1$ where $t=\lceil\frac{p}{6}\rceil-1$. Then $|N[T']|=3(t+1)\ge\lceil\frac{p}{2}\rceil$. T' is an IMD-set of G and $\gamma_M^{-1}(G)=t+1$. Also, T' contains a cut vertex u and the induced subgraph $\langle V-T'\rangle$ is disconnected implies that T' is an ISMD-set of G and $\gamma_{SM}^{-1}(G)=|T'|=t+1$. Hence $\gamma_M^{-1}(G)=\gamma_{SM}^{-1}(G)$. If $d(u)\ge 3$ and $u\in V-T$ then the set $T'=\{u,u_1,\dots,u_t\}\subseteq V-T$ with $d(u_i,u_j)\ge 3$, for $i\ne j$ and $u_i,u_j\in T'$ and $|T'|=t+1$, where $t\ge\lceil\frac{p}{8}\rceil-1$. Then $|N[T']|\ge 4(t+1)\ge\lceil\frac{p}{2}\rceil$ T' is an IMD-set of G and $\gamma_M^{-1}(G)=t+1$. Since T' includes the cut vertex u, $\langle V-T'\rangle$ is disconnected and $\gamma_{SM}^{-1}(G)=|T'|=t+1$. Hence, in all degrees of a cut vertex u, if the induced subgraph $\langle V-T\rangle$ contains a cut vertex 'u' then $\gamma_{SM}^{-1}(G)=\gamma_M^{-1}(G)$. □

Theorem 4.3. *Let T be a minimum MD set of a connected graph G. If the induced subgraph $\langle V-T\rangle$ does not contain cut vertex then $\gamma_{SM}^{-1}(G)\ge\kappa(G)$ where $\kappa(G)$ is a vertex connectivity of G.*

Proof. Let T be a γ_M-set of a connected graph G with p vertices. Let $S\subseteq V-D$ and $S=\{u_1,u_2,\dots,u_t\}$ be a vertex cut of G. Hence, $\langle V-S\rangle$ is disconnected with atleast two components g_1 and g_2 and each vertex is the end vertex of every edge connecting the components g_1 and g_2. Therefore the vertex connectivity number $\kappa(G)=|S|=t$.

Case (i): If $|N[S]|\ge\lceil\frac{p}{2}\rceil$ and $S\subseteq V-T$, then S is an inverse split majority dominating set of G and

$$\gamma_{SM}^{-1}(G)=|s|=\kappa(G). \tag{4.1}$$

Case (ii): If $|N[S]|<\lceil\frac{p}{2}\rceil$ then choose a sub set $T_1=\{S\}\cup\{S_1\}$, where $S=\{u_1,u_2,\dots,u_t\}$ is a vertex cut and $S_1=\{v_1,v_2,\dots,v_{t2}\}\subseteq V-T$ such that $|N[T_1]|\ge\lceil\frac{p}{2}\rceil$ and $|T_1|=t=t_1+t_2$. If $\gamma_{SM}^{-1}>\kappa(G)=|S|$. Since T_1 includes a vertex cut $\langle V-T_1\rangle$ is disconnected. Since $S\subseteq V-T$ and $S_1\subseteq V-T$, $T_1\subseteq V-T$ and $|T_1|=t=t_1+t_2$. Then, T_1 is an ISMD-set of G and $|S|\subseteq|T_1|$. Therefore $|S|=\kappa(G)<\gamma_{SM}^{-1}(G)=|T_1|$. Thus $\gamma_{SM}^{-1}(G)>\kappa(G)$. Hence from case (i) and (ii), we obtain $\gamma_{SM}^{-1}(G)\ge\kappa(G)$, where $k(G)$ is the vertex connectivity of G. This bound is sharp if $G=C_{19}$. By the result, $\gamma_{SM}^{-1}(C_{19})=\lceil\frac{p}{6}\rceil=4$ and $k(G)=2$. Hence $\gamma_{SM}^{-1}(G)>\kappa(G)$. Also, for Petersen graph, $k(G)=3$ and $\gamma_{SM}^{-1}(G)=3$. □

5. Bounds of $\gamma_{SM}^{-1}(G)$

Theorem 5.1. *For any tree $T\ne K_{1,p-1}$, $\lceil\frac{p}{8}\rceil\le\gamma_{SM}^{-1}(G)\le\lceil\frac{p}{4}\rceil+1$.*

Proof. The theorem is proved by induction on the number of pendants 'e'. Since every tree T has atleast two pendants, if $e = 2$ then $t = P_p$. By Proposition 3.2, $\gamma_{SM}^{-1}(P_p) = \lceil \frac{p}{6} \rceil > \lceil \frac{p}{8} \rceil$. If $e = 3$, then the tree t has the structure a caterpillar or a double star. By Theorem 3.7, $\gamma_{SM}^{-1}(T) = \lceil \frac{p}{8} \rceil$ and if t is a double star, $\gamma_{SM}^{-1}(D_{r,s})$. If $e = 4$, then t is a binary tree and by Theorem 3.7 $\gamma_{SM}^{-1}(t) = \lceil \frac{p}{8} \rceil$. This result is true for $e = 2,3,4,\dots,p-3$. If $e = p-2$ then $t = D_{r,s}$, a double star with $(r+s) = (p-2)$. By Result 3.1(vii). Then $\gamma_{SM}^{-1}(G) = 1$, if $r = s$. The lower bound is sharp if t is a caterpillar with p vertices. The upper bound exists if $t = D_{1,12}$. Then t is a double star with $p = 15$. Let $t = \{v\}$ be MD-set $S = \{u, v_1, v_2, v_3, v_4\} \subseteq V - t$, where $d(u) = 2$ and v_i's are pendants such that $|N[S]| \ge \lceil \frac{p}{2} \rceil$. S is an ISMD-set of G and $\gamma_{SM}^{-1}(G) = |S| = 5 = \lceil \frac{p}{4} \rceil + 1$. □

Theorem 5.2. *Let G be a connected graph with vertices. If H is a connected spanning subgraph of G then $\gamma_{SM}^{-1}(G) \le \gamma_{SM}^{-1}(H)$.*

Theorem 5.3. *If a connected graph $G \ne K_p$, a complete graph with p vertices, $\left\lceil \frac{p}{2(\Delta+1)} \right\rceil \le \gamma_{SM}^{-1}(G) \le p-2$. The bounds are sharp.*

Proof. Since $\gamma_M^{-1}(G) \le \gamma_{SM}^{-1}(G)$ and $\gamma_M^{-1}(G) \ge \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. Then $\gamma_{SM}^{-1}(G) \ge \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. Next, inequality is proved by induction on $\Delta(G)$. If $\Delta = 2$, then $G = C_p$, a cycle or P_p, a path with p vertices. By Proposition 3.2, $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{6} \rceil = \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. If $\Delta = 3$, then G is a caterpillar with one pendant at each vertex of the path. By the proposition, $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{8} \rceil = \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. This result is true for all $\Delta = 2,3,\dots,(p-3)$. Suppose $\Delta = p-2$, then any single vertex is a majority dominating set and IMD set of G and $\gamma_M(G) = \gamma_M^{-1}(G) = 1$. Since G has no cut vertex, $\gamma_{SM}^{-1}(G) \ge k(G)$ and $k(G) \le \Delta(G) = p-2$. Hence, $\gamma_{SM}^{-1}(G) \le (p-2)$. If $\Delta = p-1$ and $G \ne k_P$, then $G = k_P - \{e\}$. $\gamma_{SM}^{-1}(G) = k(G)$ and $k(G) = p-2$ and $\gamma_{SM}^{-1}(G) = (p-2)$. Hence, $\left\lceil \frac{p}{2(\Delta+1)} \right\rceil \le \gamma_{SM}^{-1}(G) \le (p-2)$. These bounds are sharp for $G = C_p$, cycle, $G = K_p - \{e\}$. □

Theorem 5.4. *If any tree t, $\gamma_{SM}^{-1}(t) \le \lceil \frac{p}{2} \rceil - d + 1$ where d is the degree of a cut vertex in $(V - T)$.*

6. Characterization Theorem for Minimal Inverse Split Majority Dominating Set

Theorem 6.1. *Let T be a γ_M-set of a connected graph G and $(V-T)$ has a cut vertex. Then the ISMD-set $T' \subseteq V - T$ is minimal if and only if for each $u \in T'$, either the following condition (a) or (b) holds.*

(a) (i) *If $|N[T']| > \lceil \frac{p}{2} \rceil$, $|Pn[u,T']| > |N[T']| - \lceil \frac{p}{2} \rceil$, and*
(ii) *$\rangle(V - T') \cup \{u_i\}\rangle$ is connected, for all $u_i \in T'$.*

(b) (i) *If $|N[T']| = \lceil \frac{p}{2} \rceil$, either u is an isolate of T' or $Pn[u,T'] \cap (V - T') \ne \phi$, and*
(ii) *$\langle (V - T') \cup \{u_i\}\rangle$ is connected, for all $u_i \in T'$.*

Proof. Let T be a γ_M-set of connected graph G and $(V-T)$ has a cut vertex.

Assume that $T' \subseteq V - T$ is a minimal ISMD-set of G with respect to T. (6.1)

Case (i): Let $u \in T'$. Since T' is minimal and $|N[T']| \geq \lceil \frac{p}{2} \rceil$, the set $(T' - \{u\})$ is not an inverse split majority dominating set of G.

$$\text{Then either } |N[T' - \{u\}]| < \left\lceil \frac{p}{2} \right\rceil \text{ or } \langle V - (T' - \{u\}) \rangle \text{ is connected.} \tag{6.2}$$

Let $|N[T']| > \lceil \frac{p}{2} \rceil$. By (6.2), $|N[T' - \{u\}]| < \lceil \frac{p}{2} \rceil$, then $|Pn[u, T']| - |N[T']| - |N[T' - \{u\}]|$ and $|N[T']| - |Pn[u, T']| < \lceil \frac{p}{2} \rceil$. Hence $|Pn[u, T']| > |N[T']| - \lceil \frac{p}{2} \rceil$, for some $u \in T'$. Hence the condition (a)(i) holds. If $T' = \{u_1\} \subseteq (V - T)$ and $\langle V - T \rangle$ has a cut vertex u_1 then $\langle (V - T') \cup \{u_1\} \rangle$ is connected, for some $u \in T'$. If $T' = \{u_1, \ldots, u_i\}$, $i \geq 2$ and $T' \subseteq (V - T)$ contains a vertex cut 'S' with $|S| \geq 2$ and $|N[T']| \geq \lceil \frac{p}{2} \rceil$ then the induced subgraph $\langle V - T' \rangle$ is disconnected with atleast two components. Now, if add all vertices $u_i \in T$ to $(V - T')$ then $\langle (V - T') \cup \{u_i\} \rangle$ would be connected in G. Hence the condition (a)(ii) holds.

Case (ii): Let

$$|N[T']| = \left\lceil \frac{p}{2} \right\rceil . \tag{6.3}$$

and $T' \subseteq (V - T)$ and $u \in T'$. Suppose u is neither an isolate of T' nor u has a private neighbour in $\langle V - T' \rangle$. Then $Pn[u, T'] = \phi$. Since $|N[T' - \{u\}]| = |N[T']| - |Pn[u, T']|$, $|N[T' - \{u\}]| = |N[T']| = \lceil \frac{p}{2} \rceil$. It implies that $(T' - \{u\}) \subseteq (V - T)$ is an inverse split majority dominating set of G, which is a contradiction to (6.1). Also, by the above arguments, $\langle (V - T') \cup \{u_i\} \rangle$ connected in G, for all $u_i \in T'$.

Conversely, suppose one of the above conditions (a) or (b) is true. Let T' be an inverse split majority dominating set of G. Then prove that T' is a minimal ISMD set of G. Suppose T' is not minimal. Then $T_1 = (T' - \{u\}) \subseteq (V - T)$ is an ISMD-set of G, for some $u \in T'$. It implies that

$$|N[T_1]| \geq \left\lceil \frac{p}{2} \right\rceil \text{ and } \langle V - T_1 \rangle \text{ is disconnected.} \tag{6.4}$$

Case (i): Suppose the condition (a)(i) holds, for some $u \in T'$. Then $|N[T']| > \lceil \frac{p}{2} \rceil$ and $|Pn[u, T']| > |N[T']| - \lceil \frac{p}{2} \rceil$. Since $|Pn[u, D']| = |N[T']| - |N[T' - \{u\}]|$ and $T_1 = T' - \{u\}$, $|N[T']| - |N[T_1]| = |Pn[u, T']| > N[T'] - \lceil \frac{p}{2} \rceil$. It implies that $|N[T_1]| < \lceil \frac{p}{2} \rceil$, which is a contradiction to the assumption (6.4). Also, if the condition (a)(ii) holds then $\langle (V - T') \cup \{u_i\} \rangle$ is connected, for some $u_i \in T'$. If for any $u_1 \in T'$ and u_1 is a cut vertex in $(V - T)$, then $\langle (V - T') \cup \{u_i\} \rangle = \langle V - (T' \cup \{u_1\}) \rangle$ is connected. It implies that $\langle V - T_1 \rangle$ is connected for any $u_1 \in T'$, which is a contradiction to (6.4). Hence T' is a minimal ISMD-set

Case (ii): Suppose the condition (b)(i) holds for some $u \in D'$. Then $|N[T']| = \lceil \frac{p}{2} \rceil$ and either u is an isolate of T' or $Pn[u, T'] \cap (V - T') \neq \phi$. If u is an isolate of T' then $u \in Pn[u, T]$ and $|Pn[u, T']| \geq 1$. If $|Pn[u, T']| \cap (V - T') \neq \phi$ then $|Pn[u, T']| \geq 2$. Since $|Pn[u, T']| = |N[T']| - |N[T_1]|$, $\lceil \frac{p}{2} \rceil - |N[T_1]| \geq 2$. Then $|N[T_1]| \leq \lceil \frac{p}{2} \rceil - 2$, which is contradiction to (6.4). Also, by the above arguments the $\langle V - T_1 \rangle$ is connected, for any $u_1 \in T'$, which is a contradiction to (6.4). Hence T' is a minimal ISMD-Set of G. □

7. Conclusion

We have introduced an inverse split majority domination and we have investigated $\gamma_{SM}^{-1}(G)$ for some classes of graphs then the characterization theorem for minimal ISMD-set and bounds $\gamma_{SM}^{-1}(G)$ are established. The relationship between $\gamma_{SM}^{-1}(G)$ with other inverse domination parameter is also discussed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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