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Research Article

Characteristics of Hyperideals in Ternary Semihyperrings

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Abstract. Ternary semihyperring is an algebraic structure with one binary hyper operation and ternary multiplication. In this paper, we give some properties of hyperideals in ternary semihyperring. We introduce the notion of simple, (0-)simple ternary semihyperring and characterize the minimality and maximality of hyperideals in ternary semihyperring. The relationship between them is investigated in ternary semihyperring extending and generalizing the analogous results for ternary semirings.

Keywords. Semihyperring, Ternary semihyperring, Hyperideal, Minimal and maximal hyperideal, (0-)simple ternary semihyperring

Mathematics Subject Classification (2020). 20N20, 16Y99

1. Introduction

In mathematics 'algebraic structures' play a vital role with applications in various disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc. In the 19th century, Ternary algebraic operations were considered by many mathematicians such as Cayley [4] who introduced the notion of 'cubic matrix' which in turn was generalized by Gelfand *et al*. [21] in 1990. Some significant physical applications of Ternary structures in Nambu mechanics, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics, supersymmetric theories, Yang-Baxter equation, etc. can be seen in [2, 8, 24, 25, 35, 36]. The notion of an n-ary group was introduced by W. Dörnte [15] in 1928. The concepts of n-ary algebras, i.e., sets with one n-ary operation, seems to be going back to Miller's article [23]. In the theory of automata [22] are used n-ary systems satisfying some associative law, some others n-ary systems are applied in the theory of quantum groups and combinatorics [34]. Kerner in [24] described different applications of ternary structures in physics. In physics there are used n-ary structures as n-ary Filippov algebras (see [32]) and n-Lie algebras (see [36]). Some n-ary structures have application in coding theory, cryptology and in the theory of (t,m,s)-nets [26]. Ternary semigroups are universal algebras with one associative operation. In 1932 the theory of ternary algebraic system was introduced by Lehmer [27]. He studied certain algebraic systems called triplexes which are commutative ternary groups. S. Banach (cf. [28]) introduced the notion of ternary semigroups. Sioson [33] introduced ideal theory in ternary semigroups and characterized regular ternary semigroups in 1965. Dudek *et al*. [16, 17] studied the ideals in n-ary semigroups. In 1995, Dixit and Dewan [14] introduced and studied some properties of ideals and quasi-(bi-)ideals in ternary semigroups. In 1934 Hyperstructure theory was introduced, when F. Marty [29] defined hypergroups based on the notion of hyper operation. Nowadays, a number of different hyperstructures are widely studied by many mathematicians from the theoretical point of view and for their applications to many subjects of pure and applied mathematics. In an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on hyper-structure theory (see [6,7,13,37]). A recent book on hyperstructures [7] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relation and hypergraphs. The book entitled 'Hyperring Theory and Applications' of Davvaz and Leoreanu-Fotea [13] is devoted to the study of hyperring theory. Several types of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures, e-hyperstructures and transposition hypergroups. Some basic notions about semihypergroup theory can be seen in [1, 5, 10–12, 19, 20, 31].

Davvaz in [9] introduced the notion of ternary semihyperrings which is a generalization of semihyperrings [3] and also a generalization of ternary semirings [18]. In a ternary semihyperring, multiplication is a ternary operation and addition is a hyperoperation. Also, the notion of ternary semihyperrings is a generalization of semirings. Our main purpose of this paper is to introduce the notions of simple, (0-)simple ternary semihyperring and characterize the minimality and maximality of hyperideals in ternary semihyperring.

2. Preliminaries

Definition 2.1 ([9])**.** Let H be a non-empty set and $\circ : H \times H \to \wp^*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a hypergroupoid.

For any two non-empty subsets A and B of H and $x \in H$ we have

$$A \circ B = \bigcap_{a \in A, b \in B} a \circ b, \; A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2 ([9])**.** A ternary hyper grouped is called the pair $(H, [\ ])$ if H_1, H_2, H_3 are the non-empty subsets of H then, we define

$$[H_1 H_2 H_3] = \bigcup_{h_1 \in H_1, h_2 \in H_2, h_3 \in H_3} [h_1 h_2 h_3].$$

Definition 2.3 ([9])**.** A non-empty set H is called ternary semihyperring if for all $h_1, h_2, h_3, h_4, h_5 \in H$ and $(H, \oplus)$ is a commutative semi hyper group and the ternary multiplication [ ] satisfies the following conditions:

(i) $[[h_1 h_2 h_3] h_4 h_5] = [h_1 [h_2 h_3 h_4] h_5] = [h_1 h_2 [h_3 h_4 h_5]]$,

(ii) $[(h_1 \oplus h_2) h_3 h_4] = [h_1 h_3 h_4] \oplus [h_2 h_3 h_4]$,

(iii) $[h_1 (h_2 \oplus h_3) h_4] = [h_1 h_2 h_4] \oplus [h_1 h_3 h_4]$,

(iv) $[h_1 h_2 (h_3 \oplus h_4)] = [h_1 h_2 h_3] \oplus [h_1 h_2 h_4]$.

Definition 2.4 ([12])**.** A ternary hyper semi ring H is said to have a zero element if there exist an element $0 \in H$ such that for all $h_1, h_2 \in H$,

$$[0 h_1 h_2] = [h_1 0 h_2] = [h_1 h_2 0] = \{0\}.$$

Definition 2.5 ([9])**.** An element e of ternary hyper semi ring H is called an identity if $[h_1 h_1 e] = [h_1 e h_1] = [e h_1 h_1] = \{h_1\}$ for all $h_1 \in H$ and it is clear that $[e e h_1] = [e h_1 e] = [h_1 e e] = \{h_1\}$.

Definition 2.6 ([9])**.** A non empty additive sub semi hyper group I of a ternary semi hyper ring H is called

(i) A left hyper ideal of H if $[HHI] \subseteq I$.

(ii) A lateral hyper ideal of H if $[HIH] \subseteq I$.

(iii) A right hyper ideal of H if $[IHH] \subseteq I$.

If I is both left as well as right hyper ideal of H, then I is called a two sided hyper ideal of H. If I is a left, a lateral, a right hyper ideal of H then I is called a hyper ideal of H.

3. (0)-Simple Ternary Semihyperrings

In this section we 'introduce' and 'characterize' the '(0-) simple ternary semihyperrings' . Some properties of them are investigated in terms of 'hyperideals' .

Definition 3.1. Let H be a ternary semi hyper ring with zero, then H is known as simple if H has no proper hyperideals.

Example 3.2. Let $H = \{a,b,c,d,e,f\}$ and $[x,y,z] = (x * y) * z$ for all $x,y,z \in H$, where $\oplus$ and $*$ are defined as follows:

$\oplus$	a	b	c	b	e	f
a	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
b	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$
c	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	H-d	H-d	H-d	H-d	H-d	H-d
e	H-e	H-e	H-e	H-e	H-e	H-e
f	H-f	H-f	H-f	H-f	H-f	H-f

$*$	a	b	c	b	e	f
a	a	b	c	c	e	e
b	b	b	b	b	f	f
c	c	b	c	b	c	c
b	b	b	b	b	b	b
e	e	f	c	c	e	f
f	f	f	b	b	f	f

Then $(H, \oplus, [\ ])$ is a ternary semihyperring. There is no proper hyperideal of H and hence H is simple.

It is clear that if H is a ternary semihyperring with zero, then every hyperideal of H contains a zero element.

Definition 3.3. Let $(H, \oplus, [\ ])$ be a ternary semihyperring with zero. H is called 0-simple if it is has no nonzero proper hyperideal and $[HHH] \neq \{0\}$.

Remark 3.4. Let $(H, \oplus, [\ ])$ be a 'ternary semi hyper ring' for every element $h \in H$ then the 'hyper ideal generated' by h are respectively shown by $J(h) = \langle h \rangle = \{h\} \cup [HHh] \cup [HhH] \cup [H[HhH]H] \cup [hHH]$.

Lemma 3.5. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring. For any nonempty subset A of H, $[HHA] \cup [HHAHH] \cup [HAH] \cup [AHH] \cup A$ is the smallest hyperideal of H containing A.*

Lemma 3.6. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring. For any nonempty subset A of H, $[HHA] \cup [HHAHH] \cup [HAH] \cup [AHH]$ is the hyperideal of H.*

Theorem 3.7. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring without zero. Then the following statements are equivalent.*

(i) *H is simple,*

(ii) $\forall\ a \in H$, $[HHh] \cup [HhH] \cup [H[HhH]H] \cup [hHH] = H$,

(iii) $\forall\ a \in H$, $\langle h \rangle = H$.

Proof. (i)⇒(ii): Let H be a simple, by Lemma 3.6, we have

$$\forall\ a \in H,\ [HHh] \cup [HhH] \cup [H[HhH]H] \cup [hHH] = H.$$

(ii)⇒(iii): By Lemma 3.5, $\langle h \rangle = [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cup \{h\} = H \cup \{h\} = H$.

(iii)⇒(iv): Let A be a hyper ideal of H as well as $h \in A$. Then $H = \langle h \rangle \subseteq A \subseteq H$ implies that $A = H$. Therefore, H is simple. □

Theorem 3.8. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring with zero. Then the following statements are true.*

(i) *If H is a 0-simple. Then $\forall\ h \in H \setminus \{0\}$, $\langle h \rangle = H$.*

(ii) *If $\forall\ h \in H \setminus \{0\}$, $\langle h \rangle = H$. Then either $[HHH] = \{0\}$ or H is 0-simple.*

Proof. (i): Let H be a 0-simple. Then $\forall\ h \in H \setminus \{0\}$, $\langle h \rangle$ is non-zero hyperideal of H. Therefore, $\forall$ $h \in H \setminus \{0\}$, $\langle h \rangle = H$.

(ii): Let us assume that $\forall\ h \in H \setminus \{0\}$, $\langle h \rangle = H$. Then either $[HHH] \neq \{0\}$.
Let A be a non zero hyperideal of H. Let $h \in A \setminus \{0\} \Rightarrow \langle h \rangle = H \subseteq A \subseteq H$.
Therefore, $A = H$. Hence H is a 0-simple. □

Theorem 3.9. *The nonempty intersection of a family of hyper-filters of a ternary semihyperring H is also a hyper-filter of H.*

Theorem 3.10. *The 'Union' of family of 'hyper ideals of a ternary semi hyper ring' H is a 'hyper ideal' of H.*

Theorem 3.11. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring and A be a hyper ideal of H. Let T is a ternary subsemihyperring. Then the following statements are true.*

(i) *If T is a simple such that $T \cap A \neq \emptyset$, then $T \subseteq A$.*

(ii) *If T is a 0-simple such that $T \setminus \{0\} \cap A \neq \emptyset$, then $T \subseteq A$.*

Proof. (i): Let us assume that T is simple such that $T \cap A \neq \emptyset$. Let $h \in T \cap A$. By Lemma 3.6, $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cap T$ is a hyperideal of T. Then we have $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cap T = T \Rightarrow T \subseteq [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \subseteq [HHA] \cup [HHAHH] \cup [HAH] \cup [AHH] \subseteq A$. Therefore, $T \subseteq A$.

(ii): We assume that T is 0-simple such that $T \setminus \{0\} \cap A \neq \emptyset$. Let $h \in T \setminus \{0\} \cap A$. By Lemma 3.5, and Theorem 3.8(i), we get

$$\begin{aligned} T = \langle h \rangle &= \{h\} \cup [HHh] \cup [HhH] \cup [H[HhH]H] \cup [hHH] \cap T \\ &\subseteq \langle h \rangle = \{h\} \cup [HHh] \cup [HhH] \cup [H[HhH]H] \cup [hHH] \subseteq \langle h \rangle \\ &\subseteq A. \end{aligned}$$

Therefore, $T \subseteq A$. □

Theorem 3.12. *In any ternary semihyperring H, the following are equivalent.*

(i) *Principal hyperideals of H form a chain.*

(ii) *Hyperideals of H form a chain.*

Proof. (i)⇒(ii): Suppose that principal hyperideals of H form a chain.
Let A, B be two hyperideals of H. Suppose if possible $A \nsubseteq B$, $B \nsubseteq A$.
Then there exists $a \in A \setminus B$ and $b \in B \setminus A$.
$a \in A \Rightarrow \langle a \rangle \subseteq A$ and $b \in B \Rightarrow \langle b \rangle \subseteq B$.
Since principal hyperideals form a chain, either $\langle a \rangle \subseteq \langle b \rangle$ or $\langle b \rangle \subseteq \langle a \rangle$.
If $\langle a \rangle \subseteq \langle b \rangle$, then $a \in \langle b \rangle \subseteq B$. It is a contradiction.
If $\langle b \rangle \subseteq \langle a \rangle$, then $b \in \langle a \rangle \subseteq A$. It is also a contradiction.
Therefore, either $A \subseteq B$ or $B \subseteq A$ and hence hyperideals from a chain.

(ii)⇒(i): Suppose that hyperideals of H form a chain.
Then clearly principal hyperideal of H form a chain. □

4. Minimal and Maximal Hyperideals of Ternary Semihyperrings

In this section, we give some properties of (0-)minimal hyperideals and (0)-maximal hyperideals of ternary semihyperrings and investigate the relationship between the (0-)minimal hyperideals, (0)-maximal hyperideals and the (0)-simple ternary semihyperrings.

Definition 4.1. Let $(H, \oplus, [\ ])$ be a ternary semihyperring without zero. A hyper ideal A of H is known as *minimal hyperideal* of H if there is no hyperideal B of H such that $B \subseteq A$.

Definition 4.2. Let $(H, \oplus, [\ ])$ be a ternary semihyperring with zero. A hyper ideal A of H is known as 0-*minimal hyperideal* of H if there is no non zero hyperideal B of H such that $B \subseteq A$.
(or)
Let $(H, \oplus, [\ ])$ be a ternary semihyperring with zero. A hyper ideal A of H is known as 0-*minimal hyperideal* of H if for every hyperideal B of H such that $B \subseteq A$, we get $B = \{0\}$.

Theorem 4.3. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring without zero and A be a hyperideal of H. Then the following statements are true.*

(i) *If A is a minimal hyperideal without zero of H, if and only if A is a simple.*

(ii) *If A is a minimal hyperideal of H with zero, then A is a* (0-)*simple.*

Proof. (i): Let A is a minimal hyperideal of H without zero. Let B is a hyperideal of A. Then we get $[AAB] \cup [AABAA] \cup [ABA] \cup [BAA] \cap B$ is a hyperideal of B. Then we have $[AAB] \cup [AABAA] \cup [ABA] \cup [BAA] \cap B = B \Rightarrow A \subseteq [AAB] \cup [AABAA] \cup [ABA] \cup [BAA] \subseteq B$. Therefore, $A \subseteq B$ and hence $A = B$. Therefore, A is a simple.

Conversely, suppose that A is a simple and B is hyperideal of H such that $B \subseteq A$. Then we get $B \cap A \neq \emptyset$. By Theorem 3.11(i) we have $A \subseteq B \Rightarrow A = B$. Therefore, A is the minimal hyperideal of H.

(ii): The proof is similar to (i). □

Theorem 4.4. *Let $(H,\oplus,[\ ])$ be a ternary semihyperring with zero and A be a non zero hyperideal of H. Then the following statements are true.*

(i) *If A is a 0-minimal hyperideal of H. Then either there exist a nonzero hyperideal B of A such that $[AAB]\cup[AABAA]\cup[ABA]\cup[BAA]=\{0\}$ or A is a 0-simple.*

(ii) *If A is a 0-simple, then A is a 0-minimal hyperideal of H.*

Proof. The proof is similar to the proof of Theorem 4.3(i) and Theorem 3.11(ii). □

Theorem 4.5. *Let $(H,\oplus,[\ ])$ be a ternary semihyperring without zero having proper hyperideals. Then every proper hyperideal of H is minimal if and only if H contains exactly one proper hyperideal of H or H contains exactly two proper hyperideals A_1, A_2 such that $A_1\cup A_2=H$ and $A_1\cap A_2=\emptyset$.*

Proof. Suppose that every proper hyperideal of H is minimal and A be a proper hyperideal of H. Then A be a minimal hyperideal of H. Then we get the following cases:

Case 1: $\forall\ a\in H\setminus A$, $H=\langle a\rangle$. If B is also proper hyperideal of H and $B\neq A$, then since A is minimal hyperideal, we get $B\setminus A\neq\emptyset$. Hence $\exists\ a\in B\setminus A\subseteq H\setminus A$. Therefore, $H=\langle a\rangle\subseteq B\subseteq H$, so $B=H$. This is a contradiction and hence $A=B$. Therefore, in this case A is unique proper hyperideal of H.

Case 2: $\exists\ a\in H\setminus A$, $H\neq\langle a\rangle$. We have $\langle a\rangle\neq A$ and $\langle a\rangle$ is a minimal hyperideal of H. By Theorem 3.10, $\langle a\rangle\cup A$ is a hyperideal of H. Since $A\subset\langle a\rangle\cup A$. Hence by hypothesis we get $\langle a\rangle\cup A=H$. Here $\langle a\rangle\cap A\subseteq\langle a\rangle$ and $\langle a\rangle$ is the minimal hyperideal of H. Therefore, $\langle a\rangle\cap A=\emptyset$. Let B be the any arbitrary proper hyperideal of H, then B is a minimal hyperideal of H. We obtain $B=B\cap H=P\cap(\langle a\rangle\cup A)=(P\cap\langle a\rangle)\cup(P\cap A)$. If $P\cap A\neq\emptyset$. Since B and $\langle a\rangle$ are minimal hyperideals of H. We get $B=\langle a\rangle$. In this case H contains exactly two proper hyperideals A and $\langle a\rangle$ such that $\langle a\rangle\cup A=H$ and $\langle a\rangle\cap A=\emptyset$.
Converse part is obvious. □

Theorem 4.6. *Let $(H,\oplus,[\ ])$ be a ternary semihyperring with zero having nonzero proper hyperideals. Then every nonzero proper hyperideal of H is 0-minimal if and only if H contains exactly one nonzero proper hyperideal of H or H contains exactly two nonzero proper hyperideals A_1, A_2 such that $A_1\cup A_2=H$ and $A_1\cap A_2=\{0\}$.*

Proof. Proof is similar to the proof of Theorem 4.5. □

Definition 4.7. Let $(H,\oplus,[\ ])$ be a ternary semihyperring. A hyper ideal A of H is known as *maximal hyperideal* of H if for every hyperideal B of H such that $A\subseteq B$ we have $B=H$.

Theorem 4.8. *Let $(H,\oplus,[])$ be a ternary semihyperring with zero having proper hyperideals. Then every proper hyperideal of H is maximal if and only if H contains exactly one proper* *hyperideal of H or H contains exactly two proper hyperideals A_1, A_2 such that $A_1 \cup A_2 = H$ and $A_1 \cap A_2 = \emptyset$.*

Proof. Suppose that every proper hyperideal of H is maximal and A be a proper hyperideal of H. Then A be a maximal hyperideal of H. Then we get the following cases:

Case 1: $\forall\ a \in H \setminus A$, $H = \langle a \rangle$. If B is also proper hyperideal of H and $B \neq A$, then since A is maximal hyperideal, we get $B \setminus A \neq \emptyset$. Hence $\exists\ a \in B \setminus A \subseteq H \setminus A$. Therefore, $H = \langle a \rangle \subseteq B \subseteq H$, so $B = H$. This is a contradiction and hence $A = B$. Therefore in this case A is unique proper hyperideal of H.

Case 2: $\exists\ a \in H \setminus A$, $H \neq \langle a \rangle$. We have $\langle a \rangle \neq A$ and $\langle a \rangle$ is a maximal hyperideal of H. By Theorem 3.10, $\langle a \rangle \cup A$ is a hyperideal of H. Since $A \subset \langle a \rangle \cup A$ and A is a maximal hyperideal of H. Hence we get $\langle a \rangle \cup A = H$. Here $\langle a \rangle \cap A \subseteq \langle a \rangle$ and by hypothesis we have $\langle a \rangle \cap A = \emptyset$. Let B be the any arbitrary proper hyperideal of H, then B is a maximal hyperideal of H. We obtain $B = B \cap H = P \cap (\langle a \rangle \cup A) = (P \cap \langle a \rangle) \cup (P \cap A)$. If $P \cap A \neq \emptyset$. Since $B \cap \langle a \rangle$ and $\langle a \rangle$ are maximal hyperideals of H. We get $B = \langle a \rangle$. In this case H contains exactly two proper hyperideals A and $\langle a \rangle$ such that $\langle a \rangle \cup A = H$ and $\langle a \rangle \cap A = \emptyset$.
Converse part is obvious. □

Theorem 4.9. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring with zero having nonzero proper hyperideals. Then every nonzero proper hyperideal of H is maximal if and only if H contains exactly one nonzero proper hyperideal of H or H contains exactly two nonzero proper hyperideals A_1, A_2 such that $A_1 \cup A_2 = H$ and $A_1 \cap A_2 = \{0\}$.*

Proof. Proof is similar to the proof of Theorem 4.8. □

Theorem 4.10. *Let $(H, \oplus, [\ ])$ be a ternary semihyperring. A proper hyperideal A of H is maximal if and only if*

(i) *$H \setminus A = \{h\}$ and $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \subseteq A$ for some $h \in H$, or*

(ii) *$H \setminus A \subseteq [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$ for all $h \in H \setminus A$.*

Proof. Suppose A is a maximal hyperideal of H. Then the following cases are arising:

Case 1: $\exists\ h \in H \setminus A \ni [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \subseteq A$. By Lemma 3.5,

$$\begin{aligned} A \cup \{h\} &= A \cup [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cup \{h\} \\ &= A \cup \{[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cup \{h\}\} = A \cup \langle h \rangle. \end{aligned}$$

Since $A \cup \langle h \rangle$ is a hyperideal of H, $A \cup \{h\}$ is a hyperideal of H. Here A is a maximal hyperideal of H as well as $A \subseteq A \cup \{h\}$. We get $A \cup \{h\} = H$. Therefore $H \setminus A = \{h\}$.

Case 2: For all $h \in H \setminus A$, $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \nsubseteq A$. Since $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$ is a hyperideal of H. By Lemma 3.6, and Theorem 3.10, $A \cup [HHh] \cup [HHhHH] \cup$

$[HhH]\cup[hHH]$ is a hyperideal of H as well as $A\subseteq A\cup[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]$. Since A is maximal hyperideal of H and hence $A\cup[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]=H$. Therefore, for all $h\in H\setminus A$ we get $H\setminus A\subseteq[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]$.

Conversely, suppose that B is a hyperideal of H such that $A\subset B$. Then $B\setminus A\neq\emptyset$. If $H\setminus A=\{h\}$ and $[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]\subseteq A$ for some $h\in H$. Then $B\setminus A\subseteq H\setminus A=\{h\}$. Thus $B\setminus A=\{h\}$ and hence $B=A\cup\{h\}=H$. Therefore, A is a maximal hyperideal of H. If $H\setminus A\subseteq[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]$ for all $h\in H\setminus A$. Then $H\setminus A\subseteq[HHx]\cup[HHxHH]\cup[HxH]\cup[xHH]\subseteq[HHB]\cup[HHBHH]\cup[HBH]\cup[BHH]\subseteq B forall x\in B\setminus A$. Therefore, $H=H\setminus A\cup A\subseteq B\subseteq H$ and hence $B=H$. Hence A is a maximal hyperideal of H. □

Note 4.11. Let $(H,\oplus,[\ ])$ be a ternary semihyperring. Let $\mathfrak{U}$ indicate union of all proper hyperideals of H.

Lemma 4.12. *Let $(H,\oplus,[\ ])$ be a ternary semihyperring. Then $\mathfrak{U}=H$ if and only if $\langle h\rangle\neq H$ $\forall$ $h\in H$.*

Theorem 4.13. *Let $(H,\oplus,[\ ])$ be a ternary semihyperring without zero. Then only one of the following statements is satisfied:*

(i) *H is simple,*

(ii) *$\forall$ $h\in H$, $\langle h\rangle\neq H$,*

(iii) *$\exists$ $h\in H\ni\langle h\rangle=H$, $h\notin[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]\subseteq\mathfrak{U}=H\setminus\{h\}$, and $\mathfrak{U}$ is the unique maximal hyperideal of H.*

(iv) *$H\setminus\mathfrak{U}=\{h\in H:[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]=H\}$ and $\mathfrak{U}$ is the unique maximal hyperideal of H.*

Proof. Suppose that H is not simple. Then there exist a proper hyperideal of H implies that $\mathfrak{U}$ is a hyperideal of H, then we get following two cases:

Case 1: $\mathfrak{U}=H$.
Lemma 4.12, implies that $\forall$ $h\in H$, $\langle h\rangle\neq H$ and hence statement (ii) is satisfied.

Case 2: $\mathfrak{U}\neq H$.
We get $\mathfrak{U}$ is the maximal hyperideal of H. Suppose A is the maximal hyperideal of H. Then since A is a proper hyperideal of H, we have $A\subseteq\mathfrak{U}\subseteq H$. Since A is a maximal hyperideal of H, we obtained $A=\mathfrak{U}$. Hence $\mathfrak{U}$ is the unique maximal hyperideal of H. By Theorem 4.10, gives

(a) $H\setminus\mathfrak{U}=\{h\}$ and $[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]\subseteq\mathfrak{U}$ for some $h\in H$, or

(b) $\forall$ $h\in H\setminus\mathfrak{U}$, $H\setminus\mathfrak{U}\subseteq[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]$.

Suppose that, $H\setminus\mathfrak{U}=\{h\}$ and $[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]\subseteq\mathfrak{U}$ for some $h\in H$. Then $[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]\subseteq\mathfrak{U}=H\setminus\{h\}$. Since $h\notin\mathfrak{U}$, we get $\langle h\rangle=H$. If $h\in[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]$, then $\{h\}\subseteq[HHh]\cup[HHhHH]\cup[HhH]\cup[hHH]$. By

Lemma 3.5, implies $H = \langle h \rangle = [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cup \{h\} \subseteq [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \subseteq [HH\mathfrak{U}] \cup [HH\mathfrak{U}HH] \cup [H\mathfrak{U}H] \cup [\mathfrak{U}HH] \cup \mathfrak{U} = \mathfrak{U} \subseteq H$. Therefore, we get $H = \mathfrak{U}$. Which is impossible and hence $h \notin [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$ and so statement (iii) is satisfied.

Let us suppose that for all $h \in H \setminus \mathfrak{U}$, $H \setminus \mathfrak{U} \subseteq [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$. Let $h \in H \setminus \mathfrak{U}$, then $h \in [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$. So $\{h\} \subseteq [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$. Lemma 3.5, implies $\langle h \rangle = [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cup \{h\} = HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$. Since $h \notin \mathfrak{U}$. We get $\langle h \rangle = H$. Therefore, $H = \langle h \rangle = [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH]$.

Conversely, let $h \in H$ such that $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] = H$. Let $h \in \mathfrak{U}$, then $\langle h \rangle \subseteq \mathfrak{U} \subset H$. By Lemma 3.5, implies $\langle h \rangle = [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \cup \{h\} = H \cup \{h\} = H$. It is a contradiction and hence we have $h \in H \setminus \mathfrak{U}$ and implies that $H \setminus \mathfrak{U} = \{h \in H : [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] = H\}$ and hence the statement (iv) satisfied. Therefore, this completes the proof. □

Theorem 4.14. *Let* $(H, \oplus, [\ ])$ *be a ternary semihyperring with zero and* $[HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \neq \{0\}$. *Then only one of the following statements is satisfied:*

(i) *H is 0-simple,*

(ii) $\forall\ h \in H$, $\langle h \rangle \neq H$,

(iii) $\exists\ h \in H \ni \langle h \rangle = H$, $h \notin [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] \subseteq \mathfrak{U} = H \setminus \{h\}$ *and $\mathfrak{U}$ is the unique maximal hyperideal of H.*

(iv) $H \setminus \mathfrak{U} = \{h \in H : [HHh] \cup [HHhHH] \cup [HhH] \cup [hHH] = H\}$ *and $\mathfrak{U}$ is the unique maximal hyperideal of H.*

Proof. The proof is similar to the proof of Theorem 4.13. □

5. Conclusion

We introduce the notion of simple, (0-)simple and characterize the minimality and maximality of hyperideals in ternary semihyperrings. The relationship between them is investigated in ternary semihyperrings extending and generalizing the analogues results for ternary semirings.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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