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Research Article

A New Root-Finding Method for Univariate Non-Linear Transcendental Equations With Quadratic Convergence

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Abstract. This paper presents a new algorithm to find a non-zero real root of the non-linear transcendental equations. The proposed method is based on the combination of the inverse sine series and Newton-Raphson method. Implementation of the proposed method in MATLAB is applied to different problems to ensure the methods applicability. Error calculation has been done for available existing methods and the proposed method. The suggested method is evaluated using a number of numerical examples and the results indicate that the proposed method is effective than well-known methods. The proposed method's convergence is discussed, and it is shown to be quadratically convergence.

Keywords. Non-linear equations; Quadratic convergence; Root-finding; Iteration method

Mathematics Subject Classification (2020). 65H04; 65H05

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1. Introduction

Numerical root-finding methods are very important for solving non-linear transcendental equations, which have wide range of applications in science and engineering. The root finding methods have enormous applications in many fields such as the optimization of photovoltaic system's maximum power point tracking as well as the selection of appropriate digital pulse width modulation and converter parameters [5]. These numerical methods are used to solve nonlinear differential equations, evaluate gradient retention durations in liquid chromatography, weather forecasting, mechanical motions, circuit analysis, analysis of state equations for a real gas, optimization, and a variety of other engineering design procedures.

Existing iterative approaches for solving non-linear transcendental equations often require one or more initial estimates for the chosen root. These approaches must meet a variety of criteria, including convergence qualities, efficiency, and numerical stability. The bisection and regula falsi procedures are universally convergent iterative methods for evaluating a simple root of a nonlinear equation by repeating a technique until the root is found. But their asymptotic convergence rate of iterative sequence $(X_n-\alpha)$, where α be a simple root, is linear. The renowned Newton's method and its variants are iterative formulae are mostly quadratic convergent.

In the last quarter of the nineteenth century, Grossman and Katz presented multiplicative calculus in [6], and the value of multiplicative calculus has only lately been understood by scholars from several disciplines. For finding a root of a univariate non-linear equation, the well-known Newton's method and its variants are employed [10, 11, 14]. However, if the initial approximation point is far from the root or the derivative vanishes in the region of the root, these approaches may fail to converge. Chen and Li [1, 3] obtained quadratic convergence of the sequence of points and diameters by combining the bisection method with a series of exponential iteration methods. Wu [16] and Zhu [17] created a new class of quadratic convergence iteration algorithms that do not rely on derivatives. Wu [16] proposed a new Algorithm NA for containing simple roots of nonlinear equations, as well as a class of novel iterative formulae with greater order of convergence. Algorithm NA has nice asymptotic quadratic convergence of both the iterative sequence $\{X_n\}$ and the sequence of the diameters $\{(b_n-a_n)\}$. Kou [8] proposed a series of improved Newton-like methods that eliminate the stringent $f'(x)\neq 0$ placed on Newton-Raphson method in the vicinity of the needed root. New root-finding methods based on exponential and sine series with quadratic convergence were proposed [9, 15], which generated roots in fewer iterations than conventional techniques.

2. Proposed Method

The proposed formula using inverse sine series is presented as

$$x_{n+1}=x_n\left[1+\arcsin\left(\frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))}\right)\right],\quad n=0,1,2,\dots. \tag{2.1}$$

2.1 Convergence Analysis

Theorem 1. *Suppose $\alpha\neq 0$ is a real exact root of $f(x)$ and θ is a sufficiently small neighborhood of α. Let $f''(x)$ exist and $f'(x)\neq 0$ in θ. Then the proposed iterative formula in equation* (2.1) *generates a quadratically convergent series of iterations.*

Proof. The given proposed formula for new approximation is

$$x_{n+1}=x_n\left[1+\arcsin\left(\frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))}\right)\right],\quad n=0,1,2,\dots \tag{2.2}$$

and using the fact,

$$\lim_{x_n\to\alpha}\left[1+\arcsin\left(\frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))}\right)\right]=1.$$

Hence $x_{n+1} = \alpha$.
We know that

$$\arcsin(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \dots.$$

By using the above expansion,

$$\arcsin\left(\frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))}\right) = \frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))} + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))}\right)^3 + \dots.$$

Substituting in equation (2.2),

$$x_{n+1} = x_n\left[1 + \frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))} + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))}\right)^3 + \dots\right]. \tag{2.3}$$

Using finite forward difference formula we know that

$$f'(x_n) = \frac{f(x_n)-f(x_{n-1})}{h}.$$

which implies to

$$f(x_n)-f(x_{n-1}) = hf'(x_n) \text{ and } \frac{(x_{n-1}-x_n)f(x_n)}{x_n(f(x_n)-f(x_{n-1}))} = \frac{-hf(x_n)}{x_n h f'(x_n)} = \frac{-f(x_n)}{x_n f'(x_n)}.$$

Now, equation (2.3) reduces to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{x_n^2}\left(\frac{-f(x_n)}{f'(x_n)}\right)^3 - \dots. \tag{2.4}$$

Equation (2.4) reduces to the Newton-Raphson method, which has quadratic convergence in the first two terms by neglecting $\left(\frac{-f(x_n)}{f'(x_n)}\right)^3$ and higher order terms. Therefore, the order of convergence of proposed method is ≥ 2. □

2.2 Step by Step Process of Proposed Method

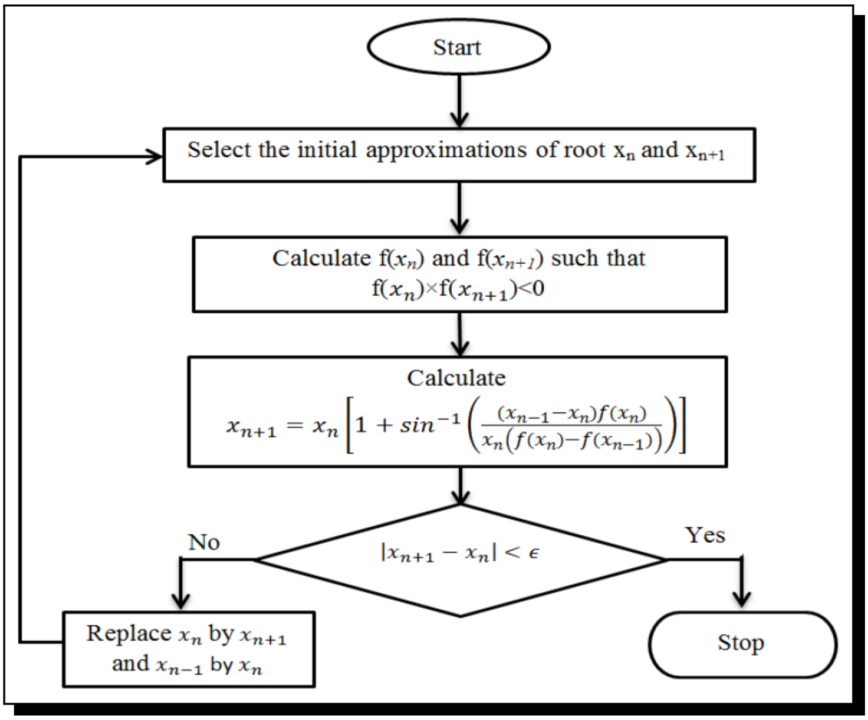

Figure 1. Flow chart of the proposed method

Step 1: Identify the initial approximations of the root x_0 and x_1 such that $f(x_0)\times f(x_1)<0$.

Step 2: Using the proposed method (i.e., eq. (2.1)) find the next approximations.

Step 3: Repeat the above *Step* 2 until we get the desired accuracy.

2.3 MATLAB Code for the Proposed Method

In this section we are going to present MATLAB code for the proposed method as follows:

```
a=input('Given Function/nonlinear equation:','s');
f=inline(a);
x(1)=input('Enter first approximate root i.e., x0 value:');
x(2)=input('Enter second approximate root i.e., x1 value:');
n=input('Enter allowed Error:');
itr=0;
for i=3:100
t1=fxn(x(i-1))*(x(i-2)-x(i-1));
t2=x(i-1)*(fxn(x(i-1))-fxn(x(i-2)));
  x(i)=x(i-1)*(1+asin(t1/t2));
  disp(x(i));
  if abs((x(i)-x(i-1))/x(i))*100<n
    root=x(i);
    disp(itr);
    disp(x(i));
    error=abs((x(i)-x(i-1))/x(i))*100;
  break
end
```

3. Comparing With Existing Results

The performance of the proposed method (2.1), is compared with available results in the literature [4, 9, 13, 15]. Table 1 and Table 2, shows that the proposed method (PM) is more effective than the existing method proposed by Chin and Li [4], Mahesh *et al*. [9], Venkat *et al*. [15] and Srinivasarao [13] in terms of number of iterations and the number of function evaluations (NFE).

The numerical results obtained are shown in Tables 1-2.

Example 1. $f_1(x)=11x^{11}-1$, with $x_0=1$ exact root is $8.04133e-01$.

Example 2. $f_2(x)=xe^{-x}-0.1$, with $x_0=0.1$ exact root is $1.11833e-01$.

Example 3. $f_3(x)=x-e^{\sin x}+1$, with $x_0=2$ exact root is $1.69681e+00$.

Example 4. $f_4(x)=\log(x)$, with $x_0=0.5$ exact root is $1.00000e+00$.

Table 1. Comparing number of iterations and function evaluations with Chin and Li [4], Mahesh *et al.* [9] and Venkat *et al.* [15]

Example	Chin and Li [4]	NFE	Mahesh *et al.* [9]	NFE	Venkat *et al.* [15]	NFE	PM	NFE
1	8	32	6	24	7	56	8	24
2	6	24	7	28	3	24	5	15
3	11	44	6	24	5	40	6	18
4	7	28	-	-	5	40	5	15

Table 2. Comparing number of iterations and function evaluations with Srinivasarao [13]

S. No	Function	$[a,b]$	Exact root	Srinivasarao [13]	NFE	PM	NFE
1	$f(x)=x^6-x-1$	$[1,1.5]$	1.134724138	8	32	6	18
2	$f(x)=e^x-x-2$	$[1,2]$	1.146193221	8	32	6	18
3	$f(x)=8-4.5(x-\sin x)$	$[2,3]$	2.430465741	7	28	5	15
4	$f(x)=xe^{-x}-0.1$	$[0,0.1]$	0.111832561	6	24	5	15

4. Numerical Examples

In this section the proposed method is executed on some numerical examples to describe its efficiency by comparing with standard methods such as bisection, regula falsi and secant methods. All the numerical computations are carried out with an accuracy of 10^{-15} through MATLAB. These examples are used by many researchers in the literature [2, 4, 7, 12]. The numerical results obtained are shown in Tables 3-7.

Example 5. $f(x)=x^6-x-1$, with $x_0=1$ and $x_1=1.5$.

Example 6. $f(x)=e^x-x-2$, with $x_0=1$ and $x_1=2$.

Example 7. $f(x)=8-4.5(x-\sin x)$, with $x_0=2$ and $x_1=3$.

Example 8. $f(x)=xe^x-0.1$, with $x_0=0$ and $x_1=0.1$.

Table 3. Comparing number of iterations with classical methods

Example	Exact root	Bisection method		Regula falsi method		Secant method		PM	
		n	$\lvert f(x_n)\rvert$	n	$\lvert f(x_n)\rvert$	n	$\lvert f(x_n)\rvert$	n	$\lvert f(x_n)\rvert$
4	1.13472413	28	9.971623e-10	37	2.530625e-09	37	2.530628e-09	7	8.881784e-16
5	1.14619322	35	4.363132e-11	30	2.439448e-11	30	2.220446e-14	6	0.000000e+0
6	2.43046574	29	5.066743e-09	10	2.293010e-11	10	3.095728e-10	5	9.3614005e-13
7	0.09127652	29	1.804810e-10	5	3.791055e-14	5	3.710565e-14	4	6.2089465e-14

From Table 3, the results show that the proposed method is more effective in terms of number of iterations and the $|f(x_n)|$ value in the proposed method is closer to zero than the classical Bisection, regula falsi and Secant methods.

Table 4. Comparing number of iterations of the proposed method with various existing methods for $f(x) = x^6 - x - 1$ with $x_0 = 1$ and $x_1 = 1.5$

n	Bisection method	n	Regula falsi method	n	Secant method	n	PM
1	1.25	1	1.050552922	1	1.050552922	1	1.043540604
2	1.125	2	1.083627074	2	1.083627074	2	1.079152185
3	1.1875	3	1.104301085	3	1.104301085	3	1.149706584
4	1.15625	4	1.116832665	4	1.116832665	4	1.132610005
5	1.140625	5	1.124281662	5	1.124281662	5	1.134648787
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
28	1.134724138	37	1.134724138	37	1.134724138	7	1.134724138

Table 5. Comparing number of iterations of the proposed method with various existing methods for $f(x) = e^x - x - 2$, with $x_0 = 1$ and $x_1 = 2$

n	Bisection method	n	Regula falsi method	n	Secant method	n	PM
1	1.5	1	1.076746253	1	1.076746253	1	1.040343396
2	1.25	2	1.113782264	2	1.113782264	2	1.096404007
3	1.125	3	1.131195342	3	1.131195342	3	1.150334852
4	1.1875	4	1.139280803	4	1.139280803	4	1.146039271
5	1.15625	5	1.143013246	5	1.143013246	5	1.146192754
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
35	1.146193221	30	1.146193221	30	1.146193221	6	1.146193221

Table 6. Comparing number of iterations of the proposed method with various existing methods for $f(x) = 8 - 4.5(x - \sin x)$, with $x_0 = 2$ and $x_1 = 3$

n	Bisection method	n	Regula falsi method	n	Secant method	n	PM
1	2.25	1	2.427272523	1	2.427272523	1	2.384264259
2	2.375	2	2.430228705	2	2.430228705	2	2.426935578
3	2.4375	3	2.430448182	3	2.430448182	3	2.430496895
4	2.40625	4	2.430464441	4	2.430464441	4	2.430465721
5	2.421875	5	2.430465645	5	2.430465645		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	2.430465741	10	2.430465741	10	2.430465741	5	2.430465741

Table 7. Comparing number of iterations of the proposed method with various existing methods for $f(x) = xe^x - 0.1$, with $x_0 = 0$ and $x_1 = 0.1$

n	Bisection method	n	Regula falsi method	n	Secant method	n	PM
1	0.05	1	0.091269924	1	0.091269924	1	0.090469319
2	0.075	2	0.091276472	2	0.091276472	2	0.091269815
3	0.0875	3	0.091276526	3	0.091276526	3	0.091276532
4	0.09375	4	0.091276527	4	0.091276527		
5	0.090625						
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	0.091276527	5	0.091276527	5	0.091276527	4	0.091276527

5. Conclusion

In this paper, a new method is proposed to compute an approximate positive real root of a given non-linear algebraic/transcendental equation. This method is based on combination of an inverse sine series and Newton-Raphson method. The rate of convergence of the proposed method is discussed and found to be quadratic. This proposed method is implemented in MATLAB and the numerical computation results show that the proposed method is very effective in terms of reducing number of iterations with minimum error and the less number of function evaluations. To show the efficiency of the proposed method, the results are compared with available classical methods like bisection, regula falsi, secant methods. The proposed method has obvious practical utility from a real-world standpoint.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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