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The Faedo-Galerkin Method for the Relativistic Boltzmann Equation

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Abstract We prove the existence and uniqueness of solution to the relativistic Boltzmann equation locally in time. We clarify the choice of the function spaces and we establish step by step all the essential energy estimations leading to the existence theorem.

1. Introduction

In this paper we consider the relativistic Boltzmann equation which is one of the basic equations of the kinetic theory. This equation rules the dynamics of a kind of particles subject to mutual collisions, by determining their *distribution function*, which is a non-negative real-valued function of both the position and the momentum of the particles. Physically, this function is interpreted as the *probability of the presence density* of the particles in a given volume, during their collisional evolution. We consider the case of instantaneous, localized, binary and elastic collisions. Here the distribution function is determined by the Boltzmann equation through a non-linear operator called the collision operator. The operator acts only on the momentum of the particles, and describes, at any time; at each point where two particles collide with each other, the effects of the behaviour imposed by the collision to the distribution function, also taking in account the fact that the momentum of each particle is not the same, before and after the collision, only the sum of their two momenta being preserved.

Several authors studied local and global in time existence theorems of the relativistic Boltzmann equation as:

D. Bancel in [4], D. Bancel and Y. Choquet-Bruhat in [5]; R.T. Glassey and W. Strauss in [6] who obtained a global result in the case of data near to that of an equilibrium solution with non-zero density.

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More recently, N. Noutcheguеme and D. Dongho obtained in [7] a global existence theorem, *but the method used for the investigation of the local existence theorem for the relativistic Boltzmann equation, through characteristics is not quite clear*. N. Noutcheguеme; D. Dongho and E. Takou in [8]; N. Noutcheguеme and R. Ayissi in [9] have used the same method.

Now P.B. Mucha proved a local existence theorem in [3], but some hypotheses and proofs, *particularly the hypotheses and the proof concerning the theorem in appendix, which is however fundamental, are not so clear.*

The objectives of the present work in the particular case of the Bianchi type I space-time are:

- firstly to prove the local existence in time and uniqueness of solution to the relativistic Boltzmann equation; clarifying things in the method used by Mucha, explaining the choice made for the function spaces, demonstrating completely the main propositions and theorems, adding essential conditions necessary to obtain those theorems;
- secondly to give a correct method in our own case, of solving the relativistic Boltzmann equation which ignores the method of characteristics heavily used by us in [7, 8, 9] as in several other works.

The paper is organized as follows:

In section 2, we introduce the space-time and we give the unknown functions.

In section 3, we describe the relativistic Boltzmann equation.

In section 4, we introduce the function spaces and we give the energy estimations.

In section 5, we prove the main existence theorem.

2. The Back-Ground Space-Time and the Unknown Functions

Greek indexes $\alpha, \beta, \gamma, \dots$ range from 0 to 3, and Latin indexes $i, j, k, \dots$ from 1 to 3. We adopt the Einstein summation convention:

$$A^\alpha B_\alpha = \sum_\alpha A^\alpha B_\alpha .$$

We consider the collisional evolution of a kind of fast moving massive and charged particles in the time-oriented Bianchi type 1 space-time $(\mathbb{R}^4, g)$, and denote by $x^\alpha = (x^0, x^i) = (t, x^i)$ the usual coordinates in $\mathbb{R}^4$, where $x^0 = t$ represents the time and (x^i) the space; g stands for the given metric tensor of Lorentzian signature $(-,+,+,+)$ which writes:

$$g = -(dt)^2 + a^2(t)(dx^1)^2 + b^2(t)((dx^2)^2 + (dx^3)^2) \tag{1}$$

where $a > 0$, $b > 0$ are two continuously differentiable functions on $\mathbb{R}$, whose variable is denoted t.

The expression of the Levi-Civita connection ∇ associated to g, which is:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}[\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}],$$

gives directly:

$$\begin{cases} \Gamma^1_{10} = \frac{\dot{a}}{a};\ \Gamma^2_{20} = \frac{\dot{b}}{b};\ \Gamma^3_{30} = \frac{\dot{b}}{b};\ \Gamma^0_{11} = a\dot{a};\ \Gamma^0_{22} = b\dot{b} = \Gamma^0_{33} \\ \Gamma^{\lambda}_{\alpha\beta} = 0 \quad \text{otherwise}, \end{cases} \tag{2}$$

where the dot stands for the derivative with respect to t. Recall that $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$.

We require the assumption that $\frac{\dot{a}}{a}$ and $\frac{\dot{b}}{b}$ are bounded. This implies that there exists a constant $C > 0$ such that:

$$\left|\frac{\dot{a}}{a}\right| \leq C, \qquad \left|\frac{\dot{b}}{b}\right| \leq C. \tag{3}$$

As a direct consequence, we have for $t \in \mathbb{R}^+$:

$$a(t) \leq a_0 e^{Ct};\ b(t) \leq b_0 e^{Ct};\ \frac{1}{a}(t) \leq \frac{1}{a_0} e^{Ct};\ \frac{1}{b}(t) \leq \frac{1}{b_0} e^{Ct} \tag{4}$$

where $a_0 = a(0);\ b_0 = b(0)$.

The massive particles have a rest mass $m > 0$, normalized to the unity, i.e., $m = 1$. We denote by $T(\mathbb{R}^4)$ the tangent bundle of $\mathbb{R}^4$ with coordinates (x^α, p^β), where $p = (p^\beta) = (p^0, \overline{p})$ stands for the momentum of each particle and $\overline{p} = (p^i)$, $i = 1, 2, 3$. Really the charged particles move on the future sheet of the mass-shell or the mass hyperboloid $P(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $P_x(p) : g_x(p, p) = g_{\alpha\beta} p^\alpha p^\beta = -1$ or equivalently, using expression (1) of g:

$$p^0 = \sqrt{1 + a^2(p^1)^2 + b^2((p^2)^2 + (p^3)^2)}, \tag{5}$$

where the choice $p^0 > 0$ symbolizes the fact that, naturally, the particles eject towards the future.

Setting:

$$\varrho = \sqrt{\sum_{i=1}^{3} (p^i)^2} = \varrho(\overline{p}),$$

if $\varrho > 1$, the relations (4) and (5) also show that in any interval $[0, T]$, $T > 0$:

$$Ap^0 \leq \varrho \leq Bp^0, \tag{6}$$

where $A = A(T) > 0$, $B = B(T) > 0$ are constants.

The invariant volume element in $P_x(p)$ reads:

$$\omega_p = |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p^0},$$

where

$$|g| = |\det g_{\alpha\beta}|.$$

We denote by f the distribution function which measures the probability of the presence of particles in the plasma. f is a non-negative unknown real-valued function of both the position (x^α) and the 4-momentum of the particles $p = (p^\alpha)$, so:

$$f : T(\mathbb{R}^4) \approx \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^+, \quad (x^\alpha, p^\alpha) \longmapsto f(x^\alpha, p^\alpha) \in \mathbb{R}^+.$$

We define a scalar product on $\mathbb{R}^3$ by setting for $p = (p^0, \overline{p}) = (p^0, p^i)$ and $q = (q^0, \overline{q}) = (q^0, q^i)$:

$$\overline{p} \cdot \overline{q} = a^2 p^1 q^1 + b^2 (p^2 q^2 + p^3 q^3). \tag{7}$$

In this paper we consider the homogeneous case for which f depends only on the time $x^0 = t$ and $\overline{p}$. According to the Laplace law, the fast moving and charged particles create an unknown electromagnetic field F which is a 2-closed antisymmetric form and locally writes:

$$F = F_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

So in the homogeneous case we consider:

$$F_{\alpha\beta} : \mathbb{R} \to \mathbb{R}, \quad t \longmapsto F_{\alpha\beta}(t) \in \mathbb{R}.$$

In the presence of the electromagnetic field F, the trajectories

$$s \longmapsto (x^\alpha(s), p^\alpha(s))$$

of the charged particles are no longer the geodesics of space-time $(\mathbb{R}^4, g)$, but the solutions of the differential system:

$$\frac{dx^\alpha}{ds} = p^\alpha; \quad \frac{dp^\alpha}{ds} = P^\alpha \tag{8}$$

where:

$$P^\alpha = P(F, f) = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + e p^\beta F^\alpha_\beta, \tag{9}$$

where $e = e(t)$ denotes the charge density of particles.

Notice that the differential system (8) shows that the vectors field $X(F)$ defined locally by:

$$X(F) = (p^\alpha, P^\alpha(F)) \tag{10}$$

where P^α is given by (9), is tangent to the trajectories.

The charged particles also create a current $J = (J^\beta)$, $\beta = 0, 1, 2, 3$, called the Maxwell current we take in the form:

$$J^\beta = \int_{\mathbb{R}^3} p^\beta f \omega_p - e u^\beta \tag{11}$$

in which $u = (u^\beta)$ is a unit future pointing time-like vector, tangent to the time axis at any point, which means that $u^0 = 1$, $u^i = u_i = 0$, $i = 1, 2, 3$. The particles are then supposed to be spatially at rest.

The electromagnetic field $F = (F^{0i}, F_{ij})$, where F^{0i} and F_{ij} stand for the electric and magnetic parts respectively, is subject to the Maxwell equations which write:

$$\nabla_\alpha F^{\alpha\beta} = J^\beta, \tag{12}$$

and called first group of the Maxwell equations to which we add the second group formed of Bianchi identities:

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0. \tag{13}$$

Now the well-known identity $\nabla_\alpha \nabla_\beta F^{\alpha\beta} = 0$ imposes, given (12) that the current j^β is always subject to the conservation law:

$$\nabla_\beta J^\beta = 0. \tag{14}$$

However using $\beta = 0$ in (12), we obtain:

$$\nabla_\alpha F^{\alpha 0} = \partial_\alpha F^{\alpha 0} + \Gamma^\alpha_{\alpha\lambda} F^{\alpha 0} + \Gamma^0_{\alpha\lambda} F^{\alpha\lambda} = 0,$$

since $F = F(t)$, $F^{\alpha\lambda} = -F^{\lambda\alpha}$, and by (2) $\Gamma^\alpha_{\alpha i} = 0$.
So (12) implies that:

$$J^0 = 0. \tag{15}$$

By (15), the expression (11) of J^β in which we set $\beta = 0$ then allows to compute e and gives, since $u^0 = 1$:

$$e(t) = \int_{\mathbb{R}^3} a b^2 f \, d\overline{p}, \tag{16}$$

which shows that f determines e.

3. The Boltzmann Equation in f

The relativistic Boltzmann equation in f, for charged particles in the Bianchi type 1 space-time can be written:

$$L_X f = Q(f, f) \tag{17}$$

where L_X is the Lie derivative of f with respect to the vectors field $X(F)$ defined by (10) and $Q(f, f)$ the collision operator we now introduce.

According to Lichnerowicz and Chernikov, we consider a scheme, in which, at a given position (t, x^i), only two particles collide each other, without destroying each one, the collision affecting only the momentum of each particle, which changes after shock, only the sum of the two momenta being preserved. If p, q stand for the two momenta before the shock, and p', q' for the two momenta after the shock, then we have:

$$p + q = p' + q'.$$

The collision operator Q is then defined, using functions f and g on $\mathbb{R}^3$ and the above notations, by:

$$Q(f,g) = Q^+(f,g) - Q^-(f,g) \tag{18}$$

where:

$$Q^+(f,g) = \int_{\mathbb{R}^3} \omega_{\overline{q}} \int_{S^2} f(\overline{p}')g(\overline{q}')\sigma(t,\overline{p},\overline{q},\overline{p}',\overline{q}',\Omega)d\Omega, \tag{19}$$

$$Q^-(f,g) = \int_{\mathbb{R}^3} \omega_{\overline{q}} \int_{S^2} f(\overline{p})g(\overline{q})\sigma(t,\overline{p},\overline{q},\overline{p}',\overline{q}',\Omega)d\Omega \tag{20}$$

whose elements we now introduce step by step, specifying properties and hypotheses we adopt:

- S^2 is the unit sphere of $\mathbb{R}^3$, whose area element is denoted $d\Omega$;
- σ is a non-negative continuous real-valued function of all its arguments, called the *collision kernel* or the *cross-section* of the collisions, on which we require the boundedness and Lipschitz continuity assumptions, in which $C_1 > 0$ is a constant:

$$\begin{cases} 0 \leq \sigma(t,\overline{p},\overline{q},\Omega) \leq C_1 \\ |\sigma(t,\overline{p}_1,\overline{q},\overline{p}',\overline{q}',\Omega) - \sigma(t,\overline{p}_2,\overline{q},\overline{p}',\overline{q}',\Omega)| \leq C_1\|\overline{p}_1 - \overline{p}_2\| \end{cases} \tag{21}$$

  where $\|\overline{p}\| = \left(\sum_{i=1}^{3}(p^i)^2\right)^{\frac{1}{2}} = \varrho$ is the norm in $\mathbb{R}^3$.
- The conservation law $p + q = p' + q'$ splits into:

$$p^0 + q^0 = p'^0 + q'^0, \tag{22}$$

$$\overline{p} + \overline{q} = \overline{p}' + \overline{q}'. \tag{23}$$

  (22) expresses, using (5), the conservation of the quantity:

$$\tilde{e} = \sqrt{1 + a^2(p^1)^2 + b^2((p^2)^2 + (p^3)^2)} + \sqrt{1 + a^2(q^1)^2 + b^2((q^2)^2 + (q^3)^2)} \tag{24}$$

  called the elementary energy of the unit rest mass particles; we can interpret (23) by setting, following R.T. Glassey in [6]:

$$\begin{cases} \overline{p}' = \overline{p} + c(\overline{p},\overline{q},\Omega)\Omega, \\ \overline{q}' = \overline{q} - c(\overline{p},\overline{q},\Omega)\Omega \end{cases} \qquad (\Omega \in S^2) \tag{25}$$

  in which $c(\overline{p},\overline{q},\Omega)$ is a real-valued function. We prove, by a direct calculation, using (5) to express p'^0, q'^0 in terms of $\overline{p}'$, $\overline{q}'$ and next (25) to express $\overline{p}'$, $\overline{q}'$ in terms of $\overline{p}$, $\overline{q}$, that equation (22) leads to a quadratic equation in c, which solves to give the only non trivial solution:

$$c(\overline{p},\overline{q},\Omega) = \frac{2p^0q^0\tilde{e}\Omega \cdot (\hat{\overline{q}} - \hat{\overline{p}})}{(\tilde{e})^2 - [\Omega \cdot (\overline{p} + \overline{q})]^2} \tag{26}$$

in which $\hat{\overline{p}} = \dfrac{\overline{p}}{p^0}$, $\tilde{e}$ is given by (24) and the dot $(\cdot)$ is the scalar product defined by (7).

It then appears, using (25) that the functions in the integrals (19) and (20) depend only on $\overline{p}$, $\overline{q}$, Ω, and that these integrals with respect to $\overline{q}$ and Ω give functions $Q^+(f,g)$ and $Q^-(f,g)$ of the single variable $\overline{p}$.

Now using the usual properties of the determinants, the jacobian of the change of variables $(\overline{p},\overline{q}) \mapsto (\overline{p}',\overline{q}')$ defined by (25) is computed to be:

$$\frac{\partial(p',q')}{\partial(p,q)} = -\frac{p'^0 q'^0}{p^0 q^0}. \tag{27}$$

Since $f = f(t,\overline{p})$, using (5), the Boltzmann equation (17) takes the form:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0}Q(f,f). \tag{28}$$

Next, let us introduce the subgroup G of $\mathcal{O}_3$ defined by:

$$G = \left\{ N_{\varepsilon,\theta} \in \mathcal{O}_3,\ N_{\varepsilon,\theta} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \varepsilon, \theta \in \mathbb{R},\ \varepsilon^2 = 1 \right\}. \tag{29}$$

We require in all what follows that the initial datum $f_0 = f(0;\cdot)$ *of the distribution function* f *is invariant under* G, or equivalently:

$$f_0(N\overline{p}) = f_0(\overline{p}); \quad \text{for all } N \in G, \text{ for all } \overline{p} \in \mathbb{R}^3. \tag{30}$$

It is prove in [7] that if f_0 is invariant under G then so will be the solution f of the Boltzmann equation satisfying $f_0(\overline{p}) = f(0,\overline{p})$.

Using (12), (13), (30), we proved in [9] that

$$F^{0i} = \frac{a_0 b_0^2}{a b^2} F^{0i}(0), \quad F_{ij} = F_{ij}(0). \tag{31}$$

4. Function Spaces and Energy Estimations

We define now the function spaces in which we are searching the solution to the Boltzmann equation.

We also establish some useful energy estimations.

Definition 1. $L_2^1(0,T,\mathbb{R}^3)$
We define

$$L_2^1(\mathbb{R}^3) = \{g : \mathbb{R}^3 \to \mathbb{R}, (1+\varrho)g \in L^1(\mathbb{R}^3)\}$$

where $\varrho = \|\overline{p}\|$, $\overline{p} = (p^i) \in \mathbb{R}^3$.

Let $T > 0$ be given, then

$$L_2^1(0,T,\mathbb{R}^3) = \{h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}, h(t,\cdot) \in L_2^1(\mathbb{R}^3)\}$$

$L_2^1(0,T,\mathbb{R}^3)$ is a Banach space endowed with the norm:

$$|||h|||_{L_2^1(0,T,\mathbb{R}^3)} = \sup_{t\in[0,T]} \|(1+\varrho)h(t,\cdot)\|_{L^1(\mathbb{R}^3)}.$$

Definition 2. $H_s^m(0,T,\mathbb{R}^3)$
Let $T>0$, $m\in\mathbb{N}$, $s\in\mathbb{R}$ be given.

We define $H_s^m(\mathbb{R}^3)$ as

$$H_s^m(\mathbb{R}^3) = \{h:\mathbb{R}^3\to\mathbb{R}, (1+\varrho)^{s+|\beta|}\partial_{\overline{p}}^{\beta} h \in L^2(\mathbb{R}^3), |\beta|\le m\}.$$

$H_s^m(\mathbb{R}^3)$ will be endowed with the norm

$$\|h\|_{H_s^m(\mathbb{R}^3)} = \max_{0\le|\beta|\le m} \|(1+\varrho)^{s+|\beta|}\partial_{\overline{p}}^{\beta} h\|_{L^2(\mathbb{R}^3)}.$$

$\overline{H}_s^m(\mathbb{R}^3)$ will be the completion of $H_s^m(\mathbb{R}^3)$ in the norm $\|\cdot\|_{H_s^m(\mathbb{R}^3)}$.
A function $y\in H_s^m(0,T,\mathbb{R}^3)$ if for all $t\in[0,T]$ and all $|\beta|\le m$

$$(1+\varrho)^{s+|\beta|}\partial_{\overline{p}}^{\beta} y(t,\cdot)\in L^2(\mathbb{R}^3)$$

and

$$t\mapsto (1+\varrho)^{s+|\beta|}\partial_{\overline{p}}^{\beta} y(t,\cdot)$$

is a continuous function from $[0,T]$ to $L^2(\mathbb{R}^3)$.

Endowed with the norm

$$|||y|||_{H_s^m(0,T,\mathbb{R}^3)} = \max_{0\le|\beta|\le m}\ \sup_{t\in[0,T]} \|(1+\varrho)^{s+|\beta|}\partial_{\overline{p}}^{\beta} y(t,\cdot)\|_{L^2(\mathbb{R}^3)},$$

$H_s^m(0,T,\mathbb{R}^3)$ is a Banach space.

For $r>0$ be given, we define

$$H_{s,r}^m = \{y\in H_s^m(0,T,\mathbb{R}^3),\ y\ge 0,\ p\cdot p,\ \|y\|_{H_s^m(0,T,\mathbb{R}^3)}\le r\}.$$

Endowed with the induced distance by the norm $|||\cdot|||_{H_s^m(0,T,\mathbb{R}^3)}$, $H_{s,r}^m$ is a complete metric subspace of $H_s^m(0,T,\mathbb{R}^3)$.

Remark 1. If $m=0$, then $y\in H_s^m(\mathbb{R}^3)\Longleftrightarrow(1+\varrho)^s y\in L^2(\mathbb{R}^3)$, so $H_s^0(\mathbb{R}^3)$ will be denoted $L_s^2(\mathbb{R}^3)$.

Remark 2. The reasons for the choice of the function space $H_d^m(\mathbb{R}^3)$ for $m=3$ and $d>\dfrac{5}{2}$.

The objective of the present work being the existence of solution to the Boltzmann equation (28), we are searching a function $f=f(t,\overline{p})$ which is continuously differentiable, in particular we can search $f=f(t,\cdot)$ belonging to the space $\mathscr{C}_b^1(\mathbb{R}^3)$.

We want to use the Faedo-Galerkin method which is applied for separable Hilbert spaces. That is the case for the Sobolev spaces $H^m(\mathbb{R}^3)$, $m\in\mathbb{N}$.

We need then to find an integer m such that

$$H^m(\mathbb{R}^3) \hookrightarrow \mathscr{C}_b^1(\mathbb{R}^3).$$

But we know by the Sobolev theorems that

$$W_p^m(\mathbb{R}^n) \hookrightarrow \mathscr{E}^k(\mathbb{R}^n), \quad m > k + \frac{n}{p}.$$

Since in our case we have $n = 3$, $p = 2$, $k = 1 (W_2^m = H^m)$, we must choose m such that

$$m > 1 + \frac{3}{2} = \frac{5}{2}.$$

The smallest integer m satisfying $m > \dfrac{5}{2}$ is naturally $m = 3$.
Consequently we have

$$H_d^3(\mathbb{R}^3) \hookrightarrow H^3(\mathbb{R}^3) \hookrightarrow \mathscr{C}_b^1(\mathbb{R}^3).$$

Furthermore if

$$d > \frac{5}{2}$$

then

$$H_d^m(\mathbb{R}^3) \hookrightarrow L_d^2(\mathbb{R}^3) \hookrightarrow L_2^1(\mathbb{R}^3). \tag{32}$$

It then results that

$$H_d^m(\mathbb{R}^3) \cap L_2^1(\mathbb{R}^3) = H_d^m(\mathbb{R}^3).$$

In fact if $f \in H_d^m(\mathbb{R}^3)$, then

$$\begin{aligned} \|f\|_{L_2^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} (1+\varrho)|f| d\overline{p} \\ &= \int_{\mathbb{R}^3} (1+\varrho)^{1-d+d} |f| d\overline{p} \\ &= \int_{\mathbb{R}^3} (1+\varrho)^{1-d} (1+\varrho)^d |f| d\overline{p}. \end{aligned}$$

So by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|f\|_{L_2^1(\mathbb{R}^3)} &\leq \left(\int_{\mathbb{R}^3} (1+\varrho)^{2-2d} d\overline{p} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (1+\varrho)^{2d} |f|^2 d\overline{p} \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^3} (1+\varrho)^{2-2d} d\overline{p} \right)^{\frac{1}{2}} \|f\|_{L_d^2(\mathbb{R}^3)}. \end{aligned}$$

But using polar coordinates, the integral $\int_{\mathbb{R}^3} (1+\varrho)^{2-2d} d\overline{p}$ converges if and only if it is the case for the integral $\int_0^{+\infty} (1+r)^{2-2d} r^2 dr$.

Since $\int_0^{+\infty} (1+r)^{2-2d} r^2 dr \underset{r \to +\infty}{\sim} \int_0^{+\infty} r^{4-2d} dr$, using also the fact that $\int_0^{+\infty} r^{4-2d} dr$ is convergent if in the primitive containing r^{5-2d}.

We have $5-2d<0$, we conclude that if

$$d>\frac{5}{2}$$

then

$$\|f\|_{L^1_2(\mathbb{R}^3)}\leq C\|f\|_{L^2_d(\mathbb{R}^3)}\leq C\|f\|_{H^3_d(\mathbb{R}^3)}.$$

We can now state the following result which in what follows will be fundamental

Proposition 1. *Let $d>\frac{5}{2}$, $\|\sigma\|_{L^1(\mathbb{R}^3\times S^2)}\in L^\infty(\mathbb{R}^3)$ and $(\partial^\beta\sigma)(1+|\overline{p}|)^{|\beta|-1}\in L^\infty(\mathbb{R}^3\times\mathbb{R}^3\times S^2)$, $|\beta|\leq 3$ be given.*
If $f,g\in H^3_d(\mathbb{R}^3)$ then $\frac{1}{p^0}Q(f,g)\in H^3_d(\mathbb{R}^3)$ and we have

$$\left\|\frac{1}{p^0}Q(f,g)\right\|_{H^3_d(\mathbb{R}^3)}\leq C\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)} \tag{33}$$

where $C=C(T)>0$.

Before the proof of Proposition 1, the following lemma will be helpful.

Lemma 1. *There exists a real number $T>0$ such that*

$$(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2>2. \tag{34}$$

Furthermore we have

$$\begin{cases}(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2\geq(1-a^2-b^2)(p^0)^2,\\ (\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2\geq(1-a^2-b^2)(q^0)^2,\\ (\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2\geq(1-a^2-b^2)p^0q^0\end{cases} \tag{35}$$

and the function $(\overline{p},\overline{q},\Omega)\mapsto D^\beta_{\overline{p}}c(\overline{p},\overline{q},\Omega)$, $1\leq|\beta|\leq 3$ is bounded.

Proof. We have $\Omega(\overline{p}+\overline{q})\leq|\Omega|\times|(\overline{p}+\overline{q})|$.
It follows after computation that

$$(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2\geq 2+(1-a^2-b^2)(|\overline{p}|^2+|\overline{q}|^2+2|\overline{p}||\overline{q}|).$$

Accordingly $1-a^2-b^2\geq 0\Rightarrow(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2\geq 2$.

Using (3), the inequality $1-a^2-b^2\geq 0$ then writes

$$(a_0^2+b_0^2)\exp(2CT)\leq 1.$$

It remains to prove (35). We have

$$(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2\geq 1+(1-a^2-b^2)(|\overline{p}|^2+|\overline{q}|^2+2|\overline{p}||\overline{q}|).$$

Using the above inequalities and $1-a^2-b^2\leq 1$ gives immediately (35).

Lastly we have to show that $(\overline{p},\overline{q},\Omega)\mapsto D^\beta_{\overline{p}}c(\overline{p},\overline{q},\Omega)$, $|\beta|\leq 3$ is bounded.

- If $|\beta|=1$, we must evaluate $\frac{\partial c(\overline{p},\overline{q},\Omega)}{\partial p^i}$, $i=1,2,3$.

Setting $c(\overline{p},\overline{q},\Omega)=\frac{2p^0q^0\tilde{e}\Omega\cdot(\hat{\overline{q}}-\hat{\overline{p}})}{(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2}=\frac{A}{B}$,

$$A=2p^0q^0\tilde{e}\Omega\cdot(\hat{\overline{q}}-\hat{\overline{p}}),$$

$$B=(\tilde{e})^2-[\Omega\cdot(\overline{p}+\overline{q})]^2,$$

we have

$$\frac{\partial c(\overline{p},\overline{q},\Omega)}{\partial p^i}=\frac{B\frac{\partial A}{\partial p^i}-A\frac{\partial B}{\partial p^i}}{B^2}=\frac{\frac{\partial A}{\partial p^i}}{B}-\frac{A\frac{\partial B}{\partial p^i}}{B^2}.$$

We obtain after computations

$$\begin{cases}|\frac{\partial A}{\partial p^i}|\leq C(T)[p^0q^0+(q^0)^2]\\ |A\frac{\partial B}{\partial p^i}|\leq C(T)[(p^0q^0)^2+p^0(q^0)^3+q^0(p^0)^3].\end{cases}$$

Combining the above inequalities with (35) leads clearly to $|\frac{\partial c(\overline{p},\overline{q},\Omega)}{\partial p^i}|\leq C(T)$.

- For $\left|\beta\right|=2$, we have

$$\partial^2_{p^ip^j}c(\overline{p},\overline{q},\Omega)=\frac{\partial^2_{p^ip^j}A}{B}-\frac{\partial_{p^i}A\partial_{p^j}B}{B^2}-\frac{\partial_{p^j}A\partial_{p^i}B}{B^2}-\frac{A\partial^2_{p^ip^j}B}{B^2}+\frac{2A\partial^2_{p^ip^j}B}{B^3}.$$

Since

$$\begin{cases}|A|\leq C(T)[q^0(p^0)^2+p^0(q^0)^2],|\partial_{p^i}B|\leq C(T)[p^0+q^0]\\ |\partial^2_{p^ip^j}A|\leq C(T)[p^0q^0+(q^0)^2],|\partial^2_{p^ip^j}B|\leq C(T)[p^0+q^0],\end{cases}$$

we conclude using (35) that $|\partial^2_{p^ip^j}c(\overline{p},\overline{q},\Omega)|\leq C(T)$.

- For $|\beta|=3$, we prove by similar calculations that

$$|\partial^3_{p^ip^jp^k}c(\overline{p},\overline{q},\Omega)|\leq C(T),\quad i,j,k=1,2,3.$$

This ends the proof of Lemma 1. □

Proof of Proposition 1. Let $f,g\in H^3_d(\mathbb{R}^3)$ be given. Let us show that for all $\beta\in\mathbb{N}^3$, $|\beta|\leq 3$,

$$\|(1+\varrho)^{d+|\beta|}\partial^\beta(\frac{1}{p^0}Q^+(f,g))\|_{L^2(\mathbb{R}^3)}\leq C\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)}.$$

- For $|\beta|=0$, we have

$$\begin{aligned}&\left\|(1+\varrho)^d\frac{1}{p^0}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)}\\ &=\int_{\mathbb{R}^3}d\overline{p}\left[\frac{1}{p^0}(1+\varrho)^d|g|^{\frac{1}{2}}\int_{\mathbb{R}^3}d\overline{q}\frac{1}{q^0}\int_{S^2}d\Omega|f(\overline{p}')||g(\overline{q}')|\sigma\right]^2\\ &\leq C(T)\int_{\mathbb{R}^3}d\overline{p}\frac{1}{(p^0)^2}(1+\varrho)^{2d}\left[\int_{\mathbb{R}^3}d\overline{q}\frac{1}{q^0}\int_{S^2}d\Omega|f(\overline{p}')||g(\overline{q}')|\sigma\right]^2\\ &=C(T)\int_{\mathbb{R}^3}d\overline{p}\frac{1}{(p^0)^2}(1+\varrho)^{2d}\left[\iint_{\mathbb{R}^3\times S^2}d\overline{q}d\Omega\frac{1}{q^0}|f(\overline{p}')||g(\overline{q}')|\sigma^{\frac{1}{2}}\sigma^{\frac{1}{2}}\right]^2.\end{aligned}\tag{36}$$

Applying now the Schwarz inequality we obtain:

$$\left\|(1+\varrho)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T) \int_{\mathbb{R}^3} d\overline{p} \frac{1}{(p^0)^2}(1+\varrho)^{2d} \times \left[\iint_{\mathbb{R}^3\times S^2} d\overline{q} d\Omega \frac{|f(\overline{p}')|^2 |g(\overline{q}')|^2}{(q^0)^2}\right] \times \left[\iint_{\mathbb{R}^3\times S^2} d\overline{q} d\Omega \sigma\right]$$

Given that $(\frac{1}{p^0})^2 \leq \frac{1}{p^0}$, $\iint_{\mathbb{R}^3\times S^2} d\overline{q} d\Omega\sigma = \|\sigma\|_{L^1(\mathbb{R}^3\times S^2)}$, we find:

$$\left\|(1+\varrho)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T) \int_{\mathbb{R}^3} d\overline{p} \frac{1}{p^0} \|\sigma\|_{L^1(\mathbb{R}^3\times S^2)} (1+\varrho)^{2d} \times \left[\int_{\mathbb{R}^3} d\overline{q} \frac{1}{q^0} \int_{S^2} |f(\overline{p}')|^2 |g(\overline{q}')|^2 \sigma d\Omega\right].$$

By hypothesis $\|\sigma\|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty(\mathbb{R}^3)$, by (21), σ is bounded, thus:

$$\left\|(1+\varrho)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T) \int_{\mathbb{R}^3} d\overline{p} \frac{1}{p^0} (1+\varrho)^{2d} \times \left[\int_{\mathbb{R}^3} d\overline{q} \frac{1}{q^0} \int_{S^2} |f(\overline{p}')|^2 |g(\overline{q}')|^2 d\Omega\right]. \tag{37}$$

Since

$$(1+\varrho)^2 = \left(1+\sqrt{\frac{1}{a^2}(ap^1)^2 + \frac{1}{b^2}(bp^2)^2 + \frac{1}{b^2}(bp^3)^2}\right)^2 \leq 2\left(1+\frac{1}{a^2}+\frac{1}{b^2}\right)(a^2(p^1)^2 + b^2(p)^2 + b^2(p^3)^2) \leq C(T)(p^0)^2$$

it follows, invoking (22) that

$$(1+\varrho) \leq C(T)p^0 \leq C(T)(p^0+q^0) = C(T)(p^{'0}+q'^0).$$

Now

$$p'^0 = \sqrt{1+a^2(p'^1)^2 + b^2((p'^2)^2 + (p'^3)^2)} \leq C(T)(1+|\overline{p'}|),$$

so

$$(1+\varrho) \leq C(T)(1+|\overline{p'}|)(1+|\overline{q'}|).$$

Using the above inequality, (37) yields:

$$\left\|(1+\varrho)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T) \int_{S^2} d\Omega \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{d\overline{p} d\overline{q}}{p^0 q^0} (1+|\overline{p'}|)^{2d} |f(\overline{p}')|^2 (1+|\overline{q'}|)^{2d} |g(\overline{q}')|^2.$$

However by (27) we have $\dfrac{\partial(p',q')}{\partial(p,q)} = -\dfrac{p'^0 q'^0}{p^0 q^0}$, and since $\dfrac{1}{p'^0}, \dfrac{1}{q'^0} < 1$, it follows that

$$\left\|(1+\varrho)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\int_{S^2} d\Omega \int_{\mathbb{R}^3} d\overline{p'}(1+|\overline{p'}|)^{2d}|f(\overline{p}')|^2 \int_{\mathbb{R}^3} d\overline{q'}(1+|\overline{q'}|)^{2d}|g(\overline{p}')|^2.$$

Consequently

$$\left\|(1+\varrho)^d \frac{1}{p^0} Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^2_d(\mathbb{R}^3)}\|g\|_{L^2_d(\mathbb{R}^3)} \leq C\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)} \tag{38}$$

where $C = 4\pi C(T)$.

• For $|\beta| = 1$, it suffices to estimate $\|(1+\varrho)^{d+1}\partial_{p^i}(\frac{1}{p^0}Q^+(f,g))\|_{L^2(\mathbb{R}^3)}$.
We have

$$(1+\varrho)^{d+1}\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right) = (1+\varrho)^{d+1}\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right) - \frac{g_{ii}p^i}{(p^0)^2}(1+\varrho)^d\frac{1}{p^0}Q^+(f,g).$$

So

$$\left\|(1+\varrho)^{d+1}\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)} \leq \left\|(1+\varrho)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)} + C(T)\left\|(1+\varrho)^d\frac{1}{p^0}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)}.$$

Now

$$\left\|(1+\varrho)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} d\overline{p}\left[\frac{1}{p^0}(1+\varrho)^{d+1}|g|^{\frac{1}{2}}\int_{\mathbb{R}^3} d\overline{q}\frac{1}{q^0}\int_{S^2} d\Omega|\partial_{p^i}(f(t,\overline{p}')g(t,\overline{q}')\sigma)|\right]^2.$$

But

$$|\partial_{p^i}(f(t,\overline{p}')g(t,\overline{q}')\sigma)| \leq |\partial_{p^i}p'^i\partial_{p'^i}f(\overline{p}')g(\overline{q}')\sigma| + |\partial_{p^i}q'^i\partial_{q'^i}gf(\overline{p}')(\overline{q}')\sigma| + |f(t,\overline{p}')g(t,\overline{q}')\partial_{p^i}\sigma|,$$

so it is sufficient to estimate each of the three quantities

$$u = \int_{\mathbb{R}^3} d\overline{p}\left[\frac{1}{p^0}(1+\varrho)^{d+1}\int_{\mathbb{R}^3} d\overline{q}\frac{1}{q^0}\int_{S^2} d\Omega|\partial_{p^i}p'^i\partial_{p'^i}f(\overline{p}')g(\overline{q}')\sigma|\right]^2,$$

$$v = \int_{\mathbb{R}^3} d\overline{p}\left[\frac{1}{p^0}(1+\varrho)^{d+1}\int_{\mathbb{R}^3} d\overline{q}\frac{1}{q^0}\int_{S^2} d\Omega|\partial_{p^i}q'^i f(\overline{p}')\partial_{q'^i}g(\overline{q}')\sigma|\right]^2,$$

$$w = \int_{\mathbb{R}^3} d\overline{p}\left[\frac{1}{p^0}(1+\varrho)^{d+1}\int_{\mathbb{R}^3} d\overline{q}\frac{1}{q^0}\int_{S^2} d\Omega|f(\overline{p}')g(\overline{q}')\partial_{p^i}\sigma|\right]^2.$$

Estimation of u. We have using the Cauchy-Schwarz inequality

$$u \leq C(T) \int_{\mathbb{R}^3} d\overline{p} \frac{1}{(p^0)^2} (1+\varrho)^{2d+2} \left\{ \left(\iint_{\mathbb{R}^3 \times S^2} \sigma d\overline{q} d\Omega \right) \times \left(\iint_{\mathbb{R}^3 \times S^2} \frac{|\partial_{p'^i} f(\overline{p}')|^2 |g(\overline{q}')|^2}{q^0} |\partial_{p^i} p'^i|^2 \sigma d\overline{q} d\Omega \right) \right\}.$$

By hypothesis σ, $(\overline{p}, \overline{q}, \Omega) \mapsto \partial_{p^i} \overline{p}'$ are bounded.

In addition $\iint_{\mathbb{R}^3 \times S^2} \sigma d\overline{q} d\Omega = \|\sigma\|_{L^1(\mathbb{R}^3 \times S2)} \in L^\infty(\mathbb{R}^3)$, so

$$u \leq C(T) \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (1+\varrho)^{2d+2} |\partial_{p'^i} f(\overline{p}')|^2 |g(\overline{q}')|^2 \frac{d\overline{p} d\overline{q} d\Omega}{(p^0 q^0)^2}.$$

Since by (27) we have $d\overline{p} d\overline{q} = \frac{p^0 q^0 d\overline{p'} d\overline{q'}}{p'^0 q'^0}$, the above inequality becomes:

$$u \leq C(T) \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (1+\varrho)^{2d} |\partial_{p'^i} f(\overline{p}')|^2 |g(\overline{q}')|^2 (1+\varrho)^2 \frac{d\overline{p'} d\overline{q'} d\Omega}{p^0 q^0 p'^0 q'^0}.$$

Now $1+\varrho \leq C(T) p^0$,

$$1+\varrho \leq C(T)(1+|\overline{p'}|)(1+|\overline{q'}|) \leq C(T) p'^0 q'^0,$$

so

$$\frac{(1+\varrho)^2}{p^0 q^0 p'^0 q'^0} \leq C(T)$$

and the inequality yields

$$u \leq C(T) \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (1+\varrho)^{2d} |\partial_{p'^i} f(\overline{p}')|^2 |g(\overline{q}')|^2 d\overline{p'} d\overline{q'} d\Omega.$$

Using once more $1+\varrho \leq C(T)(1+|\overline{p'}|)(1+|\overline{q'}|)$, we find

$$u \leq C(T) \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} (1+|\overline{p'}|)^{2d+2} |\partial_{p'^i} f(\overline{p}')|^2 (1+|\overline{q'}|)^{2d} |g(\overline{q}')|^2 d\overline{p'} d\overline{q'} d\Omega$$
$$\leq 4\pi C(T) \|\partial_{\overline{p}} f\|^2_{L^2_d(\mathbb{R}^3)} \|g\|^2_{L^2_d(\mathbb{R}^3)} \leq C(T) \|f\|^2_{H^3_d(\mathbb{R}^3)} \|g\|^2_{H^3_d(\mathbb{R}^3)}.$$

Estimation of v. Similarly, $v \leq C(T) \|f\|^2_{H^3_d(\mathbb{R}^3)} \|g\|^2_{H^3_d(\mathbb{R}^3)}$.

Estimation of w. We have

$$w = \int_{\mathbb{R}^3} d\overline{p} \left(\frac{1}{p^0} \right)^2 (1+\varrho)^{2d+2} \left[\int_{S^2} \int_{\mathbb{R}^3} \frac{|f(\overline{p}') g(\overline{q}')| \, |\partial_{p^i} \sigma|^{\frac{1}{2}}}{q^0} |\partial_{p^i} \sigma|^{\frac{1}{2}} d\overline{q} d\Omega \right]^2.$$

By the Cauchy-Schwarz inequality, we obtain

$$w \leq \int_{\mathbb{R}^3} d\overline{p} \left(\frac{1}{p^0} \right)^2 (1+\varrho)^{2d+2} \left\{ \left(\iint_{\mathbb{R}^3 \times S^2} |\partial_{p^i} \sigma| d\overline{q} d\Omega \right) \times \left(\iint_{\mathbb{R}^3 \times S^2} \frac{|f(\overline{p}')|^2 |g(\overline{q}')|^2 |\partial_{p^i} \sigma|}{(q^0)^2} d\overline{q} d\Omega \right) \right\}.$$

Now invoking the hypothesis $\partial_{p^i}\sigma \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3\times S^2)$, we find

$$w \le \int_{\mathbb{S}^2}\left[\int_{\mathbb{R}^3\times S^2}\frac{|f(\overline{p}')|^2|g(\overline{q}')|^2(1+\varrho)^{2d+2}}{(p^0q^0)^2}d\overline{p}d\overline{q}\right]d\Omega.$$

But $(1+\varrho)^2 \le C(T)p^0q^0$ implies that

$$w \le \int_{\mathbb{S}^2}\left[\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f(\overline{p}')|^2|g(\overline{q}')|^2(1+\varrho)^{2d}}{p^0q^0}d\overline{p}d\overline{q}\right]d\Omega.$$

Invoking $d\overline{p}d\overline{q} = \frac{p^0q^0d\overline{p'}d\overline{q'}}{p'^0q'^0}$, $1+\varrho \le C(T)(1+|\overline{p'}|)(1+|\overline{q'}|)$, we find

$$w \le C(T)\int_{S^2}\left(\int_{\mathbb{R}^3}[(1+|\overline{p'}|)^{2d}|f(\overline{p}')|^2]d\overline{p'}\int_{\mathbb{R}^3}[(1+|\overline{q'}|)^{2d}|g(\overline{q}')|^2]d\overline{q'}\right)d\Omega.$$

Consequently

$$w \le C\|f\|^2_{L^2_d(\mathbb{R}^3)}\|g\|^2_{L^2_d(\mathbb{R}^3)} \le C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)}.$$

Combining the preceding results, we conclude that

$$\left\|(1+\varrho)^{d+1}\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)} \le C\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)}, \quad i=1,2,3,$$

- For $|\beta|=2$, we have by computations

$$\begin{aligned}(1+\varrho)^{d+2}\partial^2_{p^jp^i}\left(\frac{1}{p^0}Q^+(f,g)\right) &= \frac{-g_{jj}p^j(1+\varrho)}{(p^0)^2}\frac{(1+\varrho)^{d+1}\partial_{p^i}Q^+(f,g)}{p^0}\\ &\quad-\frac{p^i(1+\varrho)}{(p^0)^2}\frac{(1+\varrho)^{d+1}\partial_{p^j}Q^+(f,g)}{p^0}\\ &\quad-\frac{g_{ii}[\delta^j_i(p^0)^2-3p^ip^jg_{jj}](1+\varrho)^2}{(p^0)^4}\\ &\quad\times\frac{(1+\varrho)^dQ^+(f,g)}{p^0}+\frac{(1+\varrho)^{d+2}\partial^2_{p^jp^i}Q^+(f,g)}{p^0}.\end{aligned}$$

Using the estimations obtained for the cases $|\beta|=0$, $|\beta|=1$, we find

$$\begin{aligned}&\left\|(1+\varrho)^{d+2}\partial^2_{p^jp^i}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)}\\ &\quad\le C(T)\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)}+\left\|\frac{(1+\varrho)^{d+2}\partial^2_{p^jp^i}Q^+(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

The computation of $\partial^2_{p^jp^i}Q^+(f,g)$ requires the computation of

$$\partial^2_{p^jp^i}(f(\overline{p'})g(\overline{q'})\sigma(\overline{p},\overline{q},\Omega)).$$

So using the fact that $\partial^2_{p^jp^i}\overline{p'}, \partial^2_{p^jp^i}\overline{q'}$ are bounded and

$$(\partial^\beta\sigma)(1+|\overline{p}|)^{|\beta|-1} \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3\times S^2), \quad |\beta|\le 3$$

for the treatment of the terms in $\partial^2_{p^i p^j}$ appearing in $\partial^2_{p^j p^i} Q^+(f,g)$, we obtain the estimation

$$\left\| \frac{(1+\varrho)^{d+2} \partial^2_{p^j p^i} Q^+(f,g)}{p^0} \right\|_{L^2(\mathbb{R}^3)} \leq C(T) \|f\|_{H^3_d(\mathbb{R}^3)} \|g\|_{H^3_d(\mathbb{R}^3)}.$$

Finally, we find

$$\left\| (1+\varrho)^{d+2} \partial^2_{p^j p^i} \left(\frac{1}{p^0} Q^+(f,g) \right) \right\|_{L^2(\mathbb{R}^3)} \leq C(T) \|f\|_{H^3_d(\mathbb{R}^3)} \|g\|_{H^3_d(\mathbb{R}^3)}.$$

- For $|\beta| = 3$, we similarly obtain by computation using hypotheses

$$\left\| (1+\varrho)^{d+3} \partial^3_{p^k p^j p^i} \left(\frac{1}{p^0} Q^+(f,g) \right) \right\|_{L^2(\mathbb{R}^3)} \leq C(T) \|f\|_{H^3_d(\mathbb{R}^3)} \|g\|_{H^3_d(\mathbb{R}^3)}.$$

The method is similar for Q^-. Using $Q(f,g) = Q^+(f,g) - Q^-(f,g)$, we conclude for Q.

The proof of Proposition 1 is then complete. □

Remark 3. The hypothesis of Proposition 1 concerning the collision kernel σ is a supplementary hypothesis for the investigation of the solution to the Boltzmann equation.

In what is to follow, we are searching the local existence and the uniqueness of the solution to the Boltzmann equation (28) in the Banach space $\overline{H}^3_d(0,T,\mathbb{R}^3)$, $d > \frac{5}{2}$, $T > 0$.

The function f_0 is given in $\overline{H}^3_{d,r}(\mathbb{R}^3)$ and for any function $f \in \overline{H}^3_d(\mathbb{R}^3)$, we have

$$f(0,\overline{p}) = f_0.$$

We also set $F^{0i}(0) = E^i$, $F_{ij}(0) = \varphi_{ij}$, $i,j = 1,2,3$.

We now establish the local existence theorem in $H^3_d(\mathbb{R}^3)$ which shall subsist in $\overline{H}^3_d(\mathbb{R}^3)$ by completeness.

5. Main Existence Theorem

Theorem 1. *Let $\widetilde{f} \in H^3_{d,r}(\mathbb{R}^3)$ be given. Then the linearized partial differential equation*

$$\frac{\partial f}{\partial t} + \frac{\widetilde{P}^i}{p^0} \frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(\widetilde{f}, \widetilde{f}) \tag{39}$$

whose unknown is f and where $\widetilde{P}^i(\widetilde{F}, \widetilde{f}) = -\Gamma^i_{\lambda\mu} p^\lambda p^\mu + p^\beta \widetilde{F_\beta}^i (\int_{\mathbb{R}^3} \widetilde{f} a b^2 d\overline{p})$, with $f(0,\overline{p}) = f_0$ has in $H^3_d(\mathbb{R}^3)$ a unique and bounded $*$-weakly solution.

To prove this theorem, the following lemma and proposition will be very useful:

Lemma 2. *For any* $i, j, l, m = 1, 2, 3,$

$$F^{0i}, \quad \partial_{p^i}\left(\frac{\widetilde{P^i}}{p^0}\right), \quad \partial_{p^j}\left(\frac{\widetilde{P^i}}{p^0}\right), \quad \partial^2_{p^j p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)(1+|\overline{p}|), \quad \partial^3_{p^j p^l p^m}\left(\frac{\widetilde{P^i}}{p^0}\right)(1+|\overline{p}|)^2$$

are bounded.

Proof. • For F^{0i}, we set $h = ab^2$. Then

$$\dot{h} = \dot{a}b^2 + 2ab\dot{b} = ab^2\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) > 0,$$

because

$$H = \frac{1}{3}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) > 0.$$

So $ab^2 \geq a_0 b_0^2$ and $F^{0i} = \dfrac{a_0 b_0^2}{ab^2} F^{0i}(0) \Rightarrow |F^{0i}| \leq |E^i|$.

• For $\partial_{p^i}\left(\dfrac{\widetilde{P^i}}{p^0}\right)$, we have

$$\frac{\widetilde{P^i}}{p^0} = -\Gamma^i_{\lambda\mu}\frac{p^\lambda p^\mu}{p^0} + \left(F^i_0 + \frac{p^j F^i_j}{p^0}\right)\int_{\mathbb{R}^3} \widetilde{f}ab^2 d\overline{p}.$$

Thus

$$\partial_{p^i}\left(\frac{\widetilde{P^i}}{p^0}\right) = -2\Gamma^i_{0i} + \frac{\delta^j_i p^0 - p^j \partial_{p^i} p^0}{(p^0)^2} F^i_j \int_{\mathbb{R}^3} \widetilde{f}ab^2 d\overline{p}.$$

Using now

$$\partial_{p^i} p^0 = \frac{g_{ii}p^i}{p^0}, \quad F^i_j = -g^{ii}F_{ij}$$

we find

$$\partial_{p^i}\left(\frac{\widetilde{P^i}}{p^0}\right) = -2\Gamma^i_{0i} + \left(\frac{-g^{ii}F_{ij}\delta^j_i}{p^0} + \frac{F_{ij}p^i p^j}{(p^0)^3}\right)\int_{\mathbb{R}^3} \widetilde{f}ab^2 d\overline{p}.$$

But

$$\left|\frac{F_{ij}p^i p^j}{(p^0)^3}\right| \leq C a_0 b_0 T|\varphi_{ij}| = C(a_0, b_0, T, \varphi_{ij})$$

and

$$\left|\frac{g^{ii}F_{ij}\delta^j_i}{p^0}\right| \leq C(a_0, b_0, T, \varphi_{ij}), \qquad \left|\int_{\mathbb{R}^3} \widetilde{f}d\overline{p}\right| \leq Cr,$$

so finally we obtain

$$\left|\partial_{p^i}\left(\frac{\widetilde{P^i}}{p^0}\right)\right| \leq C(a_0, b_0, T, \varphi_{ij})(1+r).$$

• For $\partial_{p^j}\left(\frac{\widetilde{p^i}}{p^0}\right)$, a similar calculation where ∂_{p^i} is replaced by ∂_{p^j} gives:

$$\partial_{p^j}\left(\frac{\widetilde{P^i}}{p^0}\right) = -2\Gamma^i_{0i}\delta^j_i + \left(\frac{-g^{ii}F_{ik}\delta^k_j}{p^0} + \frac{g^{ii}g_{jj}F_{ik}p^k p^j}{(p^0)^3}\right)\int_{\mathbb{R}^3}\widetilde{f}ab^2 d\overline{p}.$$

It results that:

$$\left|\partial_{p^j}\left(\frac{\widetilde{P^i}}{p^0}\right)\right| \le C(a_0, b_0, T, \varphi_{ij})(1+r).$$

• For $\partial^2_{p^j p^l}\left(\frac{\widetilde{p^i}}{p^0}\right)\left(1+|\overline{p}|\right)$, we have after computation

$$\partial^2_{p^j pl}\left(\frac{\widetilde{P^i}}{p^0}\right) = \left[\frac{g^{ii}g_{ll}F_{ik}p^l\delta^i_k + g^{ii}g_{jj}F_{ik}\delta^l_j p^k + g^{ii}g_{jj}F_{ik}\delta^l_k p^j}{(p^0)^3} - \frac{3g^{ii}g_{jj}g_{ll}p^k p^j p^l}{(p^0)^5}\right]\int_{\mathbb{R}^3}\widetilde{f}ab^2 d\overline{p}.$$

And clearly

$$\left|\partial^2_{p^j p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)\right| \le \frac{C(a_0, b_0, T, \varphi_{ij}, r)}{(p^0)^2}, \quad 1+|\overline{p}| \le Cp^0,$$

so

$$\left|\partial^2_{p^j p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)(1+|\overline{p}|)\right| \le C(a_0, b_0, T, \varphi_{ij}, r).$$

• For the computation of $\partial^3_{p^j p^l p^m}\left(\frac{\widetilde{p^i}}{p^0}\right)(1+|\overline{p}|)^2$, we have

$$\partial^3_{p^j p^l p^m}\left(\frac{\widetilde{P^i}}{p^0}\right) = \partial_{p^m}\left[\frac{g^{ii}g_{ll}F_{ik}p^l\delta^i_k + g^{ii}g_{jj}F_{ik}\delta^l_j p^k + g^{ii}g_{jj}F_{ik}\delta^l_k p^j}{(p^0)^3} - \frac{3g^{ii}g_{jj}g_{ll}F_{ik}p^k p^j p^l}{(p^0)^5}\right]\int_{\mathbb{R}^3}\widetilde{f}ab^2 d\overline{p}$$

Since

$$\partial_{p^m}\left[\frac{p^l}{(p^0)^3}\right] = \frac{\delta^l_m}{(p^0)^3} - \frac{3g_{mm}p^m p^l}{(p^0)^5},$$

$$\partial_{p^m}\left[\frac{p^l p^j p^k}{(p^0)^5}\right] = \frac{p^l p^k\delta^j_m + p^l p^j\delta^k_m + p^j p^k\delta^l_m}{(p^0)^5} - \frac{5g_{mm}p^m p^k p^j p^l}{(p^0)^7},$$

we get

$$\left|\partial^3_{p^j p^l p^m}\left(\frac{\widetilde{P^i}}{p^0}\right)\right| \le \frac{C(a_0, b_0, T, \varphi_{ij}, r)}{(p^0)^3}.$$

But $(1+|\overline{p}|)^2 \le C(p^0)^2$, so

$$\left|\partial^3_{p^j p^l p^m}\left(\frac{\widetilde{P^i}}{p^0}\right)(1+|\overline{p}|)^2\right| \le C(a_0, b_0, T, \varphi_{ij}, r).$$

This ends the proof of Lemma 2. □

Proposition 2. *For any* $|\beta| \leq 3$*, we have*

$$\left((1+|\overline{p}|)^{d+|\beta|}\partial^{\beta}\left(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\right)\Big/(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta}f^N\right)_{L^2(\mathbb{R}^3)}$$
$$\leq C\left(\sum_{\alpha\leq\beta}\|(1+|\overline{p}|)^{d+|\alpha|}\partial^{\alpha}f^N\|_{L^2(\mathbb{R}^3)}\right)\|(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta}f^N\|_{L^2(\mathbb{R}^3)} \quad (40)$$

where $C = C(a_0, b_0, T, \varphi_{ij}, r)$.

Proof. **In what is to follow** $C = C(a_0, b_0, T, \varphi_{ij}, r)$.

- For $|\beta| = 0$ let us prove that

$$\left((1+|\overline{p}|)^d\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\Big/(1+|\overline{p}|)^d f^N\right)_{L^2(\mathbb{R}^3)} \leq C(\widetilde{f},F)\|(1+|\overline{p}|)^d f^N\|^2_{L^2(\mathbb{R}^3)}.$$

The relation

$$\partial_{p^i}[(1+|\overline{p}|)^d f^N] = \partial_{p^i}[(1+|\overline{p}|)^d]f^N + (1+|\overline{p}|)^d\partial_{p^i}f^N$$

implies that

$$\left|\left((1+|\overline{p}|)^d\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\Big/(1+|\overline{p}|)^d f^N\right)\right|$$
$$\leq \left|\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d f^N]\Big/(1+|\overline{p}|)^d f^N\right)\right|$$
$$+\left|\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d]f^N\Big/(1+|\overline{p}|)^d f^N\right)\right|.$$

But

$$\left|\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d f^N]\Big/(1+|\overline{p}|)^d f^N\right)\right|$$
$$\leq C\left[\sup_{t\in[0,T]}\left|\partial_{p^i}\left(\frac{\widetilde{P}^i}{p^0}\right)\right|\right]\|(1+|\overline{p}|)^d f^N\|^2_{L^2(\mathbb{R}^3)},$$

in fact $\mathscr{D}(\mathbb{R}^3)$ is dense in $H^3_d(\mathbb{R}^3)$ and $\left(u/\frac{\partial v}{\partial p^i}\right) = -\left(\frac{\partial u}{\partial p^i}/v\right)$ so:

$$\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d f^N]\Big/(1+|\overline{p}|)^d f^N\right)$$
$$= -\frac{1}{2}\left(\partial_{p^i}\left[\frac{\widetilde{P}^i}{p^0}\right](1+|\overline{p}|)^d f^N\Big/(1+|\overline{p}|)^d f^N\right).$$

Moreover

$$\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d]f^N = \frac{\widetilde{P}^i}{p^0}\times\frac{g_{ii}dp^i(1+|\overline{p}|)^d f^N}{|\overline{p}|(1+|\overline{p}|)}$$

implies

$$\left|\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d]f^N\Big/(1+|\overline{p}|)^d f^N\right)\right| \leq C\left[\sup_{t\in[0,T]}\left|\frac{\widetilde{P}^i}{p^0}\times\frac{g_{ii}dp^i}{|\overline{p}|(1+|\overline{p}|)}\right|\right]\|(1+|\overline{p}|)^d f^N\|^2_{L^2(\mathbb{R}^3)}.$$

Invoking Lemma 2 which implies

$$\frac{\widetilde{P}^i}{p^0}\times\frac{g_{ii}dp^i}{|\overline{p}|(1+|\overline{p}|)} = \frac{g_{ii}dp^i}{|\overline{p}|}\left(\int_{\mathbb{R}^3}\widetilde{f}ab^2d\overline{p}\right)\left[\frac{-2\Gamma^i_{0i}p^i}{(1+|\overline{p}|)}+\frac{F^i_0+\frac{p^jF^i_j}{p^0}}{(1+|\overline{p}|)}\right],$$

$$\left|\frac{\widetilde{P}^i}{p^0}\times\frac{g_{ii}dp^i}{|\overline{p}|(1+|\overline{p}|)}\right| \leq C(a_0,b_0,T,\varphi_{ij},r),$$

we conclude from the above calculations that

$$\left|\left((1+|\overline{p}|)^d\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\Big/(1+|\overline{p}|)^d f^N\right)_{L^2(\mathbb{R}^3)}\right| \leq C(\widetilde{f},F)\|(1+|\overline{p}|)^d f^N\|^2_{L^2(\mathbb{R}^3)}$$

- For $|\beta|=1$, we have:

$$\partial_{p^j}\left[\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\right] = \partial_{p^j}\left[\frac{\widetilde{P}^i}{p^0}\right]\partial_{p^i}f^N + \frac{\widetilde{P}^i}{p^0}\partial^2_{p^jp^i}f^N.$$

We additionally have

$$\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N] = (1+|\overline{p}|)^{d+1}\partial^2_{p^jp^i}f^N + \partial_{p^i}[(1+|\overline{p}|)^{d+1}]\partial_{p^j}f^N.$$

Consequently

$$\begin{aligned}&\left|\left((1+|\overline{p}|)^{d+1}\partial_{p^j}\left[\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\right]\Big/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right)\right| \\ &\quad\leq \left|\left((1+|\overline{p}|)^{d+1}\partial_{p^j}\left[\frac{\widetilde{P}^i}{p^0}\right]\frac{\partial f^N}{\partial p^i}\Big/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right)\right| \\ &\qquad+\left|\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N]\Big/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right)\right| \\ &\qquad+\left|\left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}]\partial_{p^j}f^N\Big/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right)\right|.\end{aligned}$$

Lemma 2 and $\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}]\partial_{p^j}f^N = \frac{\widetilde{P}^i}{p^0}\frac{(d+1)g_{ii}p^i(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N}{|\overline{p}|(1+|\overline{p}|)}$ imply clearly that

$$\begin{aligned}&\left|\left((1+|\overline{p}|)^{d+1}\partial_{p^j}\left[\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\right]\Big/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right)_{L^2(\mathbb{R}^3)}\right| \\ &\leq C(\|(1+|\overline{p}|)^{d+1}\partial_{p^i}f^N\|_{L^2(\mathbb{R}^3)})\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\|_{L^2(\mathbb{R}^3)} \\ &\quad+C(\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\|_{L^2(\mathbb{R}^3)})\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

- For $\left|\beta\right| = 2$ we have

$$\partial^2_{p^j p^k}\left[\frac{\widetilde{P^i}}{p^0}\frac{\partial f^N}{\partial p^i}\right] = \partial^2_{p^j p^k}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial_{p^i} f^N + \partial_{p^j}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^2_{p^i p^k} f^N + \partial_{p^k}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^2_{p^i p^j} f^N + \frac{\widetilde{P^i}}{p^0}\partial^3_{p^i p^j p^k} f^N.$$

Since by Lemma 2 $\partial^2_{p^j p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)(1+|\overline{p}|)$ is bounded, we conclude using the inequality

$$\left|\left(\frac{\widetilde{P^i}}{p^0}(1+|\overline{p}|)^{d+2}\partial^3_{p^i p^j p^k} f^N \Big/ (1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\right)_{L^2(\mathbb{R}^3)}\right| \leq C\|(1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\|^2_{L^2(\mathbb{R}^3)}$$

that

$$\left|\left((1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k}\left[\frac{\widetilde{P^i}}{p^0}\frac{\partial f^N}{\partial p^i}\right] \Big/ (1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\right)_{L^2(\mathbb{R}^3)}\right|$$
$$\leq C(\|(1+|\overline{p}|)^{d+1}\partial_{p^i} f^N\|_{L^2(\mathbb{R}^3)})\|(1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\|_{L^2(\mathbb{R}^3)}$$
$$+ C\left(\sum_{m=j,k}\|(1+|\overline{p}|)^{d+2}\partial^2_{p^i p^m} f^N\|_{L^2(\mathbb{R}^3)}\right)\|(1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\|_{L^2(\mathbb{R}^3)}$$
$$+ C(\|(1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\|_{L^2(\mathbb{R}^3)})\|(1+|\overline{p}|)^{d+2}\partial^2_{p^j p^k} f^N\|_{L^2(\mathbb{R}^3)}.$$

- For $|\beta| = 3$, we have

$$\partial^3_{p^j p^k p^l}\left(\frac{\widetilde{P^i}}{p^0}\partial_{p^i} f^N\right) = \partial^3_{p^j p^k p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial_{p^i} f^N + \partial^2_{p^j p^k}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^2_{p^i p^l} f^N + \partial^2_{p^l p^j}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^2_{p^i p^k} f^N + \partial^2_{p^k p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^2_{p^i p^j} f^N + \partial_{p^j}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^3_{p^i p^k p^l} f^N + \partial_{p^k}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^3_{p^i p^j p^l} f^N + \partial_{p^l}\left(\frac{\widetilde{P^i}}{p^0}\right)\partial^3_{p^i p^j p^k} f^N + \frac{\widetilde{P^i}}{p^0}\partial^4_{p^i p^j p^k p^l} f^N.$$

Reasoning similarly such as for the preceding steps, and using the fact that

$$\partial^3_{p^j p^l p^m}\left(\frac{\widetilde{P^i}}{p^0}\right)(1+|\overline{p}|)^2$$

is bounded, we obtain:

$$\left|\left((1+|\overline{p}|)^{d+3}\partial^3_{p^j p^k p^l}\left[\frac{\widetilde{P^i}}{p^0}\frac{\partial f^N}{\partial p^i}\right] \Big/ (1+|\overline{p}|)^{d+3}\partial^3_{p^j p^k p^l} f^N\right)_{L^2(\mathbb{R}^3)}\right|$$
$$\leq C(\|(1+|\overline{p}|)^{d+1}\partial_{p^i} f^N\|_{L^2(\mathbb{R}^3)})\|(1+|\overline{p}|)^{d+3}\partial^3_{p^j p^k p^l} f^N\|_{L^2(\mathbb{R}^3)}$$

$$+C\Big(\sum_{m=j,k,l}\|(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^m}f^N\|_{L^2(\mathbb{R}^3)}\Big)\|(1+|\overline{p}|)^{d+3}\partial^3_{p^jp^kp^l}f^N\|_{L^2(\mathbb{R}^3)}$$

$$+C\Big(\sum_{m=k,l}\|(1+|\overline{p}|)^{d+3}\partial^3_{p^ip^jp^m}f^N\|_{L^2(\mathbb{R}^3)}\Big)\|(1+|\overline{p}|)^{d+3}\partial^3_{p^jp^kp^l}f^N\|_{L^2(\mathbb{R}^3)}$$

$$+C\Big(\sum_{m=i,j}\|(1+|\overline{p}|)^{d+3}\partial^3_{p^kp^lp^m}f^N\|_{L^2(\mathbb{R}^3)}\Big)\|(1+|\overline{p}|)^{d+3}\partial^3_{p^jp^kp^l}f^N\|_{L^2(\mathbb{R}^3)}.$$

The cases $|\beta|=0,1,2,3$ prove clearly Proposition 2. □

Proof of Theorem 1. **In what follows** $C=C(a_0,b_0,T,E^i,\varphi_{ij},r)$.

We use as we have said it previously the Faedo-Galerkin method in the function space $H^3_d(\mathbb{R}^3)$.

We choose an Hilbertian orthonormal base $(w_k)\subset H^3_d(\mathbb{R}^3)$ and we take $\widetilde{f}\in H^3_{d,r}(\mathbb{R}^3)$.

We are searching a solution f of the linearized Boltzmann equation

$$\frac{\partial f}{\partial t}+\frac{\widetilde{P}^i(\widetilde{F},\widetilde{f})}{p^0}\frac{\partial f}{\partial p^i}=\frac{1}{p^0}Q(\widetilde{f},\widetilde{f}) \tag{41}$$

with

$$\widetilde{P}^i(\widetilde{F},\widetilde{f})=-\Gamma^i_{\lambda\mu}p^\lambda p^\mu+p^\beta\widetilde{F_\beta^{\ i}}\Big(\int_{\mathbb{R}^3}\widetilde{f}ab^2d\overline{p}\Big) \tag{42}$$

as a limit of sequence of approximations

$$f^N=\sum_{k=1}^N c_k(t)w_k,\quad N\in\mathbb{N}^* \tag{43}$$

where the components $c_k(t)$ are differentiable with respect to t and are given as solutions of N linear ordinary differential equations

$$(\partial_t f^N/w_k)+\Big(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}f^N/w_k\Big)=\Big(\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})/w_k\Big) \tag{44}$$

and where $(/)$ stands for the scalar product in $L^2_d(\mathbb{R}^3)$.

The initial data are

$$c_k(0)=(f_0/w_k) \tag{45}$$

in which f_0 stands for the initial datum of the Boltzmann equation (28) such as

$$f_0\in H^3_{d,r}(\mathbb{R}^3). \tag{46}$$

We are looking for an estimation of $\|f^N\|_{H^3_d(\mathbb{R}^3)}$ independent on N.

Multiplying the relation (44) by $c_k(t)$ and summing for k going to 1 from N, we find:

$$(\partial_t f^N/f^N)+\Big(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}f^N/f^N\Big)=\Big(\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})/f^N\Big). \tag{47}$$

We also have using (43) that $f^N\in H^3_d(\mathbb{R}^3)$.

We now observe that by (44)

$$\left(\partial_t f^N + \frac{\widetilde{P}^i}{p^0}\partial_{p^i} f^N - \frac{1}{p^0}Q(\widetilde{f},\widetilde{f})/w_k\right) = 0, \quad \text{for all } \ k \in \mathbb{N}^*,$$

so $V = \partial_t f^N + \frac{\widetilde{P}^i}{p^0}\partial_{p^i} f^N - \frac{1}{p^0}Q(\widetilde{f},\widetilde{f})$ is orthogonal in $H^3_d(\mathbb{R}^3)$ to the subspace generated by the base $(w_k)_{k\in\mathbb{N}^*}$ which is dense in $H^3_d(\mathbb{R}^3)$, so is V for the whole space $H^3_d(\mathbb{R}^3)$. Thus $V = 0$ and consequently

$$\frac{\partial f^N}{\partial t} + \frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i} = \frac{1}{p^0}Q(\widetilde{f},\widetilde{f}). \tag{48}$$

f^N is built as solution of (41) which verifies following (43)-(45):

$$f^N(0) = \sum_{k=1}^{N} c_k(0) w_k = \sum_{k=1}^{N} (f_0/w_k) w_k.$$

In view of (48) we have $\frac{\partial f^N}{\partial t} = -\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i} + \frac{1}{p^0}Q(\widetilde{f},\widetilde{f})$ and since $\widetilde{f} \in H^3_d(\mathbb{R}^3)$, it results using Proposition 1 that $\frac{1}{p^0}Q(\widetilde{f},\widetilde{f}) \in H^3_d(\mathbb{R}^3)$ and so

$$(1+|\overline{p}|)^{d+|\beta|}\partial^\beta\left(\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})\right) \in L^2(\mathbb{R}^3), \quad |\beta| \leq 3 \tag{49}$$

where $\partial^\beta = \partial^\beta_{\overline{p}}$.

Since by (43) $f^N \in H^3_d(\mathbb{R}^3)$, we also have:

$$(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N \in L^2(\mathbb{R}^3), \quad |\beta| \leq 3. \tag{50}$$

Consequently we can consider the scalar product in $L^2(\mathbb{R}^3)$ of the elements of (49)-(50) and we find using (48) and bilinearity that

$$\begin{aligned}
&\left((1+|\overline{p}|)^{d+|\beta|}\partial^\beta\left(\frac{\partial f^N}{\partial t}\right)\Big/(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\right)_{L^2(\mathbb{R}^3)} \\
&+\left((1+|\overline{p}|)^{d+|\beta|}\partial^\beta\left(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\right)\Big/(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\right)_{L^2(\mathbb{R}^3)} \\
&\quad =\left((1+|\overline{p}|)^{d+|\beta|}\partial^\beta\left(\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})\right)\Big/(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\right)_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{51}$$

Since

$$(1+|\overline{p}|)^{d+|\beta|}\partial^\beta\left(\frac{\partial f^N}{\partial t}\right) = \partial_t((1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N),$$

The relation (51), using Proposition 1, becomes

$$\frac{1}{2}\frac{d}{dt}\|(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\|^2_{L^2(\mathbb{R}^3)}$$

$$\leq -\left((1+|\overline{p}|)^{d+|\beta|}\partial^{\beta}\left(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}\right) \Big/ (1+|\overline{p}|)^{d+|\beta|}\partial^{\beta} f^N\right)_{L^2(\mathbb{R}^3)} + \left\|\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})\right\|_{H^3_d(\mathbb{R}^3)} \|(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta} f^N\|_{L^2(\mathbb{R}^3)}. \quad (52)$$

Reporting now the inequality (40) in (52), and simplifying we obtain:

$$\frac{d}{dt}\|(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta} f^N\|_{L^2(\mathbb{R}^3)} \leq C\left(\sum_{\alpha\leq\beta}\|(1+|\overline{p}|)^{d+|\alpha|}\partial^{\alpha} f^N\|_{L^2(\mathbb{R}^3)}\right) + C\left\|\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})\right\|_{H^3_d(\mathbb{R}^3)}. \quad (53)$$

Now rewriting (53) for $|\beta| = 0,1,2,3$ and summing the inequalities obtained gives

$$\frac{d}{dt}\left(\sum_{|\beta|\leq 3}\|(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta} f^N\|_{L^2(\mathbb{R}^3)}\right) \leq C\left(\sum_{|\beta|\leq 3}\|(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta} f^N\|_{L^2(\mathbb{R}^3)}\right) + C\left\|\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})\right\|_{H^3_d(\mathbb{R}^3)}. \quad (54)$$

By the Gronwall lemma, using Proposition 1, the above inequality yields

$$\sum_{|\beta|\leq 3}\|(1+|\overline{p}|)^{d+|\beta|}\partial^{\beta} f^N\|_{L^2(\mathbb{R}^3)} \leq C\{\|f_0\|_{H^3_d(\mathbb{R}^3)} + \|\widetilde{f}\|^2_{H^3_d(\mathbb{R}^3)}\}. \quad (55)$$

Now $f_0, \widetilde{f} \in H^3_{d,r}(\mathbb{R}^3)$, $t \in [0,T]$, $|F^{0i}| \leq |E^i|$, $F_{ij} = \varphi_{ij}$, so using (4) we obtain

$$\|f^N\|_{H^3_d(\mathbb{R}^3)} \leq C. \quad (56)$$

Accordingly, we can choose a weak convergent subsequence of the bounded sequence of approximations (f^N) in the reflexive if space $H^3_d(\mathbb{R}^3)$

$$f^{N_k} \rightharpoonup f, \quad k \to +\infty$$

where f is the unique solution of (39) in $H^3_d(\mathbb{R}^3)$ such as $f(0) = f_0$. □

Theorem 2 (The main theorem). *The Boltzmann equation* $\frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0}Q(f,f)$ *has in* $H^3_d(\mathbb{R}^3)$ *a local unique* $*$*-weak solution* f *such that* $f(0) = f_0$.

Proof. We use the Banach fixed point theorem in $H^3_d(\mathbb{R}^3)$ for the mapping:

$$\widetilde{f} \in H^3_{d,r}(\mathbb{R}^3) \longmapsto \Xi(\widetilde{f}) = f$$

where f satisfies equation (39).

• We would like firstly to prove that we can choose $\|f_0\|_{H^3_{d,r}(\mathbb{R}^3)}$ and $T > 0$ such that

$$\widetilde{f} \in H^3_{d,r}(\mathbb{R}^3) \Rightarrow \Xi(\widetilde{f}) = f \in H^3_{d,r}(\mathbb{R}^3). \quad (57)$$

In what is to follow, $C = C(a_0, b_0, r, T, E^i, \varphi_{ij})$.

Let us assume that $\widetilde{f} \in H^3_{d,r}(\mathbb{R}^3)$. Applying directly the Gronwall inequality to (53), using Proposition 1 we find

$$\sum_{|\beta|\leq 3} \|(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\|_{L^2(\mathbb{R}^3)} \leq e^C\{\|f_0\|_{H^3_d(\mathbb{R}^3)} + Cr^2 t\}.$$

Reporting the inequality the above inequality in (54) yields

$$\frac{d}{dt}\left(\sum_{|\beta|\leq 3} \|(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\|_{L^2(\mathbb{R}^3)}\right) \leq C\{e^C\{\|f_0\|_{H^3_d(\mathbb{R}^3)} + Cr^2 t\}\} + Cr^2.$$

Integrating then over $[0,t]$ and using $t \leq T$ implies that

$$\sum_{|\beta|\leq 3} \|(1+|\overline{p}|)^{d+|\beta|}\partial^\beta f^N\|_{L^2(\mathbb{R}^3)}$$
$$\leq \|f_0\|_{H^3_d(\mathbb{R}^3)} + Ce^C\left(\|f_0\|_{H^3_d(\mathbb{R}^3)}T + \frac{Cr^2T^2}{2}\right) + Cr^2 T.$$

If we take

$$\begin{cases} \|f_0\|_{H^3_d(\mathbb{R}^3)\leq \frac{r}{2}} \\ C(e^C+1)\left(\frac{rT}{2} + \frac{Cr^2T^2}{2} + Cr^2 T\right) \leq \frac{r}{2}, \end{cases}$$

then

$$\|f^N\|_{H^3_d(\mathbb{R}^3)} \leq r, \quad N \in \mathbb{N}^*.$$

Because $f^{N_k} \rightharpoonup f$, $k \to +\infty$ in $H^3_d(\mathbb{R}^3)$, it results that

$$\|f\|_{H^3_d(\mathbb{R}^3)} \leq r.$$

- Let now $\widetilde{f_1}, \widetilde{f_2} \in H^3_{d,r}(\mathbb{R}^3)$ and f_1, f_2 be two solutions of (45) such that

$$\begin{cases} \frac{\partial f_1}{\partial t} + \frac{\widetilde{P}^i(\widetilde{f_1})}{p^0}\frac{\partial f_1}{\partial p^i} = \frac{1}{p^0}Q(\widetilde{f_1},\widetilde{f_1}) \\ \frac{\partial f_2}{\partial t} + \frac{\widetilde{P}^i(\widetilde{f_2})}{p^0}\frac{\partial f_2}{\partial p^i} = \frac{1}{p^0}Q(\widetilde{f_2},\widetilde{f_2}). \end{cases}$$

Let $G = f_1 - f_2$ and $\widetilde{G} = \widetilde{f_1} - \widetilde{f_2}$.
Then we get

$$\frac{\partial G}{\partial t} + \frac{\widetilde{P}^i(\widetilde{G})}{p^0}\frac{\partial G}{\partial p^i} = \frac{1}{p^0}Q(\widetilde{f_1},\widetilde{G}) - \frac{1}{p^0}Q(\widetilde{G},\widetilde{f_2}).$$

Invoking (55), applying the Gronwall inequality, using Proposition 1 and remembering that $G(0,\overline{p}) = 0$, we obtain:

$$\|G\|_{H^3_d(\mathbb{R}^3)\leq C(a_0,b_0,r,T,E^i,\varphi_{ij})T\|\widetilde{G}\|_{H^3_d(\mathbb{R}^3)}}. \tag{58}$$

where $C = C(a_0,b_0,r,T,E^i,\varphi_{ij})T$ is a positive constant.

Since the function $T \mapsto C(a_0, b_0, r, T, E^i, \varphi_{ij})T$ is continuous at the point $T = 0$, because $C(a_0, b_0, r, T, E^i, \varphi_{ij})$ is also a polynom in T, we can choose $T > 0$ small enough such that

$$C(a_0, b_0, r, T, E^i, \varphi_{ij})T < 1. \tag{59}$$

The inequalities (57), (58) and (59) show clearly that

$$H^3_{d,r}(\mathbb{R}^3) \to H^3_{d,r}(\mathbb{R}^3) : \widetilde{f} \longmapsto \Xi(\widetilde{f}) = f$$

is a contracting mapping, so by the Banach theorem Ξ has a unique fixed point $f = \widetilde{f}$ and the proof of Theorem 1 is complete. □

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