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Research Article

Analysis of Fractional Order Differential Equation Using Laplace Transform

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Abstract. Exploring the functioning of analytical solutions of differential equations defined by the fractional order operators is one of the hottest topics in the field of research in stability problems. This article analyze the stability of a certain type of fractional order differential equation. Employing Banach contraction mapping principle, existence and uniqueness of solutions are obtained. A sufficient condition to assure the reliability of solving the fractional order differential equation by Laplace Transform method and Generalized Laplace Transform are presented. Exponential stability results for the solution is discussed using Gronwall inequality. Application of the generalized Gronwall inequality to fractional order differential equation under investigation yields new sufficient condition for stability of fractional order differential equation. Applicability of the theoretical results on stability are demonstrated with an example. In the analysis of fractional order differential equation, Laplace transform is proved to be a valid tool under certain conditions.

Keywords. Fractional order, Stability, Laplace transform, Gronwall inequality

Mathematics Subject Classification (2020). 26A33, 34A08, 34D20, 44A10

1. Introduction

Fractional calculus is the generalization of integer order calculus to any arbitrary order. It includes both derivatives and integrals with fractional orders. For the past three decades, fractional calculus is the focus of several mathematicians in view of its application in different fields of science and engineering. In addition, nowadays economists are also using the concept of fractional calculus. Researchers and academicians found that the fractional order derivatives are more suitable for the interpretation of the real world phenomena, such as population dynamics, visco-elasticity, heat conduction, finance and so on ([5,9,13,16]).

In fractional calculus, different types of operators such as Riemann-Liouville operator, Caputo operator, conformable operator and many others have been introduced by researchers to analyze the real world problem with different conditions and assumptions. Khalil *et al*. [8] proposed the conformable fractional derivative and proved the conformable fractional Leibniz rule. Abdeljawad in [1] established generalized conformable derivatives to higher orders. Bosch [4] derived new convolution properties and the extension of *Laplace Transform* (LT) for fractional order differential equation.

Recently, study on stability theory of *fractional order differential equation* (FODE) has been productive and expeditiously developing and it has drawn the attention of many researchers. To analyze FODE, different tools have been used, of which the most frequently applied method is LT method [7,18]. LT method is used to solve the differential equations where the original differential equation of time domain is transformed into algebraic equation of frequency domain. Then this can be transformed into approximate solution of the original differential equation. Laplace transform is applied when the constants are known. For instance, the study of generalized *Mittag-Leffler* (ML) stability for fractional order nonlinear dynamical systems using Laplace transform is investigated in [10,11]. Also, Sabatier *et al*. [14] established the stability conditions for fractional order system by utilizing Laplace transform. The existence and stability of fractional order difference equations with certain boundary conditions is discussed by Selvam *et al*. [17]. Also, Liang *et al*. [12], Ye *et al*. [20], and Deng *et al*. [6] have used the Laplace transform method to analyze certain qualitative properties of FODE.

Inspired by the above mentioned analysis of stability of FODE under Laplace transformation method, we propose to study the stability of the following nonlinear *fractional order initial value problem* (FIVP) of the form

$$\begin{aligned} D_0^\alpha \phi(\eta) &= \lambda\phi(\eta) + f(\eta,\phi(\eta)), \quad \eta \ge 0, \\ \phi(0) &= \mu, \end{aligned} \tag{1.1}$$

where $0 < \alpha \le 1$ is the fractional order, λ is a non-negative real constant and μ is a real constant. The nonlinear term is f and it is continuous for every $\phi \in \mathbb{R}^n$.

The core objective of the article is to establish the stability of the solution of fractional order differential equation (1.1) by employing the Laplace transform. The basic properties of LT are presented in section 2. In Section 3, the existence and uniqueness of the solutions are discussed and new sufficient condition ensuring stability is determined in Section 4. Extended work to prove the reliability of *Generalized Laplace Transform* (GLT) method for solving fractional differential equation with conformable fractional derivative in Section 5 is followed by an example and numerical illustration is provided in Section 6. Section 7 presents a conclusion for the article.

2. Prerequisites

This section recollects, some important properties that are essential to derive the main results.

Definition 2.1 ([13]). The Laplace transform for the function $F(s)$ is

$$F(s) = L[\phi(\eta)] = \int_0^\infty e^{-s\eta}\phi(\eta)d\eta, \tag{2.1}$$

where $\phi(\eta)$ is a vector-valued function.

Definition 2.2 ([9]). The Riemann-Liouville integral for the function $\phi : (0,\infty) \to \mathbb{R}^n$ of order $0 < \alpha \le 1$ is

$${}_0I_t^\alpha \phi(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta - s)^{\alpha-1}\phi(s)ds. \tag{2.2}$$

Lemma 2.1 ([11]). *Let $\mathbb{C}$ be the complex plane, for every $u > 0$, $v > 0$ and $V \in \mathbb{C}^{n\times n}$,*

$$L\{\eta^{v-1}E_{u,v}(Vt^u)\} = s^{u-v}(s^u - V)^{-1},$$

is satisfied when $\mathbb{R}(s) > \|V\|^{\frac{1}{u}}$, where the real part is denoted by $\mathbb{R}(s)$.

Proof. For $\mathbb{R}(s) > \|V\|^{\frac{1}{u}}$, from [19], we have $\sum\limits_{k=0}^{\infty} V^k s^{-(k+1)u} = (s^u - V)^{-1}$. Then

$$\begin{aligned} L[\eta^{v-1}E_{u,v}(V\eta^u)] &= L\left[\eta^{v-1}\sum_{k=0}^{\infty}\frac{(V\eta^u)^k}{\Gamma(uk+v)}\right] \\ &= \sum_{k=0}^{\infty}\frac{V^k L[\eta^{uk+v-1}]}{\Gamma(uk+v)} \\ &= s^{u-v}\sum_{k=0}^{\infty}V^k s^{-(k+1)u} \\ &= s^{u-v}(s^u - V)^{-1}. \end{aligned}$$

□

Lemma 2.2 ([12]). *If $q \ge 0, \beta > 0$. Suppose $\phi(\eta)$, $p(\eta)$ are non-negative and locally integrable on $0 \le \eta < T$ (some $T \le +\infty$) with*

$$\phi(\eta) \le p(\eta) + q\int_0^\eta (\eta - s)^{\alpha-1}\phi(s)ds, \tag{2.3}$$

on $(0,T)$. Then

$$\phi(\eta) \le p(\eta) + \theta\int_0^\eta E'_\beta(\theta(\eta - s))p(s)ds, \quad 0 \le \eta < T, \tag{2.4}$$

where

$$\theta = (q\Gamma(\beta))^{\frac{1}{\beta}}, \ E_\beta(z) = \sum_{n=0}^{\infty}\frac{z^{n\beta}}{\Gamma(n\beta+1)}, \ E'_\beta(z) = \frac{d}{dz}E_\beta(z),$$

$$E'_\beta(z) \simeq \frac{z^{\beta-1}}{\Gamma(\beta)} as\ z \to 0+, \ E'_\beta(z) \simeq \frac{1}{\beta}e^z \ \ as\ \ z \to +\infty.$$

(and $E_\beta(z) \simeq \frac{1}{\beta}e^z$ as $z \to +\infty$).

Remark. If $p(\eta) \equiv p$, (constant), then $\phi(\eta) \le pE_\beta(\theta\eta)$.

Theorem 2.1 ([12]). *If $f(\eta)$ is piece-wise continuous on the interval $[0,\infty)$ and of exponential order δ, then the LT exists for $\mathbb{R}(s) > \delta$ and it converges absolutely.*

Lemma 2.3 (Gronwall Inequality, [12]). *If*

$$x(\eta) \le \phi(\eta) + \int_{\eta_0}^{\eta} k(s)x(s)ds, \quad \eta \in [\eta_0, T),$$

where each function is continuous on $[\eta_0, T)$, $T \le +\infty$, and $k(s) \ge 0$, then $x(\eta)$ satisfies

$$x(\eta) \le \phi(\eta) + \int_{\eta_0}^{\eta} \phi(s)k(s)e^{\int_s^{\eta} k(h)dh} ds, \quad \eta \in [\eta_0, T).$$

In addition, if $\phi(\eta)$ is non-decreasing, then

$$x(\eta) \le \phi(\eta)e^{\int_{\eta_0}^{\eta} k(s)ds}, \quad \eta \in [\eta_0, T).$$

Definition 2.3 ([11]). Given (G,d) is a metric space, a function $T: G \to G$ is assumed to be a contraction mapping if there is a constant α with $0 < \alpha \le 1$, such that $\forall\ x, y \in G$,

$$d(T(x), T(y)) \le d(x, y).$$

Theorem 2.2 (Banach Contraction Principle, [11]). *Let (G,d) be a complete metric space, then every contraction has a unique fixed point.*

Definition 2.4 ([2]). The trivial solution of $D_0^{\alpha}\phi(\eta) = \lambda\phi(\eta) + f(\eta, \phi(\eta))$ with $0 < \alpha \le 1$ is stable, if for any initial value $\phi_k (k = (0,1))$, $\exists\ \epsilon > 0$ such that $\|\phi(\eta)\| < \epsilon$, $\forall\ \eta \ge \eta_0$.
In addition to being stable, the solution is asymptotically stable if $\|\phi(\eta)\| \to 0$ as $\eta \to +\infty$.

Definition 2.5 (Exponential Stability [15]). The solution of fractional order differential equation is exponentially stable if $\exists\ v > 0$ and $u \ge 0$ such that, for every solution of FODE $\phi \in S$, then the exponential inequality

$$\|\phi(\eta)\| \le u\|\phi(\eta)\|_{\infty} e^{-v\eta}$$

holds.

3. Existence and Uniqueness of Solutions

This section focuses on establishing the existence and uniqueness of the solution of eq. (1.1). Before proceeding to the theorem, we shall now define the set S

$$S = \{\phi \in G : \|\phi\| \le r \mid r \in \mathbb{R}^+\}.$$

Here G is the Banach space of continuous functions.

Theorem 3.1. *Let $\phi \in S$, then ϕ is a solution to the FIVP* (1.1) *iff it is a solution to the integral equation*

$$\phi(\eta) = \mu + \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta - \ell)^{\alpha - 1} [\lambda\phi(\ell) + f(\ell, \phi(\ell))] d\ell. \tag{3.1}$$

Proof. Applying the integral operator I^{α} to (1.1) and using the properties of the inverse operator, we attain the existence of solution for eq. (1.1). □

Theorem 3.2. *Consider the FIVP* (1.1)*, where* $f(\eta,\phi(\eta))$ *is a smooth function. If* $\lambda\phi(\eta)+f(\eta,\phi(\eta))$ *is Lipschitz function, then the FIVP* (1.1) *has a unique solution in* S*, if*

$$L < \frac{\Gamma(\alpha)}{(1-\alpha)\Gamma(\alpha)+\eta^{\alpha}}.$$

Proof. Define the norm $\|\cdot\|$ on $[0,1]$ by

$$\|\phi\| = \sup_{\eta\in(0,1)} |\phi(\eta)|, \quad \forall\ \phi \in S,$$

and consider the operator $T : S \to S$ defined by

$$T\phi(\eta) = \mu + (1-\alpha)[\lambda\phi(\eta)+f(\eta,\phi(\eta))] + \frac{\alpha}{\Gamma(\alpha)}\int_0^{\eta} \lambda\phi(s)+f(s,\phi(s))(\eta-s)^{\alpha-1}ds$$

and let $\|\lambda\phi(\eta)+f(\eta,\phi(\eta))\| \le L$.

From Theorem 3.1, finding a fixed point of T, is equivalent to finding a solution to the FIVP (1.1).

Next, for each $\phi_1(\eta),\phi_2(\eta)\in S$ and $\eta\in(0,1)$. Then

$$\begin{aligned}
|T\phi_1(\eta)-T\phi_2(\eta)| &= |(1-\alpha)[(\lambda\phi_1(\eta)+f(\eta,\phi_1(\eta)))-(\lambda\phi_2(\eta)+f(\eta,\phi_2(\eta)))]| \\
&\quad + \left|\frac{\alpha}{\Gamma(\alpha)}\int_0^{\eta}[(\lambda\phi_1(s)+f(s,\phi_1(s))(\eta-s)^{\alpha-1})]ds\right| \\
&\quad - \left|\frac{\alpha}{\Gamma(\alpha)}\int_0^{\eta}[(\lambda\phi_2(s)+f(s,\phi_2(s))(\eta-s)^{\alpha-1})]ds\right| \\
&\le (1-\alpha)L\|\phi_1-\phi_2\| + \frac{\alpha}{\Gamma(\alpha)}L\|\phi_1-\phi_2\|\left|\int_0^{\eta}(\eta-s)^{\alpha-1}ds\right| \\
&\le (1-\alpha)L\|\phi_1-\phi_2\| + \frac{\alpha}{\Gamma(\alpha)}L\|\phi_1-\phi_2\|\frac{\eta^{\alpha}}{\alpha} \\
&\le \left(\frac{(1-\alpha)\Gamma(\alpha)+\eta^{\alpha}}{\Gamma(\alpha)}\right)L\|\phi_1-\phi_2\|.
\end{aligned}$$

Since $\left(\frac{(1-\alpha)\Gamma(\alpha)+\eta^{\alpha}}{\Gamma(\alpha)}\right)L < 1$, then T is contraction. Theorem 2.2 ensures that T has a unique solution. □

4. Stability Analysis

The reliability of solving FODE using Laplace transform under certain conditions is established in this section. Some essential results on Mittag-Leffler function and Grownwall inequality are presented. Then, the solutions of FODE are estimated. The solutions are in the form of exponential order, which is essential to employ the LT. Solutions of FODE are estimated by utilizing Gronwall inequality.

Theorem 4.1. *Assume that eq.* (1.1) *has a solution* $\phi(\eta)$ *which is unique and continuous, if* $f(\eta,\phi(\eta))$ *is continuous on the interval* $[0,\infty)$ *and exponentially bounded, then their Laplace transforms exists whenever* $\phi(\eta)$ *and its derivatives are both exponentially bounded.*

Proof. Since $f(\eta,\phi(\eta))$ is exponential, there exist positive constants Q, x and y such that

$$\|f(\eta,\phi(\eta))\| \le Qe^{\eta x+\phi(\eta)y}.$$

Here $\phi(\eta)$ converges boundedly in $\mathbb{R}$ for all $\eta \ge T$.

Eq. (1.1) is written in the given integral form

$$\phi(\eta) = \mu + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\ell)^{\alpha-1}[\lambda\phi(\ell)+f(\ell,\phi(\ell))]d\ell, \quad 0 \le \eta < \infty. \tag{4.1}$$

For $\eta \ge T$, eq. (4.1) can be rewritten as

$$\phi(\eta) = \mu + \frac{1}{\Gamma(\alpha)}\int_0^T (\eta-\ell)^{\alpha-1}[\lambda\phi(\ell)+f(\ell,\phi(\ell))]d\ell + \frac{1}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}[\lambda\phi(\ell)+f(\ell,\phi(\ell))]d\ell.$$

Since, the solution $\phi(\eta)$, ($\phi(0)=\mu$) is unique and continuous on $[0,\infty)$, then $\lambda\phi(\eta)+f(\eta,\phi(\eta))$ is bounded on $[0,T]$.

That is, $\exists\ k>0$ such that $\|\lambda\phi(\eta)+f(\eta,\phi(\eta))\| \le k\ \forall\ \eta \in [0,T]$. Here k is constant.

We have

$$\|\phi(\eta)\| \le \|\mu\| + \frac{k}{\Gamma(\alpha)}\int_0^T (\eta-\ell)^{\alpha-1}d\ell + \frac{1}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}\|\lambda\phi(\ell)\|d\ell + \frac{1}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}\|f(\ell,\phi(\ell))\|d\ell.$$

Multiplication of the above expression by $e^{-\tau\eta}$ leads to

$$e^{-\tau\eta} \le e^{-\tau T},\ e^{-\tau\eta} \le e^{-\tau\ell},\ \|f(\eta,\phi(\eta))\| \le Qe^{\eta x+\phi(\eta)y} \quad \text{for } \eta \ge T.$$

Then, we obtain

$$\begin{aligned}
\|\phi(\eta)\|e^{-\tau\eta} &\le \|\mu\|e^{-\tau\eta} + \frac{ke^{-\tau\eta}}{\Gamma(\alpha)}\int_0^T (\eta-\ell)^{\alpha-1}d\ell + \frac{e^{-\tau\eta}}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}\|\lambda\phi(\ell)\|d\ell \\
&\quad + \frac{e^{-\tau\eta}}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}\|f(\ell,\phi(\ell))\|d\ell \\
&\le \|\mu\|e^{-\tau\eta} + \frac{ke^{-\tau T}}{\alpha\Gamma(\alpha)}[\eta^\alpha-(\eta-T)^\alpha] + \frac{\lambda e^{-\tau\eta}}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}\|\phi(\ell)\|d\ell \\
&\quad + \frac{e^{-\tau\eta}}{\Gamma(\alpha)}\int_T^\eta (\eta-\ell)^{\alpha-1}\|f(\ell,\phi(\ell))\|d\ell \\
&\le \|\mu\|e^{-\tau\eta} + \frac{ke^{-\tau T}}{\alpha\Gamma(\alpha)}[\eta^\alpha-(\eta-T)^\alpha] + \frac{\lambda}{\Gamma(\alpha)}\int_0^\eta (\eta-\ell)^{\alpha-1}\|\phi(\ell)\|e^{\tau(\eta-\ell)\ell}d\ell \\
&\quad + \frac{Q}{\Gamma(\alpha)}\int_0^\eta (\eta-\ell)^{\alpha-1}e^{\tau(\eta-\ell)}e^{\eta x+\phi(\eta)y}d\ell \\
&\le \|\mu\|e^{-\tau\eta} + \frac{kT^\alpha e^{-\tau T}}{\alpha\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)}\int_0^\eta s^{\alpha-1}\|\phi(\ell)\|e^{-\tau s}ds \\
&\quad + \frac{Q}{\Gamma(\alpha)}\int_0^\eta s^{\alpha-1}e^{-\tau s}e^{\eta x+\phi(\eta)y}ds \qquad (\text{take } \eta-\ell=s) \\
&\le \|\mu\|e^{-\tau\eta} + \frac{kT^\alpha e^{-\tau T}}{\alpha\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)}\int_0^\eta s^{\alpha-1}\|\phi(\ell)\|e^{-\tau s}ds \\
&\quad + \frac{Qe^{\eta x+\phi(\eta)y}}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}e^{-\tau s}ds
\end{aligned}$$

$$\leq \|\mu\|e^{-\tau\eta} + \frac{kT^{\alpha}e^{-\tau T}}{\alpha\Gamma(\alpha)} + \frac{Qe^{\eta x+\phi(\eta)y}}{\tau^{\alpha}} + \frac{\lambda}{\Gamma(\alpha)}\int_0^{\eta} s^{\alpha-1}\|\phi(\ell)\|e^{-\tau s}ds, \quad \eta \geq T.$$

Denote

$$p = \|\mu\|e^{-\tau\eta} + \frac{kT^{\alpha}e^{-\tau T}}{\alpha\Gamma(\alpha)} + \frac{Qe^{\eta x+\phi(\eta)y}}{\tau^{\alpha}}, \quad q = \frac{\lambda}{\Gamma(\alpha)}, \; r(\eta) = \|\phi(\eta)\|e^{-\tau\eta}.$$

We get

$$r(\eta) \leq p + q\int_0^{\eta} s^{\alpha-1}r(s)ds, \quad \eta \geq T.$$

By Lemma 2.2, we have

$$r(\eta) \leq pE_{\beta}(\theta\eta) = p\sum_{n=0}^{\infty}\frac{(q\Gamma(\alpha))^n\eta^{n\alpha}}{\Gamma(n\alpha+1)}, \quad \eta \geq T.$$

From the definition of M-L function

$$E_{\alpha,1}(\eta) = \sum_{n=0}^{\infty}\frac{\eta^n}{\Gamma(n\alpha+1)}, \quad \alpha,\beta > 0$$

we arrive at

$$r(\eta) \leq pE_{\alpha,1}(q\Gamma(\alpha)\eta^{\alpha}), \quad \eta \geq T. \tag{4.2}$$

The continuity of the M-L function in $\eta \geq 0$ implies that if $w \geq 0$, there is a constant C such that

$$E_{\alpha}(w\eta^{\alpha}) \leq Ce^{w^{(1/\alpha)}\eta}, \quad \eta \geq 0, \; 0 < \alpha < 2. \tag{4.3}$$

From (4.2) and (4.3), we have

$$r(\eta) \leq Ce^{(q\Gamma(\alpha))^{1/\alpha}\eta}, \quad \eta \geq T.$$

Now,

$$\Rightarrow \qquad \|\phi(\eta)\| \leq pCe^{|(q\Gamma(\alpha))^{1/\alpha}+\tau|\eta}, \quad \eta \geq T.$$

From (1.1), we obtain

$$\begin{aligned}\|D_t^{\alpha}\phi(\eta)\| &\leq \lambda\|\phi(\eta)\| + \|f(\eta,\phi(\eta))\| \leq p\lambda Ce^{[q\Gamma(\alpha)^{1/\alpha}+\tau]\eta} + Qe^{\eta x+\phi(\eta)y}\\ &\leq (p\lambda C+Q)e^{[q\Gamma(\alpha)^{1/\alpha}+\tau+\eta x+\phi(\eta)y]\eta}, \quad \eta \geq T.\end{aligned}$$

Applying LT to equation (1.1) with respect to η, we have

$$\begin{aligned}&\mathbb{L}(D_0^{\alpha}\phi(\eta)) = \mathbb{L}(\lambda\phi(\eta)) + \mathbb{L}(f(\eta,\phi(\eta))),\\ &\quad \widehat{\phi}(s) = s^{\alpha-1}(s^{\alpha}-\lambda)^{-1}\mu + \int_0^{\infty}\int_0^{\infty} f(\eta,\phi(\eta))e^{\eta x+\phi(\eta)y}dxdy\\ &\qquad = s^{\alpha-1}(s^{\alpha}-\lambda)^{-1}\mu + \int_0^{\infty} e^{\eta x}f(\eta)dx\int_0^{\infty} e^{\phi(\eta)y}f(\phi(\eta))dy\\ &\qquad = s^{\alpha-1}(s^{\alpha}-\lambda)^{-1}\mu + s^{\alpha-1}\widehat{f(s)}e^{sx} + s^{\alpha-1}\widehat{f(\phi(s))}e^{sy},\end{aligned}$$

where $\widehat{\phi}(s)$, $\widehat{f(s)}$ and $\widehat{f(\phi(s))}$ denote the Laplace transforms of $\phi(\eta)$ and $f(\eta)$, $f(\phi(\eta))$, respectively. From Lemma 2.1, we get

$$\phi(\eta) = E_{\alpha,\beta}(\eta)^{\alpha-1}\mu + \int_0^{\eta}(\eta-\ell)^{\alpha-1}E_{\alpha,\beta}(\lambda(\eta-\ell)^{\alpha})f(\ell,\phi(\ell))d\ell.$$

The proof is complete. □

Theorem 4.2. *Suppose*

$$\|(\eta-\ell)^{\alpha-1}E_{\alpha,\beta}(\lambda(\eta-\ell))^{\alpha}\| \le Ne^{-\gamma\eta}, \quad 0\le\eta<\infty,\ \gamma>0$$

and

$$\int_0^{\eta}\|f(\ell,\phi(\ell))\|d\ell \le M, \quad \text{where } M,N>0,$$

then the solution of eq. (1.1) *is exponentially stable.*

Proof. The solution of eq. (1.1) is

$$\phi(\eta)=E_{\alpha,\beta}(\eta)^{\alpha-1}\mu+\int_0^{\eta}(\eta-\ell)^{\alpha-1}E_{\alpha,\beta}(\lambda(\eta-\ell)^{\alpha})f(\ell,\phi(\ell))d\ell.$$

Then,

$$\|\phi(\eta)\| \le \|E_{\alpha,\beta}(\eta)^{\alpha-1}\|\ \|\mu\|+\int_0^{\eta}\|(\eta-\ell)^{\alpha-1}E_{\alpha,\beta}(\lambda(\eta-\ell)^{\alpha})f(\ell,\phi(\ell))\|d\ell.$$

From the boundedness, we obtain

$$\|\phi(\eta)\| \le Ne^{-\gamma\eta}\|\mu\|+\int_0^{\eta}Ne^{-\gamma(\eta-\ell)}\|f(\ell,\phi(\ell))\|d\ell. \tag{4.4}$$

Multiply by $e^{\gamma\eta}$ both sides of eq. (4.4) to get

$$e^{\gamma\eta}\|\phi(\eta)\| \le N\|\mu\|+\int_0^{\eta}Ne^{\gamma\ell}\|f(\ell,\phi(\ell))\|d\ell.$$

By Lemma 2.3, we have

$$e^{\gamma\eta}\|\phi(\eta)\| \le N\|\mu\|\exp\left(N\int_0^{\eta}\|f(\ell,\phi(\ell))\|d\ell\right). \tag{4.5}$$

Multiply by $e^{-\gamma\eta}$ both sides of eq. (4.5), we have

$$\|\phi(\eta)\| \le N\|\mu\|\exp\left(N\int_0^{\eta}\|f(\ell,\phi(\ell))\|d\ell\right)e^{-\gamma\eta}.$$

This leads to the following inequality

$$\|\phi(\eta)\| \le N\|\mu\|e^{MN-\gamma\eta}.$$

Definition 2.5 establishes the exponentially stability of the solution of eq. (1.1).
In addition, as $\eta\to\infty \Rightarrow \|\phi(\eta)\|\to 0$. This indicates the asymptotic stability of the system. □

5. Generalized Laplace Transform

In this section, essential results are presented which are needed to derive the reliability of solving FODE using generalized Laplace transform. The following definitions and theorems are essential properties of generalized Laplace transform.

Definition 5.1 ([4]). Given $\phi:(0,\infty)\to\mathbb{R}$ and $\alpha\in(0,1]$, the derivative of ϕ of order α at the point η is defined by

$$T_{\alpha}\phi(\eta)=\lim_{h\to 0}\frac{\phi(\eta)-\phi(\eta-h\eta^{1-\alpha})}{h}.$$

Here T_{α} is the particular case of $\mathfrak{P}^{\alpha}$ when $\alpha\in(0,1]$ and $T(\eta,\alpha)=\eta^{1-\alpha}$.
$\mathfrak{P}^{\alpha}$ is the conformable fractional derivative of order $\alpha\in(0,1]$.

Lemma 5.1 ([4]). *Let I be an interval $I \subseteq \mathbb{R}$, $\phi : I \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$.*

(i) *If there exists $D^{\lceil\alpha\rceil}\phi$ at the point $\eta \in I$, then ϕ is $\mathfrak{P}_T^\alpha$ differentiable at η and $\mathfrak{P}_T^\alpha\phi(\eta) = T(\eta,\alpha)^{\lceil\alpha\rceil}D^{\lceil\alpha\rceil}\phi(\eta)$.*

(ii) *If $\alpha \in (0,1]$, then ϕ is $\mathfrak{P}_T^\alpha$ differentiable at $\eta \in I$ if and only if η is differentiable at η; in this case, we have $\mathfrak{P}_T^\alpha\phi(\eta) = T(\eta,\alpha)\phi'(\eta)$.*

Definition 5.2 ([4]). Given $0 < \alpha \le 1$ and a measurable function $\phi : [0,\infty) \to \mathbb{R}$, we define its generalized Laplace transform as

$$\mathfrak{L}_T^\alpha[\phi](s) = \int_0^\infty E_\alpha(-s,\eta)\phi(\eta)\frac{d\eta}{T(\eta,\alpha)},$$

if $\mathfrak{L}_T^\alpha[|\phi|](s) < \infty$, i.e., $\frac{E_\alpha(-s,\eta)\phi(\eta)}{T(\eta,\alpha)} \in L^1([0,\infty])$.

Theorem 5.1 ([4]). *Let $\phi : [0,\infty) \to \mathbb{R}$ be a locally absolutely continuous function such that there exists $\mathfrak{L}_T^\alpha[\phi](s)$ and $\mathfrak{L}_T^\alpha[\mathfrak{P}_T^\alpha\phi](s)$ for some s and $0 < \alpha \le 1$. Then*

$$\mathfrak{L}_T^\alpha[\mathfrak{P}_T^\alpha\phi](s) = s\mathfrak{L}_T^\alpha[\phi](s) - \phi(0).$$

Theorem 5.2 ([4]). *Let $\phi : [0,\infty) \to \mathbb{R}$ be a C^1 function such that ϕ' is a locally absolutely continuous function and there exists $\mathfrak{L}_T^\alpha[\phi](s), \mathfrak{L}_T^\alpha[\mathfrak{P}_T^\alpha\phi](s)$ and $\mathfrak{L}_T^\alpha[\mathfrak{P}_T^\alpha(P_T^\alpha\phi)](s)$ for some s and $0 < \alpha \le 1$. Then*

$$\begin{aligned}\mathfrak{L}_T^\alpha[\mathfrak{P}_T^\alpha(\mathfrak{P}_T^\alpha\phi)](s) &= s^2\mathfrak{L}_T^\alpha[\phi](s) - s\phi(0) - \mathfrak{P}_T^\alpha\phi(0)\\ &= s^2\mathfrak{L}_T^\alpha[\phi](s) - s\phi(0) - T(0,\alpha)\phi'(0).\end{aligned}$$

Theorem 5.3 ([4]). *Let $\phi, \chi : [0,\infty) \to \mathbb{R}$ be functions such that there exist $\mathfrak{L}_T^\alpha[\phi](s)$ and $\mathfrak{L}_T^\alpha[\chi](s)$ for some s and $0 < \alpha \le 1$. Then*

$$\mathfrak{L}_T^\alpha[\phi * \chi](s) = \mathfrak{L}_T^\alpha[\phi](s)\mathfrak{L}_T^\alpha[\chi](s).$$

5.1 Solution of FODE by Generalized Laplace Transform

Now, consider the eq. (1.1) in the following form

$$\mathfrak{P}_T^\alpha(\mathfrak{P}_T^\alpha\phi)(\eta) = \lambda^2\phi(\eta) + f(\eta,\phi(\eta)) \tag{5.1}$$

with $\phi(0) = x$, $\mathfrak{P}_T^\alpha\phi(0) = y$. Here $x, y, \lambda \in \mathbb{R}$ and $\lambda \ne 0$.

Now applying the Generalized Laplace transform to eq. (5.1), we have

$$\mathfrak{L}_T^\alpha\left[\mathfrak{P}_T^\alpha(\mathfrak{P}_T^\alpha\phi)(\eta)(s)\right] = \lambda^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) + \mathfrak{L}_T^\alpha[f(\eta,\phi(\eta))](s).$$

Using Theorem 5.2, the above expression is transformed into

$$\begin{aligned}s^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) - s\phi(0) - \mathfrak{P}_T^\alpha\phi(0) &= \lambda^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) + \mathfrak{L}_T^\alpha[f(\eta,\phi(\eta))](s)\\ s^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) - xs - y &= \lambda^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) + \mathfrak{L}_T^\alpha[f(\eta,\phi(\eta))](s)\\ s^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) - xs - y - \lambda^2\mathfrak{L}_T^\alpha[\phi(\eta)](s) &= \mathfrak{L}_T^\alpha[f(\eta,\phi(\eta))](s)\\ \mathfrak{L}_T^\alpha[\phi(\eta)](s) &= \frac{xs+y}{s^2+\lambda^2} + \mathfrak{L}_T^\alpha[f(\eta,\phi(\eta))](s)\frac{1}{s^2+\lambda^2}.\end{aligned}$$

From Theorem 5.3 and [4, Proposition 2], we obtain

$$\phi(\eta)=x\cos\left(\lambda\int_0^\eta \frac{dq}{T(q,\alpha)}\right)+\frac{y}{\lambda}\sin\left(\lambda\int_0^\eta \frac{dq}{T(q,\alpha)}\right)+\frac{1}{\lambda}f(\eta,\phi(\eta))*\sin\left(\lambda\int_0^\eta \frac{dq}{T(q,\alpha)}\right).$$

Hence, we proved the reliability of solving fractional order differential equation using generalized Laplace transform.

6. Example

Example 6.1. Let us consider the FODE of the logistic type as

$$D^\alpha\phi(\eta)=\phi(\eta)\left[1-a\cos\phi(\eta)\right],\quad \alpha\in[0,1]. \tag{6.1}$$

The solution for (6.1) is

$$\phi(\eta)=\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\ell)^{\alpha-1}[\phi(\eta)(1-a\cos\phi(\eta))]d\ell. \tag{6.2}$$

Now

$$\begin{aligned}|\phi_1(\eta)-\phi_2(\eta)| &= |(1-\alpha)[\phi_1(\eta)(1-a\cos\phi_1(\eta))-\phi_2(\eta)(1-a\cos\phi_2(\eta))]|\\ &\quad+\left|\frac{\alpha}{\Gamma(\alpha)}\int_0^\eta[\phi_1(s)(1-a\cos\phi_1(\eta))(\eta-s)^{\alpha-1}]ds\right|\\ &\quad-\left|\frac{\alpha}{\Gamma(\alpha)}\int_0^\eta[\phi_2(s)(1-a\cos\phi_2(\eta))(\eta-s)^{\alpha-1}]ds\right|\\ &\le(1-\alpha)aV\|\phi_1-\phi_2\|+\frac{\alpha}{\Gamma(\alpha)}\|\phi_1-\phi_2\|aV\left|\int_0^\eta(\eta-s)^{\alpha-1}ds\right|\\ &\le(1-\alpha)aV\|\phi_1-\phi_2\|+\frac{\alpha}{\Gamma(\alpha)}\|\phi_1-\phi_2\|aV\frac{\eta^\alpha}{\alpha}\\ &\le\left(\frac{(1-\alpha)\Gamma(\alpha)+\eta^\alpha}{\Gamma(\alpha)}\right)aV\|\phi_1-\phi_2\|,\end{aligned}$$

$$\|\phi_1-\phi_2\|\le k\|\phi_1-\phi_2\|,\quad \text{where } k=aV\left(\frac{(1-\alpha)\Gamma(\alpha)+\eta^\alpha}{\Gamma(\alpha)}\right).$$

Hence by Theorem 4.2, it is clear that (6.1) is stable if constant $k<1$.

Table 1. Different values of k for $V\in(0,1]$ and $\alpha=0.01$ to 0.05

V	$\alpha=0.01$	$\alpha=0.02$	$\alpha=0.03$	$\alpha=0.04$	$\alpha=0.05$
	k	k	k	k	k
0.1	0.0984	0.0485	0.0318	0.0235	0.0185
0.2	0.1969	0.0969	0.0637	0.0470	0.0371
0.3	0.2953	0.1454	0.0955	0.0705	0.0556
0.4	0.3938	0.1939	0.1273	0.0941	0.0741
0.5	0.4922	0.2424	0.1591	0.1176	0.0927
0.6	0.5907	0.2908	0.1910	0.1411	0.1112
0.7	0.6891	0.3393	0.2228	0.1646	0.1298
0.8	0.7876	0.3878	0.2546	0.1881	0.1483
0.9	0.8860	0.4362	0.2865	0.2116	0.1668
1.0	0.9845	0.4847	0.3183	0.2352	0.1854

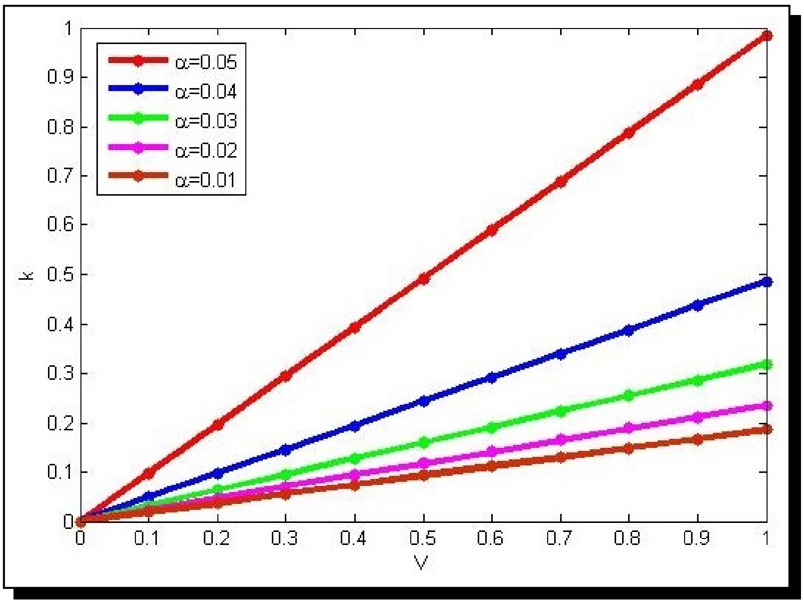

Figure 1. Numerical illustration for Example 6.1, for different values of α

Also, it has a unique solution for the values of $V \in (0,1]$, $a = 0.01$, $\eta = 0.01$ varying α from 0.01 to 0.05, the values of k are given in Table 1 and plotted in Figure 1. The curve is exponentially increasing when $V \in (0,1]$ and it is stable.

7. Conclusion

This article discussed the stability of nonlinear fractional order differential equation. To guarantee the reliability of solving the FODE by Laplace transform method under certain sufficient condition is established. Existence and Uniqueness of the solutions are derived. Sufficient conditions which ensures the stability of the FODE are derived. Extended work of solving fractional order differential equation using generalized Laplace transform is presented. An example is given to validate the analytical results. Important analytic tools used in this work are Mittag-Leffler function, Gronwall inequality and Laplace transform. Thus it is proved that the FODE can be solved by both Laplace and generalized Laplace transform and the stability of FODE under Laplace transform.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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