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Research Article

On Automata Accepting Biordered Set Languages and its Properties

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Abstract. Here we introduce the notion of finite state machines which accepts biordered set languages termed as nice automata. We begin by a basic type of automata termed as elementary automata which accepts elementary languages and give a characterization for the same. By considering the biordered set languages recognized by finite monoids we call them as finitely bsl, we see that the language L is finitely bsl if and only if there exist a nice automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$. Also we see that product is the only operation which is closed in the class of nice automata.

Keywords. Elementary automata; Finitely bsl; Nice automata

Mathematics Subject Classification (2020). 20M35; 68Q45

1. Introduction

Kleene theorem [9], a breakthrough in the study of automata theory which establishes a connection between the class of recognizable languages and the class of rational languages. Later, Myhill [14] introduced the notion of syntactic monoid — a monoid associated to each language and in 1960, Schutzenberger established an equivalence between finite state automata and finite semigroups. So it is quite relevant to study the connection between the algebraic properties of semigroups and the combinatorial properties of recognizable languages. It is well-known that the structure of algebraic systems like semigroups, rings, algebras, etc. can be

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predicted by the behaviour of its idempotents. So it is necessary to analyse the nature of the set of idempotents in an algebraic system to determine its structure. For an orthodox semigroup the set of idempotents forms a subsemigroup, but this fact cannot be extended to determine the structure of semigroups in general, but in the year 1972, Nambooripad [12] introduced the notion of biordered set to study the structure of regular semigroups. Since the structure of a semigroup is fully determined by its idempotents, the language recognised by the idempotents and in particular by the biordered set have significant role in the theory of formal languages. In fact, the languages recognised by biordered sets are known as biordered set languages [15] is developed as a combined approach of the theory of formal languages and the theory of biordered sets. Note that the regular languages in biordered set languages arises from finite biordered sets. In this paper we introduce the notion of a deterministic finite automata which accepts biordered set languages and discuss some of its properties.

2. Preliminaries

We briefly give some basic definitions and results which are required in the sequel and we refer the reader [5, 7, 8, 12, 13] for its detailed information, otherwise it is specified. A non-empty set S together with a binary operation $*$ is called a semigroup if the operation $*$ is associativeȦn element $e \in S$ with $x * e = x = e * x$ for all $x \in S$ is called the identity element for $*$ on S. We usually denote the identity element by the notation 1. A semigroup with identity element is called a monoid. A non-empty subset T of S is called a subsemigroup if for $x_1, x_2 \in T$ implies $x_1 * x_2 \in T$. A submonoid is a subsemigroup with the identity element. Let $(S, *)$ and $(S', *')$ be semigroups. A map $\varphi : S \to S'$ is called a homomorphism if for all $x, y \in S$, $(x * y)\varphi = (x\varphi) *' (y\varphi)$. If $(M, *)$ and $(M', *')$ are two monoids with identity elements 1_M and $1_{M'}$ respectively then $\varphi : M \to M'$ is called a homomorphism if we have the additional property $(1_M)\varphi = 1_{M'}$. The direct product $S \times S'$ is a semigroup under the operation $(s_1, s_1') \cdot (s_2, s_2') = (s_1 * s_2, s_1' *' s_2')$, known as the direct product of S and S'. An element $a \in S$ is called an idempotent if $a * a = a$. For a finite set X, let T_X be the set of all maps from X into X, which is a monoid under the operation composition of functions and known as the transformation monoid on X.

The notion of biordered sets was initiated by Nambooripad [12], he proved that the set of idempotents $E(S)$ of a regular semigroup S is a biordered set and later Easdown [3] proved the same for arbitrary semigroup. A set together with a partial binary operation is known as a partial algebra and the domain of the partial binary operation is denoted by D_E. Then D_E is a relation on E, and $(e, f) \in D_E$ if and only if the product $ef \in E$. On E, define $\omega^r = \{(e, f) : fe = e\}$ and $\omega^l = \{(e, f) : ef = e\}$. Recall that for any relation ρ on E the inverse ρ^{-1} of ρ is $\rho^{-1} = \{(f, e) : (e, f) \in \rho\}$.

Definition 2.1 ([12])**.** Let $E(S)$ be the set of idempotents of a semigroup S. Then the partial algebra $(E, *)$ satisfying

$$e * f \in \{e\,\rho\, f \text{ or } e\rho^{-1} f : \rho = \omega^r \text{ or } \rho = \omega^l\} \text{ for } e, f \in E \tag{2.1}$$

is called a biordered set and $e * f$ is undefined otherwise.

Several authors used the concept of biordered sets to study the structure of semigroups. For instance in [4] Easdown *et al.* proved that periodic elements of any free idempotent generated semigroup is contained in its subgroups and Dandon and Gould [2] proved that if E is a biordered set with trivial products then the free idempotent generated semigroup over E is abundant and has solvable word problem whenever E is finite. In [16] Szendrei outlined the various developments of structure theory of regular semigroups, especially using biordered sets.

Hall [6] introduced the concept of variety for regular semigroups, a class of regular semigroups, closed under homomorphic image, regular subsemigroups and direct products and Broekteeg [1] introduced the concept of variety of regular biordered set and proved that if S, S' are semigroups, then $E(S)\times E(S')$ is the biordered subset of $S\times S'$.

For a finite alphabet A, a word is a finite sequence of letters in A. There exist a unique word of length *zero*, called empty word and is denoted by 1. For each alphabet A, let A^+ be denote the set of all non-empty words over A and $A^* = A^+\cup\{1\}$. Then $A^*(A^+)$ is a monoid(semigroup) under the operation concatenation, called the free monoid(free semigroup) over A. A language over A is a subset of A^*. A language L is recognizable if there exist a finite monoid M and a homomorphism $\varphi : A^* \to M$ such that $L = P\varphi^{-1}$ where $P\subseteq M$. Let $L\subseteq A^*$ be a language. The syntactic congruence of L in A^* is the relation P_L defined on A^* by uP_Lv if and only if for all $x, y\in A^*$ $xuy\in L \Leftrightarrow xvy\in L$. The quotient monoid $M(L) = A^*/P_L$ is called the syntactic monoid of L. The canonical homomorphism from A^* to $M(L)$ is called the syntactic homomorphism of L and we denote the syntactic homomorphism of a language L by η_L. A deterministic finite automaton is a 5-tuple $\mathcal{A} = (Q, A,\delta, i,F)$ where Q is a finite set called set of states, A is the input alphabet, $\delta : Q\times A \to Q$ is called the transition function, $i\in Q$ is called the initial state and $F\subseteq Q$ is called the set of final(terminal) states. Let $\mathcal{A} = (Q, A,\delta, i,F)$ be a deterministic finite automaton. For each $u\in A^*$, the assignment $q\mapsto \delta(q,u)$ defines a function from Q into itself, denoted by μ_u. The monoid generated by $\{\mu_a : a\in A\}$ is called the transition monoid of the automaton, denoted by $\mathrm{Tr}(\mathcal{A}) = \langle \mu_a : a\in A\rangle$ and $\mathrm{Tr}(\mathcal{A})$ is a submonoid of the transformation monoid T_Q. For a transformation $\mu = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$ in T_Q, we denote it by $\mu = (a_1,a_2,\dots,a_n)$ wherever we used in the sequel.

Let $\mathcal{A} = (Q_A, A,\delta_A, i_A,F_A)$ and $\mathcal{B} = (Q_B, A,\delta_B, i_B,F_B)$ be two deterministic finite automata where Q_A and Q_B are disjoint. Define the automaton $\mathcal{C} = \mathcal{A}\cup\mathcal{B}$ as the union of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{C} = (Q_C, A,\delta_C, i_C,F_C)$ where $Q_C = Q_A\cup Q_B$, $i_C = \{i_A, i_B\}$, $F_C = F_A\cup F_B$ and $\delta_C(q_C,a) = \delta_A(q_A,a)$ or $\delta_C(q_C,a) = \delta_B(q_B,a)$ for $a\in A$. The language recognized by the automaton $\mathcal{C}$ is $L(\mathcal{C}) = L(\mathcal{A})\cup L(\mathcal{B})$. Let $\mathcal{A} = (Q, A,\delta, i,F)$ be a deterministic finite automaton. Its complement is denoted by $\mathcal{A}^c$ and defined by $\mathcal{A}^c = (Q, A,\delta, i,Q-F)$, so that $L(\mathcal{A}^c) = A^*-L(\mathcal{A})$. Let $\mathcal{A} = (Q_A, A,\delta_A, q_A,F_A)$ and $\mathcal{B} = (Q_B, A,\delta_B, q_B,F_B)$ be two deterministic finite automata. Define an automaton $\mathcal{C} = \mathcal{A}\times\mathcal{B}$ as the product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{C} = (Q_C, A,\delta_C, q_C,F_C)$ where $Q_C = Q_A\times Q_B$, $q_C = (q_A, q_B)$, $F_C = F_A\times F_B$ and $\delta_C(q_C,a) = (\delta_A(q_A,a),\delta_B(q_B,a))$ for $a\in A$.

We say that a language L is syntactically closed if for all $u,v\in L$, uP_Lv where P_L is the syntactic congruence of L and that L is an elementary language if there exist a monoid M and a morphism $\varphi : A^*\to M$ such that $L = e\varphi^{-1}$ where e is an idempotent in M. In [11] we proved that every idael language is an elementary language. If L is syntactically closed then there is a syntactic morphism $\eta_L : A^* \to M(L)$ such that $L\eta_L$ is singleton. Since L is recognised by $M(L)$ it follows that $L = L\eta_L\eta_L^{-1}$, so that $L = \{m\}\eta_L^{-1}$ for some $m \in M(L)$.

Theorem 2.2 ([10]). *Let $L \subseteq A^*$ be a language. Then L is an elementary language if and only if L is a subsemigroup of A^* and L is syntactically closed.*

Definition 2.3 ([15]). A language $L \subseteq A^*$ is said to be a biordered set language if there exist a monoid M and a surjective homomorphism $\varphi : A^* \to M$ such that $L = E\varphi^{-1}$ where E is a biordered subset of $E(M)$.

Theorem 2.4 ([15]). *Let $L \subseteq A^*$ be a biordered set language, M be a monoid and $\varphi : A^* \to M$ be a surjective morphism such that $L = L\varphi\varphi^{-1}$ and $L\varphi \subseteq E(M)$. Then $L\varphi$ is a biordered subset of $E(M)$.*

3. Elementary Automata

Here we consider a basic type of automata termed as elementary automata which leads to the notion of nice automata. First, we give a charecterisation for syntactically closed regular language.

Proposition 3.1. *Let $L \subseteq A^*$ be a language. Then L is a regular syntactically closed language if and only if there exist a deterministic finite automaton $\mathcal{A}$ accepting L such that $\mu_u = \mu_v$ for all $u, v \in L$ where $\mu_u, \mu_v \in \mathrm{Tr}(\mathcal{A})$.*

Proof. Assume that L is a regular syntactically closed language. Then there exist a finite monoid M and a homomorphism $\varphi : A^* \to M$ such that $L = m_0\varphi^{-1}$ for some $m_0 \in M$. Construct a deterministic finite automaton $\mathcal{A} = (M, A, \delta, 1, \{m_0\})$ where M is the set of states, 1(the identity element in M) is the initial state, $\{m_0\}$ the only final state and $\delta : M \times A^* \to M$ is defined by $\delta(m, w) = m \cdot (w\varphi)$ for $m \in M$ and $w \in A^*$. Then

$$
\begin{aligned}
L(\mathcal{A}) &= \{w \in A^* : \delta(1, w) = m_0\} \\
&= \{w \in A^* : 1 \cdot (w\varphi) = m_0\} \\
&= \{w \in A^* : w\varphi = m_0\} \\
&= m_0\varphi^{-1} = L
\end{aligned}
$$

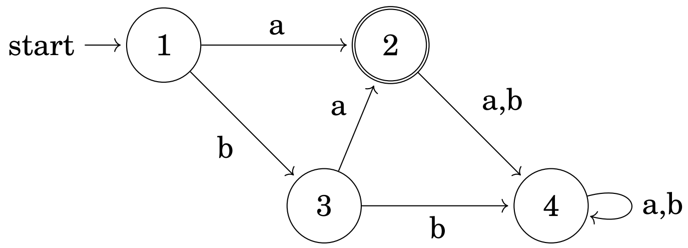
and if $u \in L$, then $u\varphi = m_0$, so that

$$(m)\mu_u = \delta(m, u) = m \cdot (u\varphi) = m \cdot m_0 = m \cdot (v\varphi) = \delta(m, v) = (m)\mu_v$$

for all $v \in L$ and all $m \in M$. Thus $\mu_u = \mu_v$ for all $u, v \in L$.

Conversely, if $\mathcal{A}$ is a deterministic finite automaton accepting L such that $\mu_u = \mu_v$ for all $u, v \in L$, then for $x, y \in A^*$ we have $\mu_{xuy} = \mu_x\mu_u\mu_y = \mu_x\mu_v\mu_y = \mu_{xvy}$, implies that $xuy \in L$ if and only if $xvy \in L$ for all $u, v \in L$. Thus L is syntactically closed. □

It follows from Proposition 3.1 that a deterministic finite automaton $\mathcal{A}$ accepting a syntactically closed regular language has single terminal state. But the converse need not be true in general. For example, consider the deterministic finite automaton $\mathcal{A} = (\{1,2,3,4\}, A, \delta, 1, \{2\})$ over the alphabet $A = \{a,b\}$ with the transition function δ is given by the following transition diagram:



Here $L(\mathcal{A}) = \{a, ba\}$ and note that $ba \in L(\mathcal{A})$, but $bba \notin L(\mathcal{A})$, that is, $L(\mathcal{A})$ is not syntactically closed.

We have seen that a language $L \subseteq A^*$ is an elementary language if $L = e\phi^{-1}$ where $\phi : A^* \to M$ is the monoid morphism and $e \in E(M)$, so that the elementary language is syntactically closed and the Proposition 3.1 ensures that the existence of a deterministic finite automaton which accepts the elementary language. The following result provides a characterization for such automata.

Proposition 3.2. *Let $L \subseteq A^*$ be a language. Then L is an elementary regular language if and only if there exist a deterministic finite automata $\mathcal{A}$ with only one final state q_f(say) accepting L such that the following holds:*

(i) $\mu_u = \mu_v$ *for all* $u, v \in L$,

(ii) $\delta(q_f, u) = q_f$ *for all* $u \in L$.

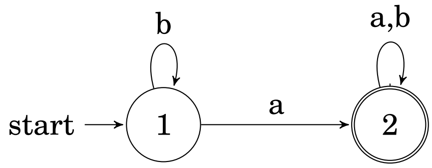
Proof. Assume that L is a elementary regular language, by Theorem 2.2 that L is syntactically closed as well as a subsemigroup of A^*. Since L is syntactically closed, there is an automaton $\mathcal{A}$ with one final state q_f(say) accepting L such that $\mu_u = \mu_v$ for all $u, v \in L$ by Proposition 3.1. Since L is a subsemigroup of A^*, we have $uv \in L$ for all $u, v \in L$. If $v \in L$, then $\delta(q_0, v) = q_f$. So $\delta(q_f, u) = \delta(\delta(q_0, v), u) = \delta(q_0, vu) = q_f$ for all $u \in L$. This is equivalent to the condition $(q_f)\mu_u = q_f$ for all $u \in L$.

Conversely, assume that there is a deterministic finite automaton $\mathcal{A}$ with only one final state q_f accepting L and satisfying (i) and (ii). By Proposition 3.1, L is a syntactically closed language. Also if $u, v \in L$, then

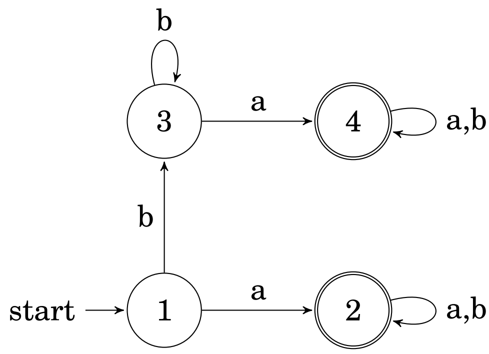
$$\delta(q_0, uv) = \delta(\delta(q_0, u), v)) = \delta(q_f, v) = q_f.$$

So $uv \in L$ and hence L is a subsemigroup of A^*. Thus by Theorem 2.2 we have that L is an elementary language. □

Example 3.3. Let $A = \{a,b\}$ and $\mathcal{A} = (\{1,2\}, A, \delta, 1, \{2\})$ be a deterministic finite automaton with the transition function δ given by the following transition diagram:



Then clearly $L(\mathcal{A}) = b^*aA^*$ and $\delta(1,b^n aw) = \delta(\delta(1,b^n),aw) = \delta(1,aw) = 2$ and $\delta(2,b^n aw) = 2$ for all $w \in A^*$ and $n \in \mathbb{N}$ where $b^n w \in L(\mathcal{A})$. That is, $\mu_u = \mu_v$ for all $u, v \in L(\mathcal{A})$. Also $\delta(2,u) = 2$ for all $u \in L(\mathcal{A})$. Thus $L(\mathcal{A})$ is an elementary language by Proposition 3.2. But we can observe that not every deterministic finite automaton recognizing elementary language have single terminal state. For example consider the deterministic finite automaton $\mathcal{A} = (\{1,2,3,4\}, A, \delta, 1, \{2,4\})$ over the alphabet $A = \{a,b\}$ whose transition function δ is given by the following transition diagram:

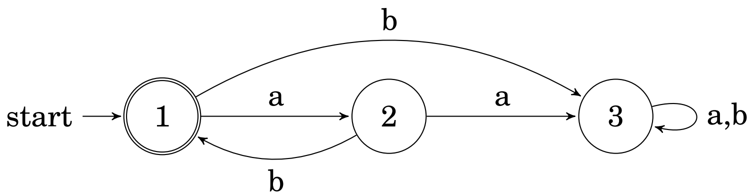


Here we can see that $L(\mathcal{A}) = b^*aA^*$ which is an elementary language, but $\mathcal{A}$ has two terminal states.

The above discussion leads to the following definition.

Definition 3.4. An automaton $\mathcal{A}$ is said to be an elementary automaton if $L(\mathcal{A})$ is an elementary language and $\mathcal{A}$ has only one terminal state.

Example 3.5. Consider a deterministic finite automaton $\mathcal{A} = (\{1,2,3\}, A, \delta, 1, \{1\})$ where $A = \{a,b\}$ with the transition function δ is given by the following transition diagram:



Here $L(\mathcal{A}) = (ab)^*$, that is, $L(\mathcal{A}) = \{w \in A^* : \delta(1,w) = 1\} = \{(1,3,3)\}\eta^{-1}$ where $\eta : A^* \to \mathrm{Tr}(\mathcal{A})$ is the transition homomorphism and we have the transition semigroup as $\mathrm{Tr}(\mathcal{A}) = \{(2,3,3),(3,1,3),(3,3,3),(1,3,3),(3,2,3)\}$, so that $L(\mathcal{A})$ is an elementary language accepted by $\mathcal{A}$ with single terminal state. Thus $\mathcal{A}$ is an elementary automaton.

Remark 3.6. It follows from the definitions that the union of elementary automata need not be an elementary automaton and the complement of an elementary automaton need not be an elementary automaton, but the product of elementary automata is an elementary automaton.

Theorem 3.7. *Minimal automaton accepting an elementary language is an elementary automaton.*

Proof. Let $L \subseteq A^*$ be an elementary language. We see that from Proposition 3.2 that there exist a deterministic finite automaton $\mathcal{A}$ with single terminal state and $L(\mathcal{A}) = L$. Since the minimization of an automaton does not increases the number of terminal states, the minimal automaton of $\mathcal{A}$ is an elementary automaton. □

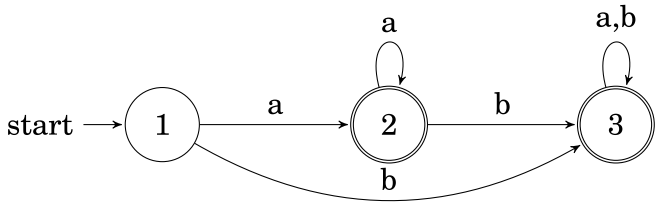
4. Nice Automata

In this section, we consider a class of automata whose transition semigroup contains biordered set, termed as nice automata and discuss its properties and its relation with elementary automata.

Definition 4.1. Let $\mathcal{A} = (Q, A, \delta, i, F)$ be a deterministic finite automaton and let $L(\mathcal{A}) = L$. We say that the automaton $\mathcal{A}$ is a nice automaton if $L\mu$ is a biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$ where $\mu : A^* \to \mathrm{Tr}(\mathcal{A})$ is the transition homomorphism of $\mathcal{A}$ and $\mathrm{Tr}(\mathcal{A})$ is the transition monoid of $\mathcal{A}$.

It follows from the definition of elementary automata that every elementary automaton is a nice automaton.

Example 4.2. Let $A = \{a, b\}$ and let $\mathcal{A} = (\{1,2,3\}, A, \delta, 1, \{2,3\})$ be an automaton with δ given by the following transition diagram:



We can see that $L(\mathcal{A}) = a^+ \cup a^+bA^* \cup bA^*$ and its transition semigroup is $\mathrm{Tr}(\mathcal{A}) = \langle \mu_a, \mu_b \rangle$ where $\mu_a = (2,2,3)$ and $\mu_b = (3,3,3)$. If $w \in L(\mathcal{A})$, then $w = a^n$, $w = a^m bu$ or $w = bu'$ for $n, m \in \mathbb{N}$ and $u, u' \in A^*$, so that $\delta(1, w) = 2$ or $\delta(1, w) = 3$ and if $\eta : A^* \to \mathrm{Tr}(\mathcal{A})$ is the transition homomorphism given by $a \mapsto \mu_a$, $b \mapsto \mu_b$, then

$$\begin{aligned} L(\mathcal{A}) &= \{w \in A^* : \delta(1,w) = 2 \text{ or } \delta(1,w) = 3\} \\ &= \{w \in A^* : \delta(1,w) = 2\} \cup \{w \in A^* : \delta(1,w) = 3\} \\ &= (\mu_a)\eta^{-1} \cup (\mu_b)\eta^{-1} \\ &= \{\mu_a, \mu_b\}\eta^{-1}, \end{aligned}$$

it follows that $L(\mathcal{A})\eta = \{\mu_a, \mu_b\}$ where $\{\mu_a, \mu_b\}$ is a biordered subset of $\mathrm{Tr}(\mathcal{A})$, hence $\mathcal{A}$ is a nice automaton.

Proposition 4.3. *Let $L \subseteq A^*$ be a language. Then L is a regular biordered set language if and only if every deterministic finite automaton accepting L is a nice automaton.*

Proof. Let $L \subseteq A^*$ be a regular biordered set language. Then there is a monoid M and a homomorphism $\phi : A^* \to M$ such that $L = E\phi^{-1}$ where E is a biordered subset of $E(M)$. Then the automaton $\mathcal{A} = (Q, A, \delta, i, F)$ constructed as in the proof of Proposition 3.1, recognizes L, hence $\mathcal{A}$ is a nice automaton.

Conversely, assume that the deterministic finite automaton $\mathcal{A}$ is a nice automaton such that $L(\mathcal{A}) = L$. Then L is regular and $L\eta$ is a biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$. Since $L = (L\eta)\eta^{-1}$, it follows that L is a regular biordered set language. □

Corollary 4.4. *$L \subseteq A^*$ is a regular biordered set language if and only if the minimal automaton of L is a nice automaton.*

Definition 4.5. If M is finite, we call such language L as finitely bsl.

By Kleene theorem [9], we can see that if the language L is finitely bsl then it is a regular biordered set language.

Proposition 4.6. *Let $L \subseteq A^*$ be a language. Then L is finitely bsl if and only if $L\eta_L$ is a finite biordered subset of $E(M(L))$.*

Proof. Suppose that L is finitely bsl, then by Definition 4.5 $L\phi$ is a finite biordered subset of $E(M)$. Let $\eta_L : A^* \to M(L)$ be the syntactic morphism. Since L is regular we have $M(L)$ is finite. Then there exist a morphism $\psi : M \to M(L)$ such that $\eta_L = \phi\psi$. Thus $L\eta_L = L(\phi\psi) = (L\phi)\psi$ is a finite subset of $E(M(L))$ because $L\phi$ is a finite biordered subset of $E(M)$). Hence $L\eta_L$ is a finite biordered subset of $E(M(L))$.

Conversely, let $L\eta_L$ be a finite biordered subset of $E(M(L))$ where $\eta_L : A^* \to M(L)$ be the syntactic morphism. Hence L is a finitely bsl, follows from the Definition 4.5. □

Proposition 4.7. *Let $L \subseteq A^*$ be a language and let $\mathcal{A}$ be a deterministic finite nice automaton such that $L(\mathcal{A}) = L$. Then L is finitely bsl.*

Proof. Suppose that $\mathcal{A}$ is a deterministic finite nice automata such that $L(\mathcal{A}) = L$. So $\mathrm{Tr}(\mathcal{A})$ is finite and $L\eta$ is a finite biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$ where $\eta : A^* \to \mathrm{Tr}(\mathcal{A})$ is the transition homomorphism of $\mathcal{A}$. Then we can find a finite monoid M such that $\psi : \mathrm{Tr}(\mathcal{A}) \to M$ is a morphism. If $\phi : A^* \to M$ is a surjective morphism, then $L\phi = L(\eta\psi^{-1}) = (L\eta)\psi^{-1}$ is a finite biordered subset $E(M)$, hence L is finitely bsl. □

Proposition 4.8. *Let $L \subseteq A^*$ be finitely bsl if and only if $\{\mu_w : w \in L(\mathcal{A})\}$ is a finite biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$ where $\mathcal{A}$ is a nice automaton.*

Proof. Suppose that $L \subseteq A^*$ is finitely bsl. By Proposition 4.6 there exist a deterministic nice automaton $\mathcal{A}$ such that $L(\mathcal{A}) = L$. So in particular $L(\mathcal{A})$ is a regular biordered set language. Since a biordered set language is the union of elementary languages [10], we have that for $u \in L(\mathcal{A})$, then u belongs to some elementary language L' and so $u^2 \in L'$. By Proposition 3.2 we can see that $\mu_{u^2} = \mu_u$. Since $\mu_{u^2} = \mu_u \mu_u$, we have μ_u is an idempotent in $\mathrm{Tr}(\mathcal{A})$ for each $u \in L(\mathcal{A})$, hence $\{\mu_w : w \in L(\mathcal{A})\}$ is a finite subset of $E(\mathrm{Tr}(\mathcal{A}))$. Again, since $L(\mathcal{A}) = \{\mu_w : w \in L(\mathcal{A})\}\eta^{-1}$ where $\eta : A^* \to \mathrm{Tr}(\mathcal{A})$ is the transition homomorphism of $\mathcal{A}$ and $L(\mathcal{A})$ is a biordered set language, it follows from Theorem 2.4 that $\{\mu_w : w \in L(\mathcal{A})\}$ is a biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$.

Conversely, let $\{\mu_w : w \in L(\mathcal{A})\}$ is a finite biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$ where $\mathcal{A}$ is a nice automaton. It follows from Proposition 4.6 we have $L(\mathcal{A}) = L$ is finitely bsl. □

Theorem 4.9. *Let $\mathcal{A}$ be a deterministic finite automaton. Then $\mathcal{A}$ is a nice automaton if and only if $L(\mathcal{A}) = \bigcup_{i=1}^{n} L_i$ for some $n \in \mathbb{N}$ where each L_i is an elementary language with $L_i \cap Ł_j = \emptyset$, $i \neq j$ and $\{\mu_u : u \in L_i, i = 1,2,\dots,n\}$ is a biordered subset of* $\mathrm{Tr}(\mathcal{A})$.

Proof. Let $\mathcal{A}$ be a nice automaton. Then $L(\mathcal{A})$ is a regular language and $\{\mu_w : w \in L(\mathcal{A})\}$ is a finite biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$. Let $\{e_1, e_2, \dots, e_n\}$ be the enumeration of $\{\mu_w : w \in L(\mathcal{A})\}$, we have $L(\mathcal{A}) = \{\mu_w : w \in L(\mathcal{A})\}\eta^{-1} = \{e_1, e_2, \dots, e_n\}\eta^{-1} = \left(\cup_{i=1}^{n} e_i\right)\eta^{-1} = \cup_{i=1}^{n} L_i$, where $L_i = e_i\eta^{-1}$ and $L_i \cap Ł_j = \emptyset$ for $i \neq j$, if not $w \in L_i$ and $w \in L_j$ implies that $e_i = w\eta = e_j$, a contradiction to the fact that $\{e_1, e_2, \dots, e_n\}$ be the enumeration of $\{\mu_w : w \in L(\mathcal{A})\}$. Thus $L(\mathcal{A})$ is the disjoint union of elementary languages.

Conversely, let $L(\mathcal{A})$ be the disjoint union of elementary languages and $\{\mu_u : u \in L_i, i = 1,2,\dots,n\}$ is a biordered subset of $\mathrm{Tr}(\mathcal{A})$. If $\{e_1, e_1, \dots, e_n\}$ is the enumeration of $\{\mu_u : u \in L_i, i = 1,2,\dots,n\}$ then we have, $L(\mathcal{A}) = \{e_1, e_2, \dots, e_n\}\eta^{-1}$. Thus $L(\mathcal{A})$ is finitely bsl, so that $\mathcal{A}$ is a nice automaton by Proposition 4.8. □

Theorem 4.10. *Any automaton accepting finitely bsl is a nice automaton.*

Proof. Let L be a finitely bsl and let $L = L(\mathcal{A})$ for some automaton $\mathcal{A}$. Since L is a regular biordered set language, it follows from Theorem 2.4 that any monoid M recognizing L recognizes it by a biordered subset of $E(M)$. In particular, L is recognized by a finite biordered subset of $E(\mathrm{Tr}(\mathcal{A}))$. Without loss of generality we assume that $E(\mathrm{Tr}(\mathcal{A})) = \{e_1, e_2, \dots, e_m\}$ and if $L_i = (e_i)\eta^{-1}$ for $i = 1,2,\dots,n$, then $L = \bigcup\limits_{i=1}^{m} L_i$ where $\eta : A^* \to \mathrm{Tr}(\mathcal{A})$ is the transition homomorphism of $\mathcal{A}$. Since each L_i is an elementary language, it is syntactically closed and hence by Proposition 3.1 we have $\mu_u = \mu_v$ for all $u, v \in L_i$ and hence it follows from Theorem 4.9 that $\mathcal{A}$ is a nice automaton. □

Theorem 4.11. *Let $\mathcal{A}$ be a nice automaton. Then*

(i) $L(\mathcal{A}) = \bigcup_{i=1}^{n} L_i$ *with* $L_i \cap L_j = \emptyset$ *if* $i \neq j$ *and* $n \in \mathbb{N}$.

(ii) *There exist an elementary automaton* $\mathcal{A} = (Q, A, \delta, i, \{f_i\})$ *accepting* L_i.

(iii) *The automaton* $\mathcal{A}' = (Q, A, \delta, i, \{f_1, f_2, \dots, f_n\})$ *accepts* $L(\mathcal{A})$. *In particular* $\mathcal{A}$ *and* $\mathcal{A}'$ *are equivalent.*

Proof. (i) Suppose $\mathcal{A}$ is a nice automaton. By Proposition 4.7, $L = L(\mathcal{A})$ is finitely bsl. So there exist a finite monoid M and a homorphism $\phi : A^* \to M$ such that $L(\mathcal{A}) = E\phi^{-1}$ where E is a biordered subset of $E(M)$. If $E = \{f_1, f_2, \dots, f_n\}$, then as in the proof of Theorem 4.9 we have $L(\mathcal{A}) = \bigcup_{i=1}^{n} L_i$ where $L_i \cap L_j = \emptyset$ if $i \neq j$ and $L_i = f_i\phi^{-1}$ for $i = 1, 2, \dots, n$.

(ii) Let $\mathcal{A}_i = (Q, A, \delta, i, \{f_i\})$ be an automaton with $Q = M$, $i = 1$ (the identity in M) and $\delta : M \times A^* \to M$ be defined by $\delta(m, w) = m \cdot (w\varphi)$. Then $\mathcal{A}_i$ is an elementary automaton and $L(\mathcal{A}_i) = L_i$ for each $i = 1, 2, \dots, n$.

(iii) We have

$$\begin{aligned} L(\mathcal{A}') &= \{w \in A^* : \delta(1, w) \in \{f_1, f_2, \dots, f_n\}\} \\ &= \{w \in A^* : 1 \cdot (w\varphi) \in \{f_1, f_2, \dots, f_n\}\} \\ &= \{w \in A^* : w\varphi \in \{f_1, f_2, \dots, f_n\}\} \\ &= \{w \in A^* : w\varphi \in E\} = E\varphi^{-1} = L(\mathcal{A}) \end{aligned}$$

Since $L(\mathcal{A}') = L(\mathcal{A})$, both $\mathcal{A}$ and $\mathcal{A}'$ are equivalent. □

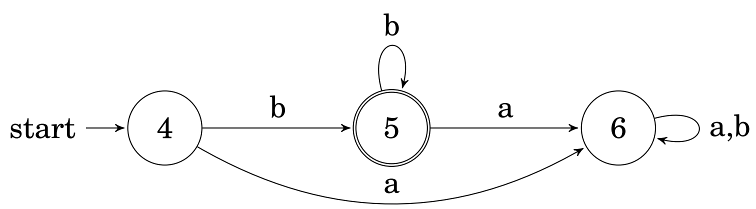
From the above theorem, we see that $\mathcal{A}' = \bigcup_{i=1}^{n} \mathcal{A}_i$ and that $\mathcal{A}'$ is equivalent to $\mathcal{A}$. We call this decomposition as the elementary decomposition of the nice automaton $\mathcal{A}$. But Theorem 4.9 shows that every nice automaton admits an elementary decomposition.

5. Operations on Nice Automata

Here we consider the operations on deterministic finite nice automata such as their union, complement and product.

5.1 Union

Consider the nice automaton $\mathcal{A} = (\{1,2,3\}, \{a, b\}, \delta, 1, \{1\})$ given in Example 3.5 and $\mathcal{B} = (\{4,5,6\}, \{a, b\}, \delta', 4, \{5\})$ be a deterministic finite automata with transition function δ' is given by the following transition diagram:



We can easily see that $\mathcal{B}$ is a nice automata. Then their union $\mathcal{A}\cup\mathcal{B}$ call it as $\mathcal{C}$ where $\mathcal{C}=(\{1,2,3,4,5,6\},\{a,b\},\Delta,\{1,4\},\{1,5\})$ is a non deterministic finite automata with transition semigroup $\mathrm{Tr}(\mathcal{C})=\langle(2,2,3,6,6,6),(3,1,3,5,5,6)\rangle$ and $L(\mathcal{C})=(ab)^+\cup b^+$. Let $\mathcal{D}$ be its equivalent deterministic finite automata with

$$\begin{aligned} L(\mathcal{D}) &= L(\mathcal{C}) \\ &= \{w\in A^*:\delta(1,w)=1,\delta(1,w)=5,\delta(4,w)=1 \text{ or } \delta(4,w)=5\} \\ &= \{(3,1,3,5,5,6),(2,3,3,6,6,6),(3,3,3,5,5,6)\}\eta^{-1} \end{aligned}$$

where $\eta:A^*\to\mathrm{Tr}(\mathcal{D})$ is the transition homomorphism. We can see that the set $\{(3,1,3,5,5,6),(2,3,3,6,6,6),(3,3,3,5,5,6)\}$ is not a subset of $E(\mathrm{Tr}(\mathcal{D}))$ due to $(3,1,3,5,5,6)$ not being an idempotent, so that $\mathcal{C}$ is not a nice automaton. Thus the operation union is not closed in the class of nice automata.

5.2 Complement

Consider the nice automaton $\mathcal{A}$ given in Example 3.5, then its complement is given by the automaton $\mathcal{A}^c=(\{1,2,3\},\{a,b\},\delta,1,\{2,3\})$ and the respective transition semigroup is $\mathrm{Tr}(\mathcal{A}^c)=\mathrm{Tr}(\mathcal{A})=\{(2,3,3),(3,1,3),(3,3,3),(1,3,3),(3,2,3)\}$. But

$$\begin{aligned} L(\mathcal{A}^c) &= \{w\in A^*:\delta(1,w)=2,\delta(1,w)=3\} \\ &= \{(2,3,3),(3,1,3),(3,3,3),(3,2,3)\}\eta^{-1} \\ &= (\mathrm{Tr}(\mathcal{A})-\{(1,3,3)\})\eta^{-1} \end{aligned}$$

where $\eta:A^*\to\mathrm{Tr}(\mathcal{A}^c)$ is the transition homomorphism and we can see that the set $\{\mathrm{Tr}(\mathcal{A})-\{(1,3,3)\}\}$ is not a biordered subset of $E(\mathrm{Tr}(\mathcal{A}^c))$ and so $\mathcal{A}^c$ is not a nice automaton. Thus the operation complement is not closed in the class of nice automata.

5.3 Product

Let $\mathcal{A}_1=(Q_1,A,\delta_1,i_1,F_1)$ and $\mathcal{A}_2=(Q_2,A,\delta_2,i_2,F_2)$ be two deterministic finite nice automata over the alphabet A. Then $L(\mathcal{A}_1)$ is finitely bsl, $\{\mu_w^{\mathcal{A}_1}:w\in L(\mathcal{A}_1)\}$ is a biordered subset of $E(\mathrm{Tr}(\mathcal{A}_1))$ and $L(\mathcal{A}_2)$ is also finitely bsl, $\{\mu_w^{\mathcal{A}_2}:w\in L(\mathcal{A}_2)\}$ is a biordered subset of $E(\mathrm{Tr}(\mathcal{A}_2))$. The product automaton is $\mathcal{C}=(Q_1\times Q_2,A,\{i_1,i_2\},\delta,F_1\times F_2)$ where $\delta\big((q_1,q_2),a\big)=\big(\delta_1(q_1,a),\delta_2(q_2,a)\big)$ for $(q_1,q_2)\in Q_1\times Q_2$ and $a\in A$. Here $L(\mathcal{C})$ is regular since

$$\begin{aligned} L(\mathcal{C}) &= \{w\in A^*:\delta\big((i_1,i_2),a\big)\in F_1\times F_2\} \\ &= \{w\in A^*:\delta_1(i_1,w)\in F_1 \text{ and } \delta_2(q_2,w)\in F_2\} \\ &= \{w\in A^*:w\in L(\mathcal{A}_1) \text{ and } w\in L(\mathcal{A}_2)\} \\ &= L(\mathcal{A}_1)\cap L(\mathcal{A}_2) \end{aligned}$$

and

$$\begin{aligned} \{\mu_w^{\mathcal{C}}:w\in L(\mathcal{C})\} &= \{\mu_w^{\mathcal{C}}:w\in L(\mathcal{A}_1)\cap L(\mathcal{A}_2)\} \\ &= \{(\mu_w^{\mathcal{A}_1},\mu_w^{\mathcal{A}_2}):w\in L(\mathcal{A}_1)\cap L(\mathcal{A}_2)\} \\ &= \{\mu_w^{\mathcal{A}_1}:w\in L(\mathcal{A}_1)\}\times\{\mu_w^{\mathcal{A}_2}:w\in L(\mathcal{A}_2)\} \end{aligned}$$

is a finite biordered subset of $E(\mathrm{Tr}(\mathcal{B}_1)) \times E(\mathrm{Tr}(\mathcal{B}_2)) = E\big(\mathrm{Tr}(\mathcal{B}_1) \times \mathrm{Tr}(\mathcal{B}_2)\big) \equiv E\big(\mathrm{Tr}(\mathcal{C})\big)$. Thus $\mathcal{C}$ is a nice automaton, hence the class of nice automata is closed under the operation product. We summarize these facts in the following theorem.

Theorem 5.1. *The class of nice automata is closed under the operation product.*

6. Conclusion

The equivalence between finite automata and recognisable languages is well-known. The concept of nice automata is introduced in this paper and it is equivalent to the class of biordered set languages. Nice automaton has a decomposition into elementary automata and the class of nice automata is closed only under product but not under union and complement. Since the class of nice automata contains several interesting class of automata such as the class of syncronizing automata, its study has several applications in theoretical computer science.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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