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Generalized f-semiperfect Modules

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Abstract In this paper we introduce generalized f-semiperfect modules as a generalization of the generalized semiperfect modules. We give various properties of the generalized f-semiperfect modules. We show that: (i) every generalized f-semiperfect module over a regular ring is f-semiperfect; (ii) for small or for finitely generated submodules L of M, the factor module $\frac{M}{L}$ is generalized f-semiperfect; (iii) If M is a projective and generalized f-semiperfect module such that every Rad-supplement in M is a direct summand of M, then every direct summand of M is a generalized f-semiperfect module; (iv) If $M = \oplus_{i\in I} M_i$ is a locally Noetherian and duo module such that $\{M_i\}_{i\in I}$ is the family of generalized f-semiperfect modules, then M is a generalized f-semiperfect module.

1. Introduction

Throughout this paper R is an associative ring with identity and all modules are unital left R-modules. Let M be an R-module. A submodule N of M is called *small* in M, written as $N \ll M$, if for every submodule N of M the equality $N + K = M$ implies that $K = M$. Let M be an R-module and let N and K be any submodules of M. K is called a *supplement* of N in M if K is minimal with respect to $N+K = M$. K is a supplement of N in M if and only if $N+K = M$ and $N\cap K \ll K$ [15]. M is called *(f-) supplemented* if every (finitely generated) submodule of M has a supplement in M (see [15]). On the other hand, M is called *amply supplemented* if, for any submodules N and K of M with $M = N + K$, K contains a supplement of N in M. Accordingly a module M is called *amply f-supplemented* if every finitely generated submodule of M satisfies same condition. It is clear that (amply) f-supplemented modules are a proper generalization of (amply) supplemented modules.

A module M is called *semilocal* if $\frac{M}{Rad(M)}$ is semisimple and a ring R is called *semilocal* if the left R-module R is semilocal [9].

Let M be an R-module. If $N, K \leq M$, $M = N + K$ and $N \cap K \subseteq Rad(K)$, then K is called Rad*-supplement* of N [5] (according to [14], generalized supplement).

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It is clear that every supplement is Rad-supplement. M is called *Rad-supplemented* (according to [14], generalized supplemented) if every submodule of M has a Rad-supplement in M, and M is called *amply Rad-supplemented* if, for any submodules N and K of M with $M = N + K$, K contains a Rad-supplement of N in M.

In [11], we introduce (amply) f-Rad-supplemented modules as a proper generalization of (amply) Rad-supplemented modules. An R-module M is called *f-Rad-supplemented* if every finitely generated submodule of M has a Rad-supplement in M, and a module M is called *amply f-Rad-supplemented* if every finitely generated submodule of M has ample Rad-supplements in M.

Let $f : P \to M$ be an epimorphism. If $Ker(f) \ll P$, then f is called *cover*, and if P is a projective module, then a cover f is called a *projective cover* [15]. Xue [16] calls f a *generalized cover* if $Ker(f) \leq Rad(P)$, and calls a generalized cover f a *generalized projective cover* if P is a projective module. In the spirit of [16], a module M is said to be *(generalized) semiperfect* if every factor module of M has a (generalized) projective cover. A module M is said to be *f-semiperfect* if, for every finitely generated submodule $U \leq M$, the factor module $\frac{M}{U}$ has a projective cover in M [15].

For the basic properties of (generalized) semiperfect modules we refer the reader to [15] and [16].

The main purpose of this paper is to develop various properties of generalized f-semiperfect modules as a proper generalization of generalized semiperfect modules. Firstly we qualify a notion of generalized f-semiperfect modules by (amply) f-Rad-supplemented modules. We show that every generalized f-semiperfect module over a left Bass ring is f-semiperfect. We obtain that a notion of generalized f-semiperfect modules coincides with a notion of f-semiperfect modules over a regular ring. We obtain that for small or finitely generated submodules of a generalized f-semiperfect module, the factor module is generalized f-semiperfect. Further we show that if every Rad-supplement in a projective module M is a direct summand, then the module M is generalized f-semiperfect if and only if every direct summand of M is generalized f-semiperfect. Furthermore we prove that a direct sum of generalized f-semiperfect modules which is a locally Noetherian duo module is generalized f-semiperfect.

2. **Generalized f-Semiperfect Modules**

Definition 2.1. Let M be an R-module. M is called *generalized f-semiperfect module* if, for every finitely generated submodule $U \leq M$, the factor module $\frac{M}{U}$ has a generalized projective cover in M.

Theorem 2.2. *Let M be an R-module and U be a finitely generated submodule of M. The following statements are equivalent.*

(i) *M is a generalized f-semiperfect module.*

(ii) *For every submodule V of M such that $M = U + V$, there exists a Rad-supplement of U, the Rad-supplement submodule contained in V and it has a generalized projective cover.*

(iii) *U has a Rad-supplement which has a generalized projective cover.*

Proof. It follows from [16, Proposition 2.1]. □

Corollary 2.3. *Let M be an R-module. The following statements are equivalent.*

(i) *M is a generalized f-semiperfect module.*

(ii) *M is amply f-Rad-supplemented by Rad-supplements which have generalized projective covers.*

(iii) *M is f-Rad-supplemented by Rad-supplements which have generalized projective covers.*

We call a module M *finitely Rad-⊕-supplemented* (or briefly *f-Rad-⊕-supplemented*) if every finitely generated submodule of M has a Rad-supplement that is a direct summand of M.

Example 2.4. Let M be a radical module. Then M is a f-Rad-⊕-supplemented module.

Proposition 2.5. *Let M be a f-Rad-⊕-supplemented module. If $Rad(M) \ll M$, then M is a finitely ⊕-supplemented module.*

Proof. Let N be a finitely generated submodule of M. Since M is a f-Rad-⊕-supplemented module, there exist submodules L and L' of M such that $M = N + L$, $N \cap L \subseteq Rad(L)$ and $M = L \oplus L'$. It follows that $N \cap L \ll M$. Since L is a direct summand of M, we have $N \cap L \ll L$. Thus M is a finitely ⊕-supplemented module. □

It is clear that f-Rad-⊕-supplemented modules are a proper generalization of generalized ⊕-supplemented (or Rad-⊕-supplemented) modules. Any f-Rad-⊕-supplemented module need not be Rad-⊕-supplemented as the following example shows.

Example 2.6. Let R be a regular ring not semisimple. Then the R-module R is f-Rad-⊕-supplemented but not Rad-⊕-supplemented.

Proposition 2.7. *Let M be a f-Rad-supplemented module such that $\frac{M}{Soc(M)}$ does not contain a maximal submodule. Then M is a f-Rad-⊕-supplemented module.*

Proof. Let N be a finitely generated submodule of M. Since M is a f-Rad-supplemented module, there exists a submodule K of M such that $M = N + K$, $N \cap K \subseteq Rad(K)$. It is clear that $\frac{M}{K}$ is finitely generated. Therefore, K is a cofinite submodule of M. By ([1, Lemma 2.7]), K is a direct summand of M. Thus M is a f-Rad-⊕-supplemented module. □

Recall from [5] that a ring R is called *left Bass* if every non-zero R-module has a maximal submodule. It is known that R is left Bass if and only if $Rad(M) \ll M$ for every non-zero R-module M.

Proposition 2.8. *Every generalized f-semiperfect module over a left Bass ring is f-semiperfect.*

Proof. Let N be a finitely generated submodule of M. By the hypothesis, $\frac{M}{N}$ has a generalized projective cover, say $\Phi : P \to \frac{M}{N}$. Since $Ker\Phi \subseteq Rad(P) \ll P$, Φ is a projective cover of $\frac{M}{N}$. Therefore M is f-semiperfect. □

A ring R is called *regular* if every finitely generated left ideal of R is a direct summand of R ([15, 3.10]). It is well known that a regular ring has zero radical.

Proposition 2.9. *Let R be a regular ring. If M is a generalized f-semiperfect R-module, then M is f-semiperfect module.*

Proof. Let U be a finitely generated submodule of M. By the hypothesis, $\frac{M}{U}$ has a generalized projective cover. Then there exists a generalized cover $f : P \to \frac{M}{U}$ such that P is projective. Note that $Rad(P) = Rad(R)P$ by ([8, Theorem 9.2.1]). Since R is regular, $Rad(R) = 0$. We have $Kerf \ll P$. Hence f is a projective cover. Thus the assertion holds. □

A ring R is called a *left V-ring* if every simple left R-module is injective. It is well known that R is a left V-ring if and only if, for every left R-module M, $Rad(M) = 0$.

Corollary 2.10. *Let R be a left V-ring and let M be an R-module. Then the following statements are equivalent.*

(i) *M is a generalized f-semiperfect module.*
(ii) *M is a f-semiperfect module.*

Proof. (i)⇒(ii) Let M be a generalized f-semiperfect module. By Corollary 2.3, M is f-Rad-supplemented by Rad-supplements which have generalized projective covers. It follows that M is f-supplemented by ([11, Corollary 3.10]). Let U be a finitely generated submodule of M. Suppose that V be any supplement submodule of U in M. Then V has a generalized projective cover $\varphi : P \to V$ with a projective module P. By the hypothesis, $Rad(P) = 0$. Thus $Ker\varphi \ll P$. It follows that $\varphi : P \to V$ is a projective cover. It can be seen analogously like for ([16, Proposition 2.1]) that M is f-semiperfect.

(ii)⇒(i) Clear. □

The following lemma states as a generalization of the well known simple fact for projective covers.

Lemma 2.11. *Let $\varphi : P \to M$ be a cover. Then M has a generalized projective cover if and only if P has a generalized projective cover.*

Proof. ($\Rightarrow$) Let $f : K \to M$ be a generalized projective cover of M. Since K is a projective module, there exists a homomorphism $g : K \to P$ such that $\varphi \circ g = f$. Since f is an epimorphism, $\varphi \circ g = f$ and φ is a cover, then g is an epimorphism. In addition $Ker g \subseteq Ker \varphi \circ g = Ker f \subseteq Rad(K)$. Thus P has a generalized projective cover $g : K \to P$.

($\Leftarrow$) Let $h : V \to P$ be a generalized projective cover of P. Since $\varphi : P \to M$ is a cover, φ is a generalized cover. By ([16, Lemma 1.1]), $\varphi \circ h : V \to M$ is a generalized cover of M. In addition V is a projective module. So $\varphi \circ h$ is a generalized projective cover of M, i.e. M has a generalized projective cover. □

Theorem 2.12. *Let M be a generalized f-semiperfect module. Then the factor module $\frac{M}{L}$ is generalized f-semiperfect for small or for finitely generated submodules L of M.*

Proof. Let L, K be submodules of M, K finitely generated. We have $\frac{\frac{M}{L}}{\frac{(L+K)}{L}} \cong \frac{M}{(L+K)}$. If L is finitely generated, $L + K$ is finitely generated. Since M is a generalized f-semiperfect module, $\frac{M}{(L+K)}$ has a generalized projective cover. So $\frac{\frac{M}{L}}{\frac{(L+K)}{L}}$ has a generalized projective cover. Thus $\frac{M}{L}$ is a generalized f-semiperfect module.

If $L \ll M$, then $\frac{(L+K)}{K} \ll \frac{M}{K}$. Since $\frac{M}{K} \to \frac{M}{(L+K)}$ is an epimorphism and $Ker\varphi = \frac{(L+K)}{K} \ll \frac{M}{K}$, φ is a small cover. By Lemma 2.11, the generalized projective cover of $\frac{M}{K}$ yields a generalized projective cover of $\frac{M}{(L+K)}$. Since K is finitely generated, $\frac{M}{K}$ has a generalized projective cover by the hypothesis. So $\frac{M}{(L+K)} \cong \frac{\frac{M}{L}}{\frac{(L+K)}{L}}$ has a generalized projective cover. Thus $\frac{M}{L}$ is a generalized f-semiperfect module. □

Corollary 2.13. *Let M be a generalized f-semiperfect module and let U be a finitely generated submodule of M such that $M = U \oplus V$. Then V is a generalized f-semiperfect module.*

Let M be an R-module. We consider the following condition.

(D3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M. By ([10, Lemma 4.6, Proposition 4.38]), every projective module has (D3).

Theorem 2.14. *Let M be a f-Rad-$\oplus$-supplemented module with* (D3). *Then every direct summand of M is a f-Rad-$\oplus$-supplemented module.*

Proof. Let N be a direct summand of M and let K be a finitely generated submodule of U. Since M is a f-Rad-$\oplus$-supplemented module, there exist submodules L and L' of M such that $M = K + L$, $K \cap L \subseteq Rad(L)$ and $M = L \oplus L'$. It follows that $N = K + (N \cap L)$ and $M = N + L$. Note that $K \cap (N \cap L) \subseteq Rad(N \cap L)$. Since M satisfies (D3), $N \cap L$ is a direct summand of M. Then there exists a submodule X of M such that $M = (N \cap L) \oplus X$. It follows that $N = (N \cap L) \oplus (N \cap X)$. Therefore N is a f-Rad-$\oplus$-supplemented module. □

Corollary 2.15. *Let M be an UC-extending module. If M is a f-Rad-$\oplus$-supplemented module, every direct summand of M is a f-Rad-$\oplus$-supplemented module.*

Theorem 2.16. *Let M be a projective module. If every Rad-supplement in M is a direct summand, then the module M is generalized f-semiperfect if and only if every direct summand of M is generalized f-semiperfect.*

Proof. Let U be a direct summand of M and K be a finitely generated submodule of U. Since M is a generalized f-semiperfect module, for every submodule L of M such that $M = K + L$, there exists a submodule L' of L with $M = K + L'$, $K \cap L' \subseteq Rad(L')$ by Theorem 2.2. By hypothesis, L' is a direct summand of M. It is clear that $U \cap L'$ is a direct summand of M. Then $U = K + (U \cap L')$ and $K \cap (U \cap L') \subseteq Rad(U \cap L')$. Thus $U \cap L'$ is a Rad-supplement of K in U. It is clear that $U \cap L'$ is projective by ([15, 18.1]). We define an identity isomorphism $I : U \cap L' \to U \cap L'$. It follows that $U \cap L'$ has a generalized projective cover. By Theorem 2.2, U is a generalized f-semiperfect module. The converse is clear. □

Theorem 2.17. *Let M be an R-module and $M = M_1 \oplus M_2$ such that M_1, M_2 are finitely generated and f-Rad-$\oplus$-supplemented modules. If M is a quasi projective module, M is a f-Rad-$\oplus$-supplemented module.*

Corollary 2.18. *A finite direct sum of f-Rad-$\oplus$-supplemented modules which are finitely generated and projective is a f-Rad-$\oplus$-supplemented module.*

Recall from [14] that a module M is called *locally Noetherian* if every finitely generated submodule is Noetherian. And recall from [15] that a submodule N of an R-module M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism of M. An R-module M is called *duo module* if every submodule of M is fully invariant.

Theorem 2.19. *Let $\{M_i\}_{i\in I}$ be the family of generalized f-semiperfect modules. If $M = \oplus_{i\in I} M_i$ is a locally Noetherian and duo module, then M is a generalized f-semiperfect module.*

Proof. Let U be a finitely generated submodule of M. Since M is locally noetherian, U is noetherian. Then $U \cap M_i$ is finitely generated. It follows that $\frac{M_i}{(U\cap M_i)}$ has a generalized projective cover for every $i \in I$. Since $f_i : V_i \to \frac{M_i}{(U\cap M_i)}$ is a generalized cover, $\oplus f_i : \oplus_{i\in I} V_i \to \oplus_{i\in I} \frac{M_i}{(U\cap M_i)}$ is a generalized cover by ([16, Lemma 1.2(2)]). It is clear that $\oplus_{i\in I} V_i$ is projective by ([15, 18.1]). Thus $\oplus f_i$ is a generalized projective cover. Since M is a duo module, then $U = \oplus_{i\in I}(U \cap M_i)$. Note that $\oplus_{i\in I} \frac{M_i}{(U\cap M_i)} \cong \frac{M}{U}$. Therefore M is a generalized f-semiperfect module. □

It is clear that every generalized semiperfect module is generalized f-semiperfect. But the converse is not always true as the following example shows this situation.

Example 2.20 (See [2]). Let F be any field. Consider the commutative ring R which is the direct product $\prod_{i=0}^{\infty} F_i$, where $F_i = F$. So R is a regular ring which is not semisimple. The left R-module R is f-Rad-supplemented but not Rad-supplemented. Let U be a finitely generated submodule of R and V be a Rad-supplement of U in R. Since R is regular, $R = U \oplus V$. It follows that left R-module R is generalized f-semiperfect by Corollary 2.3. Since R is not Rad-supplemented, R is not generalized semiperfect by ([16, Proposition 2.1]).

Proposition 2.21. *Let M be a Noetherian R-module.Then M is a generalized f-semiperfect R-module if and only if M is generalized semiperfect module.*

Proof. Since every submodule of M is finitely generated, the proof is clear. □

Theorem 2.22. *Let M be a finitely generated and projective module. If $\frac{M}{Soc(M)}$ does not contain a maximal submodule. Then the following statements are equivalent.*

(i) *M is a generalized f-semiperfect module.*
(ii) *M is a f-Rad-supplemented module.*
(iii) *M is a f-Rad-$\oplus$-supplemented module.*

Proof. (i)⇒(ii) It is clear by Corollary 2.3.

(ii)⇒(i) Let U be a finitely generated submodule of M. By the hypothesis, there exist submodules V and V' of M such that $M = U + V$, $U \cap V \subseteq Rad(V)$ and $M = V \oplus V'$. We have $\frac{M}{U} \cong \frac{V}{(U \cap V)}$. Since M is projective, then V is projective ([15, 18.1]). Then $\frac{M}{U}$ has a generalized projective cover $\varphi : V \to \frac{M}{U}$ with a projective module V. Since $\frac{M}{U}$ has a generalized projective cover, M is a generalized f-semiperfect module.

(ii)⇔(iii) It follows from Proposition 2.7. □

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